Note: This rendition of the Elements is only a progress report. I still have years of work on it to do. However, one of the main objects of this report is in finding me a wife. I need help. Even if she only cooks, cleans house, and runs interference with the social environment—it would still be of great help. Sadly, I am currently 63.

Self Note: No rants, not even mentions of linguistic failure in man today. It is not conductive nor relevant.

I have three volumes, Glyphs 1, Glyphs 2, and Glyphs 3, each corresponding to the number of variables in the equations. Each broken down into 4 chapters. Each work of glyphs over 1000 demonstrations it appears. Then I have to go back over all of it, for the first pass is exploratory, to study how things can be used in problem solving.

AUL (A Universal Language) Revision Started 2012_0916. Have also been engaged in other projects, particularly the audio-book project.

The Elements.

This work is not a replacement for the ***Elements*** of Euclid—in fact, in order to comprehend the origins of this work, one should be familiar with that work. I will, however, make as plainly as possible the concept behind the ***Elements*** of Euclid, which is also the foundation of the Platonic dialogs, in the works of Aristotle, and even certain metaphors in the Judeo-Christian Scripture. The most available source of the doctrine is in a simple definition of "thing" and is exampled even in living biology.

The human mind is one of a group of environmental acquisition systems of a living organism. It has a specific job it is evolving to accomplish and specific means of doing that job. The means is language. There are two, and only two, primitive branches of reasoning, Logics and Analogics which is derived from a simple binary distinction—"is" and "is not" and even Absolute and Relative. Eventually one will come to understand that the more correct so called theory of Relativity was not presented by Einstein, but by Euclid.

Whenever one hears of ancient beliefs in what the elements were once considered to be, one hears of earth, air, fire, and water. Why, when presented with two theories, only the one a moron would believe in becomes the one exampled is beyond me. There was another competing belief at the time which was in fact, ahead of the thinking even of today. It was a Two-Element Metaphysics. Yet it would be even safe to say that the Two-Element Metaphysics is intimated in some of the metaphors of the Judeo-Christian Scripture. The "two-witnesses" of "God" and even here,

> John 1:1 In the beginning was the Word, and the Word was with God, and the Word was God.

"was God," and "with God." If one has had an introduction to Set Theory, the same concept is expressed in another way, there are two and only two methods of constructing a set, enumeration and definition. Form and material, shape and that which is within the shape.

One might also see the Two-Element Metaphysics being referenced in such obscure passages as,

> "And he said, Hear now my words: If there be a prophet among you, I the LORD will make myself known unto him in a vision, and will speak unto him in a dream."

There are two, and only two elements of every thing, form and the material in that form—and if we are ever to do our job as mind, we must become proficient in both classes of reasoning, for Logics are derived from forms, and Analogics from material. It will turn out that for most of the history of philosophy, mankind was wrong in believing that Logics are superior to Analogics. One will come to understand that Logics are particular, where Analogics are Universal. And eventually, one will stop wondering why what men call "God" seems to communicate in a manner we find either odd, or strange, or even mad. These "visions" are analogics—a universal ways of communicating. They do not demand a particular response, but how we response reveals us at our very core. It is not that these visions are an inferior method of communicating, they are in fact far superior. It is, however, my belief, that if one cannot master Logic, they cannot master

Analogic. The defect in mental ability will be expressed in both branches—simultaneously. The defect is actually due to our position on a scale of evolution of the mind itself.

In a small part of this work, I will outline the Elements as expressed in both Logics and Analogics. The majority of this work is for a simple foundation for the study of the Logic of simple Algebra and the application of Geometry focused on proportion. One can look at the various demonstrations as "propositions," as short stories, as a dictionary of this Analogical glyph language, a dictionary of plug in functions for the language or as a method of curing insomnia. For me, I hope it is a workbook. The first pass is mainly exploratory. The second pass I hope to make will explore using what was learned for applications in problem solving.

Physical Foundation.

The most universal foundation of the Two-Element Metaphysics is contained in the definition of anything. Everything has material within some boundary. One can simply say that a thing is some container of something contained. One can call the material one element, and the boundary the other element. Everything is comprised of these two elements. It does not matter how big or how small these things are. At the foundation of true language competence is manipulating these two facts in reasoning itself. Understanding is not in the claim, no matter how fancy the words, or how involved they are, that contradiction is diction—Einsteinian gibberish is still gibberish.

One of the most important things to be able to grasp and keep in mind is that neither a things form, nor a things material, is a thing. To make a thing, these two elements must both be present—some difference, some material, in some form. This is as simple as it gets—it is at the foundation of all forms of reasoning.

Biological Foundation.

Environmental Acquisition Systems

Every living organism survives by crafting things it needs for survival from its environment. Even the act of feeding is, itself, a crafting act.

Definition: An environmental acquisition system of a living organism is that system of an organism which must acquire something from the environment and process that which it has acquired for a product that maintains and promotes the life of that organism.

Those Systems that Acquire Material.

1) The Digestive-System.
2) The Manipulative-System.
3) The Respiratory-System.

Those Systems that Acquire Form.

4) The Ocular-System.
5) The Vestibular-System.
6) The Procreative-System.
7) The Judgmental-System.

Complement.

A thing and its two elements is the foundation of our biology, and of all craft, and language is a craft. We make things by the manipulation of material and form. If a system abstracts material from a thing, then that

system must impose a new form on that material in order to make some thing. If a system abstracts form from a thing, that system must supply material to that form in order to make some thing. Language is then divided into these two primitive groups—Logics and Analogics.

Logics have forms as a given. Those forms are words, names, numbers, symbols, and the material supplied to those forms make things. If the material is not present, then we do not know what the symbols, the names, the words, *make*. It is traditional to use the word *mean*, when I just used *make*. *Mean*, however, is not the correct word. The material for Logics are experiences. Without experience—which comes from objective reality, the words cannot possibly make anything, but what is needed to be made, by definition of an environmental acquisition system is behavior that maintains and promotes the life of the body. The preceding is a critique of current educational systems. It is typical for those who are mentally defective to deliberately play a shell game with the experiences that are to be contained within the form of any particular word. One can contrast this with someone like Confucius, Plato, Christ, etc, persons who advocated that consistency, standards, were the only way to make these systems functional. It is on this level that one can see who promotes and who actually demotes functionality and thus our ability to do our job.

Analogics have material as a given. Those materials must have forms imposed upon them to make things. One of the most easily accessible analogic is that which is called geometry. The Language of Geometry is the figure itself. Thus whatever logic that is paired with geometry is determined by the subset of geometry targeted. In my work, I use elementary algebra. Analogics only require the assertion of boundaries, which are not, in of themselves differences. One can denote this simply as “A point is that which has no part.” but if the concept of the Two-Element Metaphysics does not reside within the mind, one cannot comprehend even that simple statement. One can note this lack of comprehension—this lack of linguisitic ability, when one hears such things as “a line is composed of an infinite number of points.” etc, in other words, the smallest ball bearings we can imagine. As if one can wave a knife in the air an infinite number of times and make a salad.

Another piece of rubbish is the claim that at some mythical place called infinity, parallel lines meet. Now in a two dimensional system, one only has two, and only two differences. How one conceives of them, orientation-wise, is irrelevant. Thus one can say, at some mythical place in a two-dimensional system, one dimension disappears. So why not length? Why not say that at some mythical place called infinity, length disappears and I am there now, unable to side-step the ax coming down on my own head? So many things spoken as gospel only denote not a greater linguistic ability, but a lesser one. Take another self-referential fallacy, the geometry of a sphere, etc. One must first postulate a three dimensional object in order to claim that two dimensional operations are impossible. What rubbish. Fools speak all the time, however, one should learn when to call them fools and when to leave them to their madness. Words cannot cure them.

Fallacies result in thought when concepts are not abstracted. Displacement occurs and concepts are replaced by what one does, not by what is understood. Many believe that a line is that which is drawn, when in fact, what we scribble may be called linear functions, but not lines.

Logics require standards of experience in order to create a language. Those experiences are the material encapsulated if you will, in those forms before they can make a thing. That thing one can call understanding. In Logics, the material for any form differs in accordance with a persons experiences. Logics are therefore particular ultimately to the individual unless those experiences are standardized. Standardization of experience for words do take place, for example, standards of weights and measures is one system of standardizing logic systems. Common grammars are logic systems.

Analogics require standards for the application of form in order create a language. In traditional geometry, that standard has been straight edge and compass. One can expand this to include any instrument that

renders one, and only one difference between two points. In so doing, one will note that it sets the same standard for a language that simple arithmetic does—a unit standard. One may also note, that without this concept, the geometry I posit here would not be possible. One may see that what I have done here is bring into fruition something that has been sought for since almost the creation of geometry itself. A standardized language for doing what is called mathematics.

I did not do this work because I am a mathematician. I am not. I do not do this work because I am, by definition a prophet, that would be foolish as it would be hard to find anyone who had any understanding on that score. I do it because it is my job as a mind to function in accordance with standards of language itself.

As a thing is composed of both material and form, education itself is only possible with teaching which includes both branches of reasoning together in what was once called a formal system. A logic paralleling an analogic. One can slowly instill the importance of this linguistic, this living functional duality, on rudimentary levels, by simple habits, such as involve what appear to be disparate and strange habits of diet, like chewing the cud and having cloven hoofs. Material and form, the two witnesses of any possible “God” for language, and the principles of language, is the only power a mind can ever know.

Analogics need to be studied. As mind is responsible for behavior in any creature, and since the principles of language are universal, it is these principles which are the foundation for behavior in a truly aware species. Thus moral, ethical, sane behavior is demonstrably abstracted from the principles of language itself, on earth, or in the heavens. All constitutions, all human behavior, have the same foundation. Man simply does not yet know how savage and primitive it is.

The Equation

Note: I will not apologize for that fact that common grammar has really never been taught. And that presently I will not get into it. This work is only a progress report.

In the common grammar of English, for example, we have only three possible fundamental categories of names. The names of things, the names of a things forms, and the names of a things material difference.

This also gives us two naming conventions. We can name things directly, or we can name things as a combination of material and form. Let us call the one convention where we name things directly, Subjects and the convention where we build the name of a thing by naming its parts, predicates.

Thus we have sentences which can have no subjects, no predicates, or subject and predicate. The same is true of an equation.

If I say N = 32. The sentence has no predicates. Both N and 32 are subjects. If I say $\frac{10}{2} = \frac{20}{4}$, I have a sentence with no subject, both are predicates. If I say, $R = \frac{N^2 + N}{N - 1}$, I have the subject, R and the predicate, $\frac{N^2 + N}{N - 1}$. N being the material and the structure itself the form imposed upon that material. This predicate is equal to the subject, R—which stands for Results. Thus, even in Algebra, we have two fundamentally distinct naming conventions—just as we do in common grammar. Common grammar has evolved many ways to add together these units of predication and also sentence types often into what appear to be single sentences. But what is true of any logic, one starts with a name, and the product is always a name.

An Introduction to Analogic.

We use language to do our job as mind; that job being to maintain and promote the life of the body. And since the mind manipulates virtual things, it does so by example.

> "Socrates: At the Egyptian city of Naucratis, there was a famous old god, whose name was Theuth; the bird which is called the Ibis is sacred to him, and he was the inventor of many arts, such as arithmetic and calculation and geometry and astronomy and draughts and dice, but his great discovery was the use of letters. Now in those days the god Thamus was the king of the whole country of Egypt; and he dwelt in that great city of Upper Egypt which the Hellenes call Egyptian Thebes, and the god himself is called by them Ammon. To him came Theuth and showed his inventions, desiring that the other Egyptians might be allowed to have the benefit of them; he enumerated them, and Thamus enquired about their several uses, and praised some of them and censured others, as he approved or disapproved of them. It would take a long time to repeat all that Thamus said to Theuth in praise or blame of the various arts. But when they came to letters, This, said Theuth, will make the Egyptians wiser and give them better memories; it is a specific both for the memory and for the wit. Thamus replied; O most ingenious Theuth, the parent or inventor of an art is not always the best judge of the utility or inutility of his own inventions to the users of them. And in this instance, you who are the father of letters, from a paternal love of your own children have been led to attribute to them a quality which they cannot have; for this discovery of yours will create forgetfulness in the learners souls, because they will not use their memories; they will trust to the external written characters and not remember of themselves. The specific which you have discovered is an aid not to memory, but to reminiscence, and you give your disciples not truth, but only the semblance of truth; they will be hearers of many things and will have learned nothing; they will appear to be omniscient and will generally know nothing; they will be tiresome company, having the show of wisdom without the reality." Plato

> "For it is sense-perception alone which is adequate for grasping the particulars: they cannot be objects of scientific knowledge, because neither can universals give us knowledge of them without induction, nor can we get it through induction without sense-perception." Aristotle

> "We conclude that these states of knowledge are neither innate in a determinate form, nor developed from other higher states of knowledge, but from sense-perception." Aristotle

The early Greeks were exploring the fact that there are two and only two elements of anything—a things form and a things material difference. And, so, there are two, and only two, primitive systems of reasoning. I call one Logic and the other Analogic. One of these branches produces particular systems of reasoning, the other produces universal systems of reasoning. And since the principles of language are the same throughout all of reality, which do you suppose would be used for communication? Logics, which are particular, or Analogics, which are universal?

What do the visions of the prophets have in common with dreams? What do the so called miracles of Christ have to do with either? And what do all of these have to do with Euclidean Geometry? They are all analogics.

As a thing is composed of its two elements, material and form, so too is the thing called a Formal System. A Formal System is a Logic paired with an Analogic. And, as the elements of a thing, make one and only one thing, each system of reasoning, if properly executed, can only say one and the same thing.

Within the works of Aristotle is a fundamental truth of language—one cannot say a thing both is and is not—every imaginable error in reasoning violates this principle. It is binary. Aristotle stated that if one cannot perform this function in reasoning, one is equivalent to a vegetable. Now, I will go so far as to say that one may be a Caesar Salad, but still, in the realm of reasoning, when we predicate that a thing both is and is not, they are on par with a vegetable. This is true because there are two predicates, and the name of a thing is equal to the names of that things forms and the names of the material difference in those forms. Thus we have two naming conventions in common grammar.

One may see how absurd it is to state that parallel lines meet at a mythical place called infinity. It is the same as saying that a triangle is both equal to itself and not equal to itself. Even if one could not comprehend the propositions, it should have been realized that one cannot say that one can have a 2 but not a 1, i.e. in a two dimensional system, one cannot have one of those dimensions—i.e.

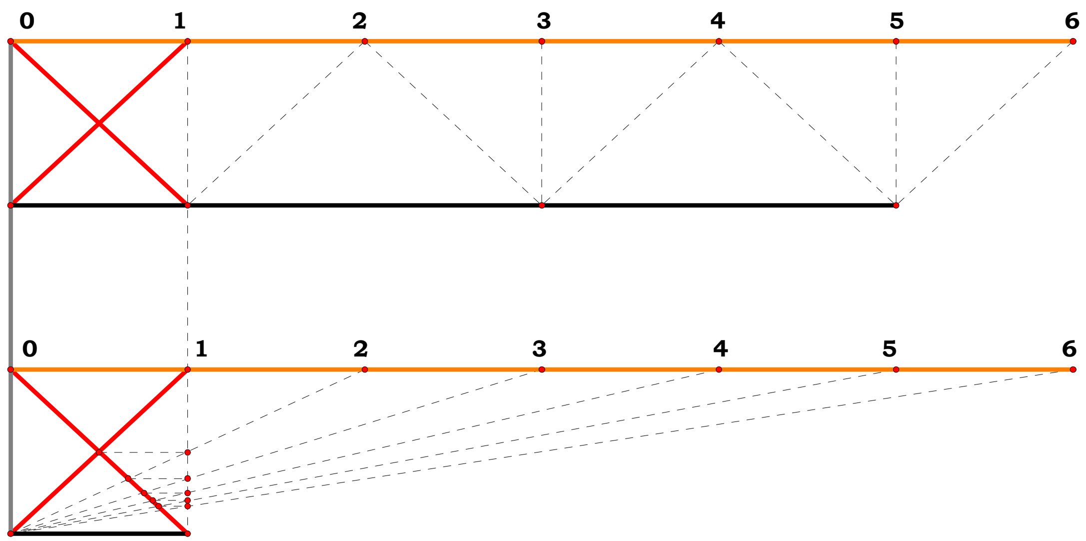
one and only one difference between two lines. But no, violating the first principles of reasoning by so-call great reasoners, is, as Aristotle pointed out, an oxymoron. There is more salad in our schools than can be measured.

Basic

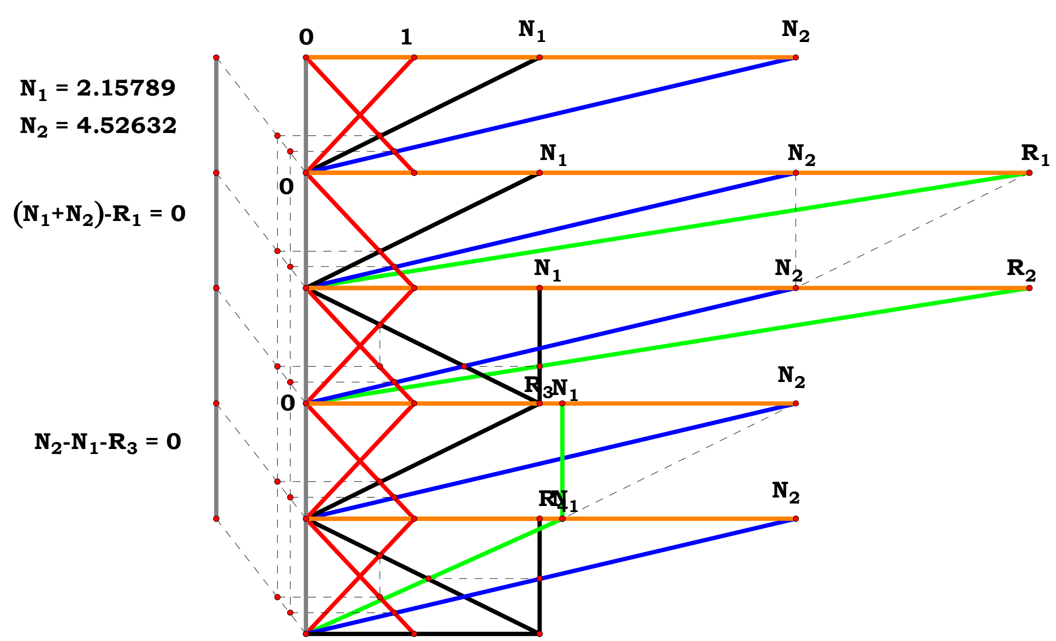
These are basic operations in the Analogic of Geometry.

Complete Induction.

or counting.



Addition and Subtraction.



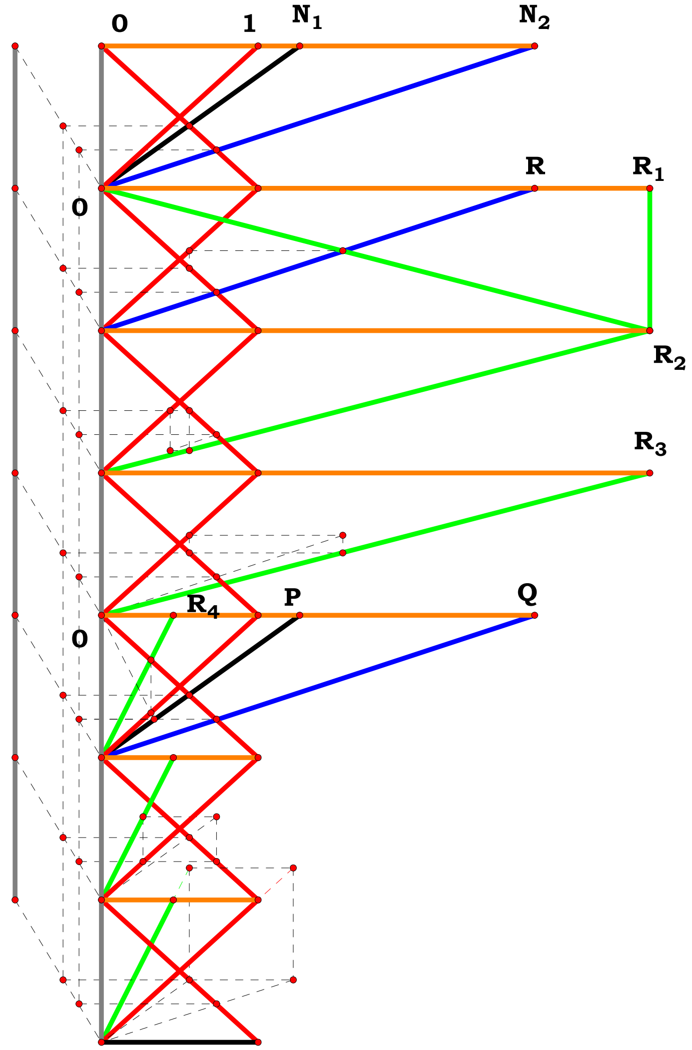
Multiplication and Division.

$N_1 = 1.26531$

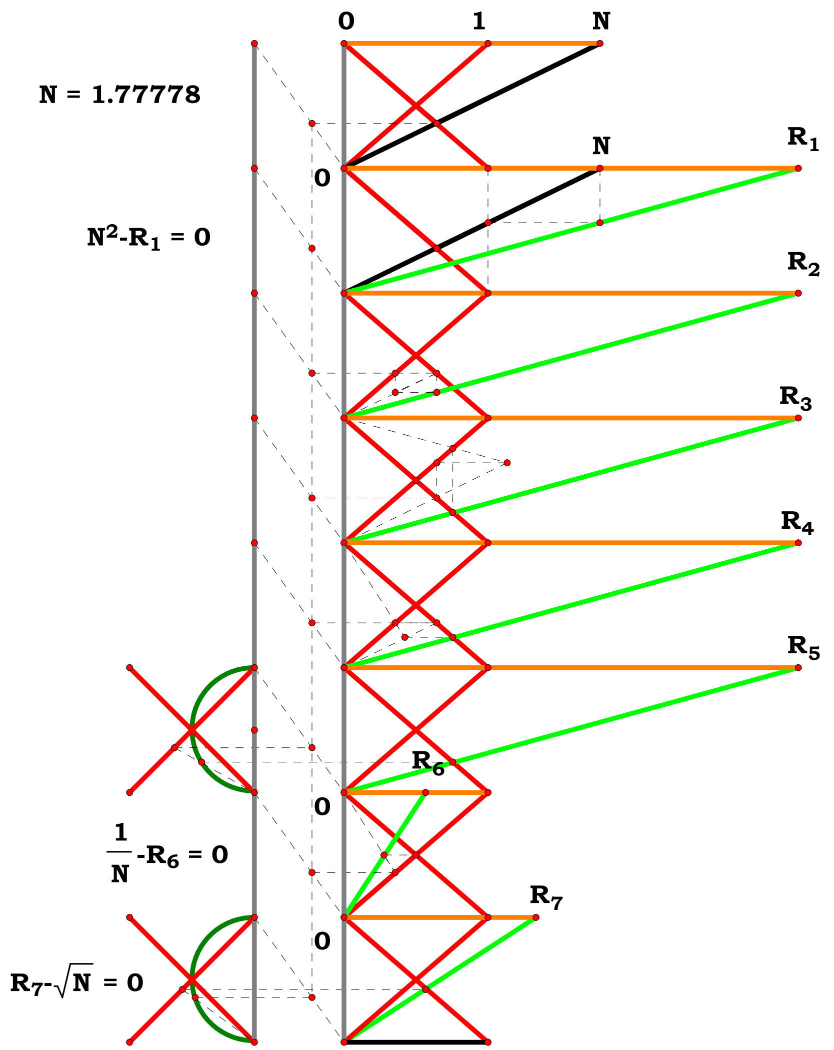
$N_2 = 2.76531$

$N_1 \cdot N_2 - R_1 = 0$

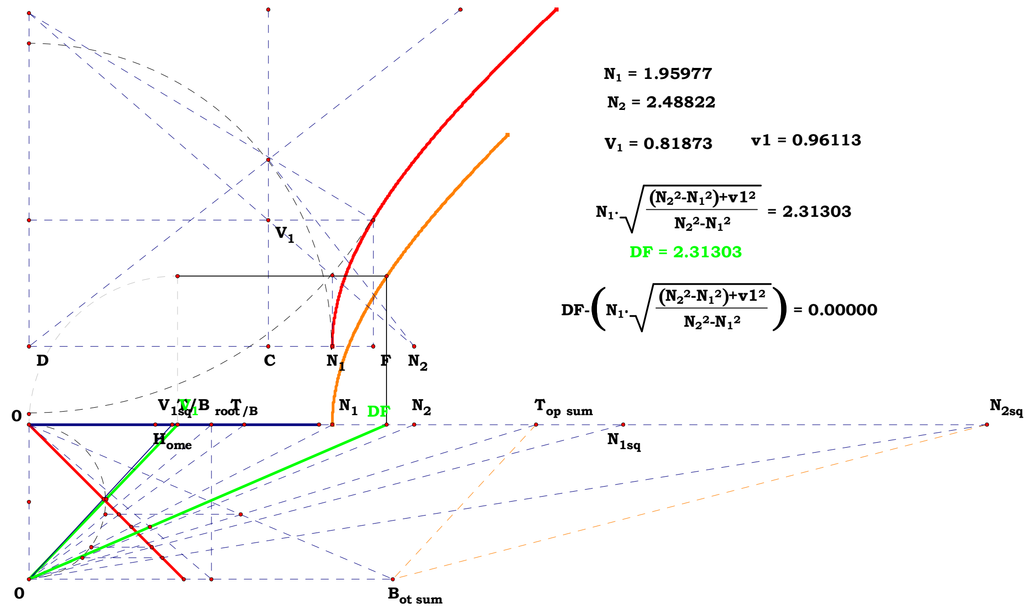
$\frac{N_1}{N_2} - R_4 = 0$



The Square, Reciprocal and Root.



With simple tools of analogic, one can construct some of the most complex equations with simple straight-edge and compass.



And one can project directly the wave or curve of the equation.

The following is chapter one of glyphs that one plug into the tree or one can use them as language themselves, for each is the same as the equation it performs.

There may be countless numbers of these plug in modules constructible. I will only work on a couple thousand. This work is scheduled for 12 chapters. This is the shortest. At this time videos on the work can be found on the Internet Archive under johnclark8659 and on YouTube under Philosopher8659.

1CST1

$N = 2.46939$

$$\frac{N}{N-1}-R_0 = 0$$

$$N-1-R_1 = 0$$

$$N^2-N-R_2 = 0$$

$$\left|\frac{N}{N-2}\right|-R_3 = 0$$

$$\left|\frac{N^2-N}{N-2}\right|-R_4 = 0$$

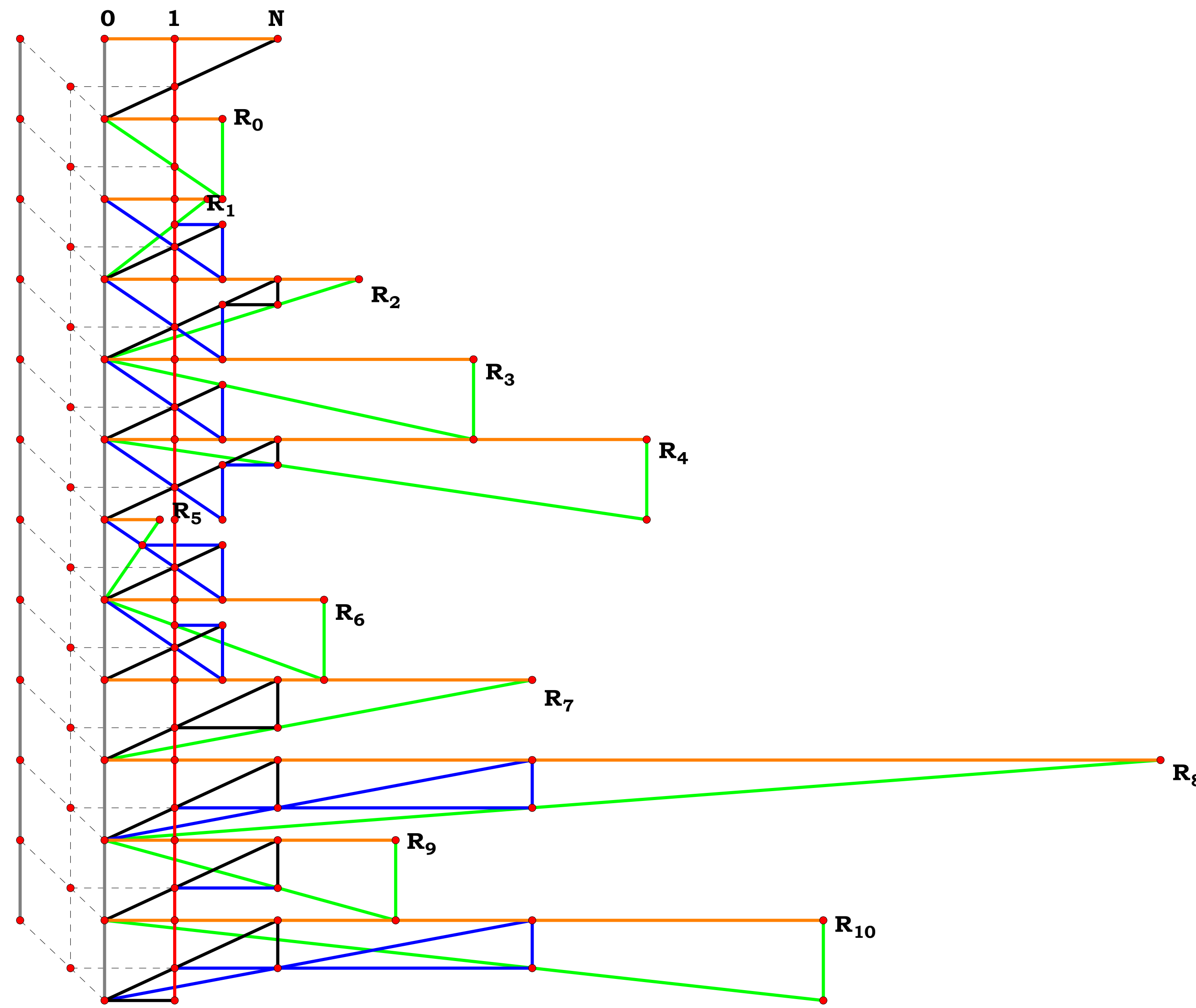
$$\left|\frac{N^2-2\cdot N}{N-1}\right|-R_5 = 0$$

$$N^2-R_7 = 0$$

$$N^3-R_8 = 0$$

$$\frac{N^2}{N-1}-R_9 = 0$$

$$\frac{N^3}{N-1}-R_{10} = 0$$



1CST2

$N = 2.81538$

$$\frac{N}{N+1}-R_0 = 0$$

$$\frac{N}{2\cdot N+1}-R_1 = 0$$

$$\frac{2\cdot N^2+N}{N^2+2\cdot N+1}-R_2 = 0$$

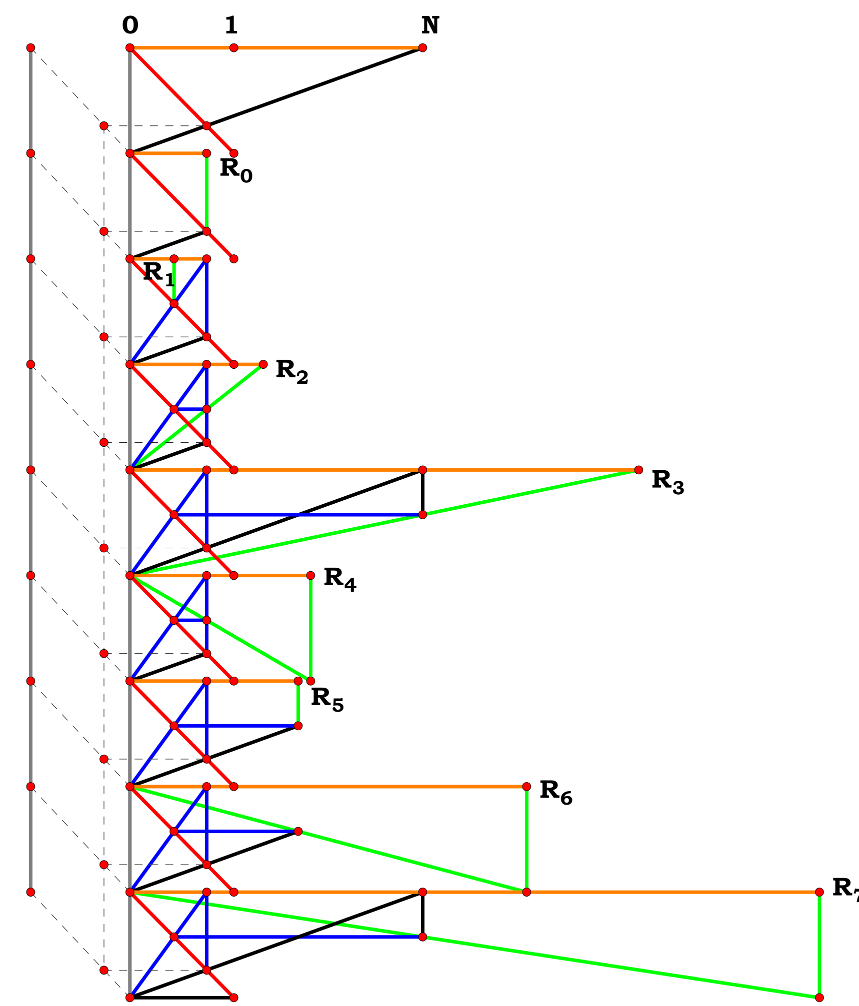
$$\frac{2\cdot N^2+N}{N+1}-R_3 = 0$$

$$\frac{2\cdot N+1}{N+1}-R_4 = 0$$

$$\frac{N^2+N}{2\cdot N+1}-R_5 = 0$$

$$(N+1)-R_6 = 0$$

$$(2\cdot N+1)-R_7 = 0$$



1CST3

$N = 1.41026$

$\frac{1}{N} - R_0 = 0$

$\frac{1}{N^2} - R_1 = 0$

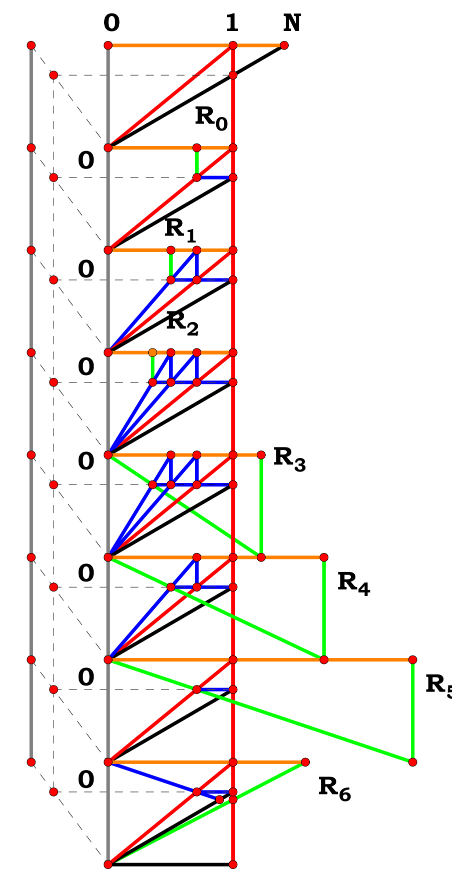
$\frac{1}{N^3} - R_2 = 0$

$\frac{1}{N^3 - N^2} - R_3 = 0$

$\frac{1}{N^2 - N} - R_4 = 0$

$\frac{1}{N - 1} - R_5 = 0$

$((N^2 - N) + 1) - R_6 = 0$



1CST4

$N = 2.34483$

$$\frac{N-1}{N} - R_0 = 0$$

$$N-1-R_1 = 0$$

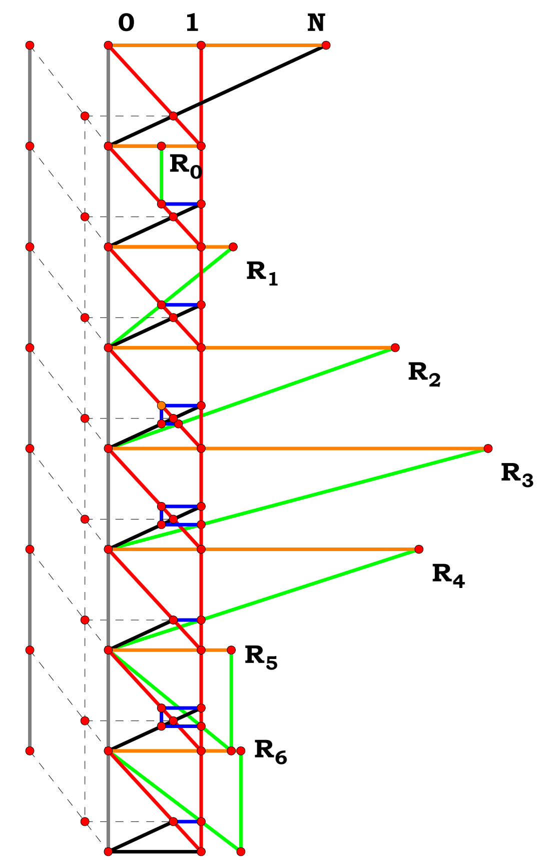
$$\frac{(N^2-N)+1}{N-1} - R_2 = 0$$

$$\frac{N^2}{N-1} - R_3 = 0$$

$$(N+1)-R_4 = 0$$

$$\frac{N^2}{(N^2-N)+1} - R_5 = 0$$

$$\frac{N+1}{N} - R_6 = 0$$



1CST5

N = 1.85938

$$\frac{1}{(N^2+N)-1}-R_0 = 0$$

$$\frac{1}{N+1}-R_1 = 0$$

$$\frac{1}{N}-R_2 = 0$$

$$\frac{N^2}{N+1}-R_3 = 0$$

$$\frac{((N^3+N^2)-N)+1}{N+1}-R_4 = 0$$

$$N^2-R_5 = 0$$

$$(N+1)-R_6 = 0$$

$$(N^2+N)-1-R_7 = 0$$

$$\frac{((N^4+N^3)-N^2)+N}{N+1}-R_8 = 0$$

$$(N^3+N^2)-N-R_9 = 0$$

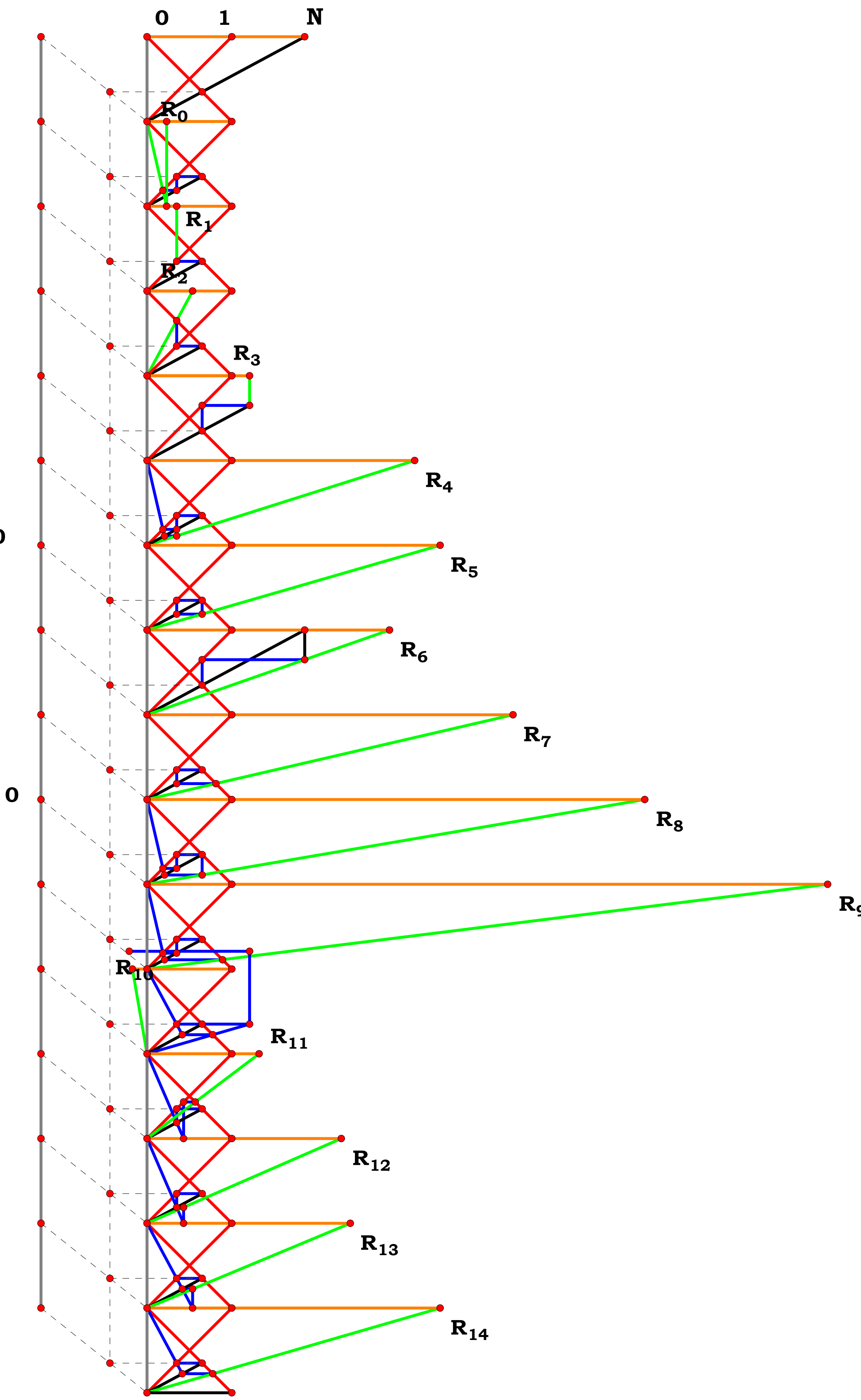
$$\left|\frac{(N+1)-N^2}{N^2}\right|-R_{10} = 0$$

$$\left|\frac{N^2-1}{N}\right|-R_{11} = 0$$

$$\frac{N^3+N^2}{(N^2+N)-1}-R_{12} = 0$$

$$\frac{N^2+1}{N}-R_{13} = 0$$

$$N^2-R_{14} = 0$$



1CST6

$N = 1.48780$

$N^2 - N - R_0 = 0$

$\frac{N+1}{N^2} - R_1 = 0$

$\frac{(N^2+N)-1}{N} - R_2 = 0$

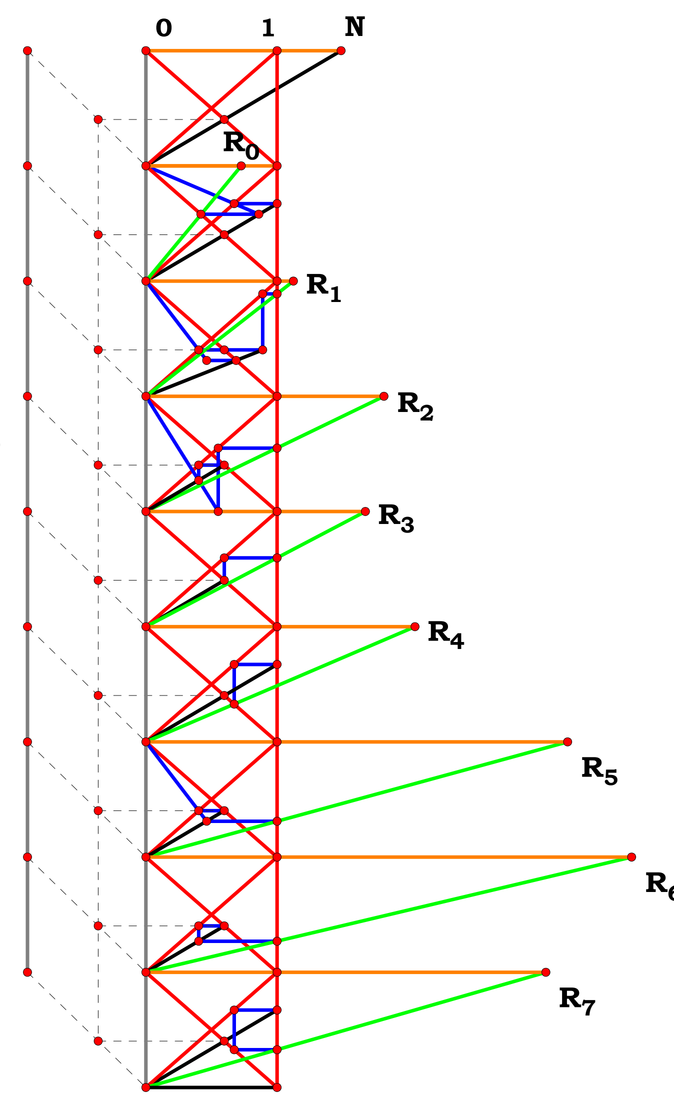
$\frac{N+1}{N} - R_3 = 0$

$\frac{1}{N-1} - R_4 = 0$

$(N^2+1) - R_5 = 0$

$(N^2+N) - R_6 = 0$

$\frac{N}{N-1} - R_7 = 0$



1CST7

N = 1.63158

$$\frac{N-1}{N^2}-R_0=0$$

$$\frac{N-1}{N}-R_1=0$$

$$\frac{N}{2\cdot N-1}-R_2=0$$

$$\frac{2\cdot N-1}{3\cdot N-2}-R_3=0$$

$$\frac{N^2}{(N^2+N)-1}-R_4=0$$

$$\frac{(N^2+N)-1}{(N^2+2\cdot N)-2}-R_5=0$$

$$\frac{(N^2+N)-1}{N^2}-R_6=0$$

$$\frac{N^2}{(N^2-N)+1}-R_7=0$$

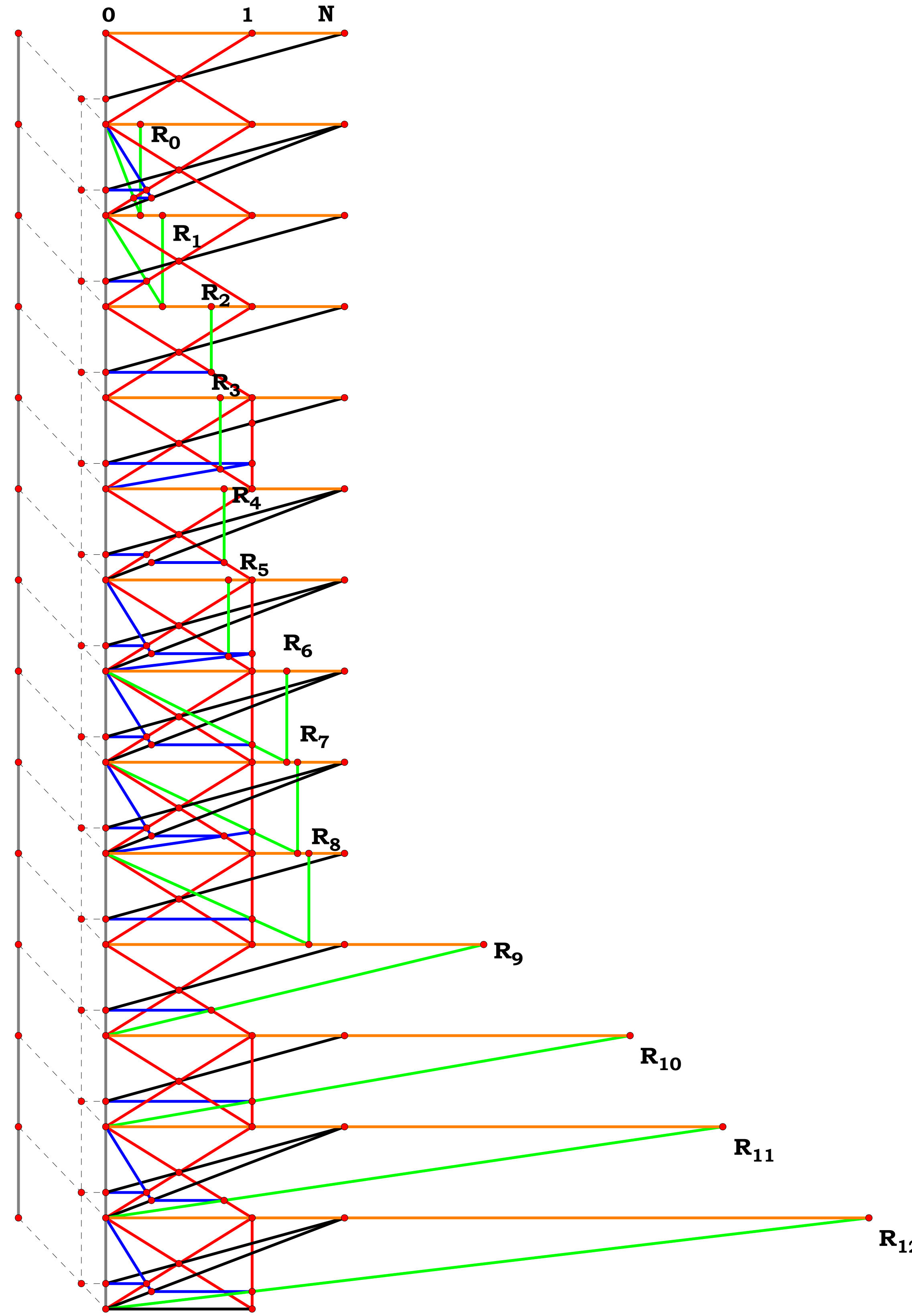
$$\frac{2\cdot N-1}{N}-R_8=0$$

$$\frac{N}{N-1}-R_9=0$$

$$\frac{2\cdot N-1}{N-1}-R_{10}=0$$

$$\frac{N^2}{N-1}-R_{11}=0$$

$$\frac{(N^2+N)-1}{N-1}-R_{12}=0$$



2SMT1

$N = 1.70833$

$\frac{N^3+N}{N^4+3\cdot N^2+1} - R_0 = 0$

$\frac{N}{N^2+1} - R_1 = 0$

$\frac{N^5+N^3}{N^6+N^4+2\cdot N^2+1} - R_2 = 0$

$\frac{1}{N} - R_3 = 0$

$\frac{N}{N^2-1} - R_4 = 0$

$\frac{N^3}{N^2+1} - R_5 = 0$

$\frac{N^2+1}{N} - R_6 = 0$

$\left|\frac{N^3+N}{N^2-1}\right| - R_7 = 0$

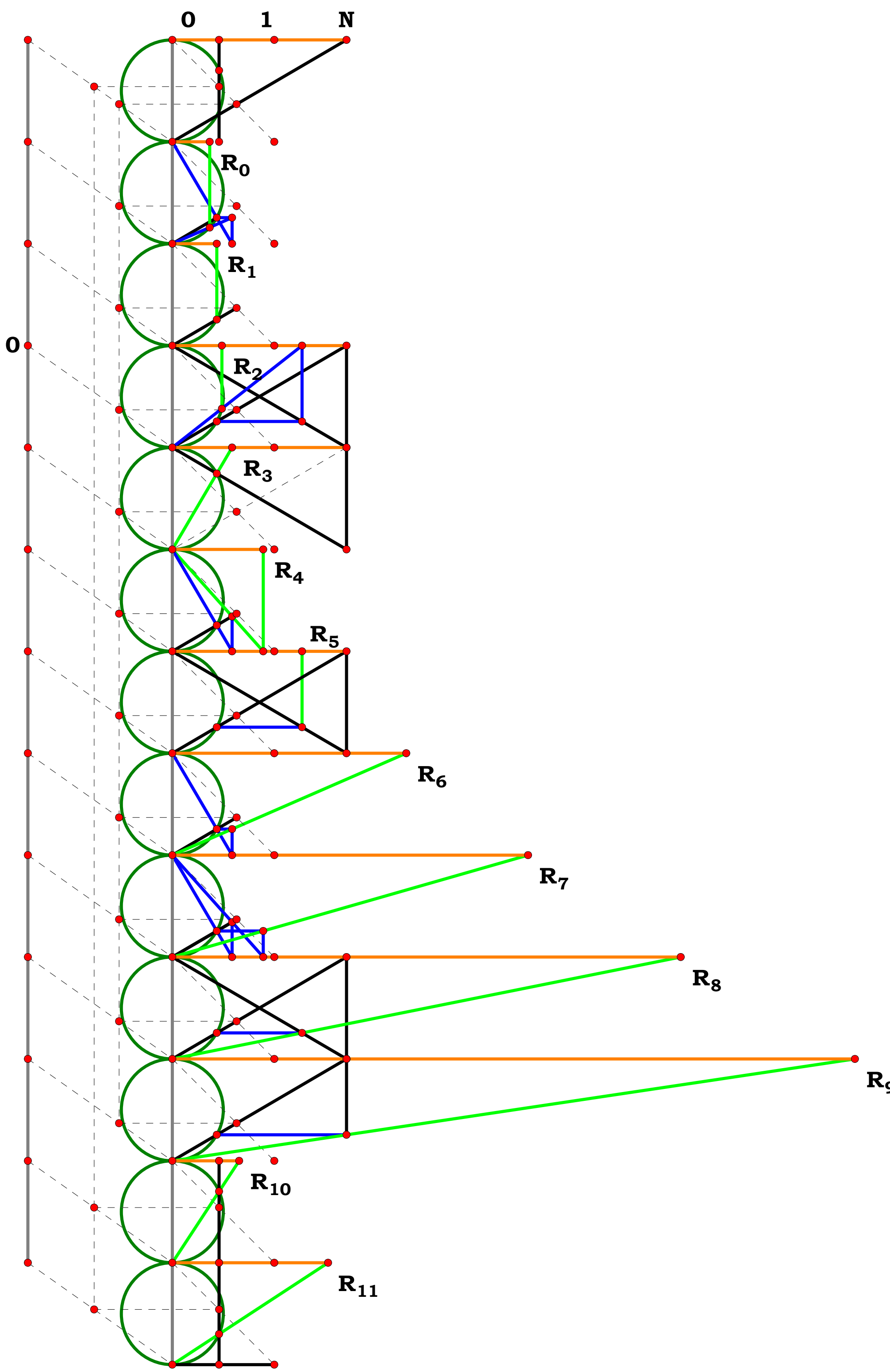
$N^3 - R_8 = 0$

$(N^3+N) - R_9 = 0$

$N = 0.45833$

$\frac{2\cdot N}{\sqrt{1-4\cdot N^2}+1} - R_{10} = 0$

$\frac{2\cdot N}{1-\sqrt{1-4\cdot N^2}} - R_{11} = 0$



2SMT2

N = 2.06250

$$\frac{N^2-N}{(2\cdot N^2-2\cdot N)+1}-R_0 = 0$$

$$\frac{N-1}{N}-R_1 = 0$$

$$\frac{\sqrt{N-1}}{N}-R_2 = 0$$

$$\frac{N}{N\cdot\sqrt{N-1}+1}-R_3 = 0$$

$$\sqrt{N-1}-R_4 = 0$$

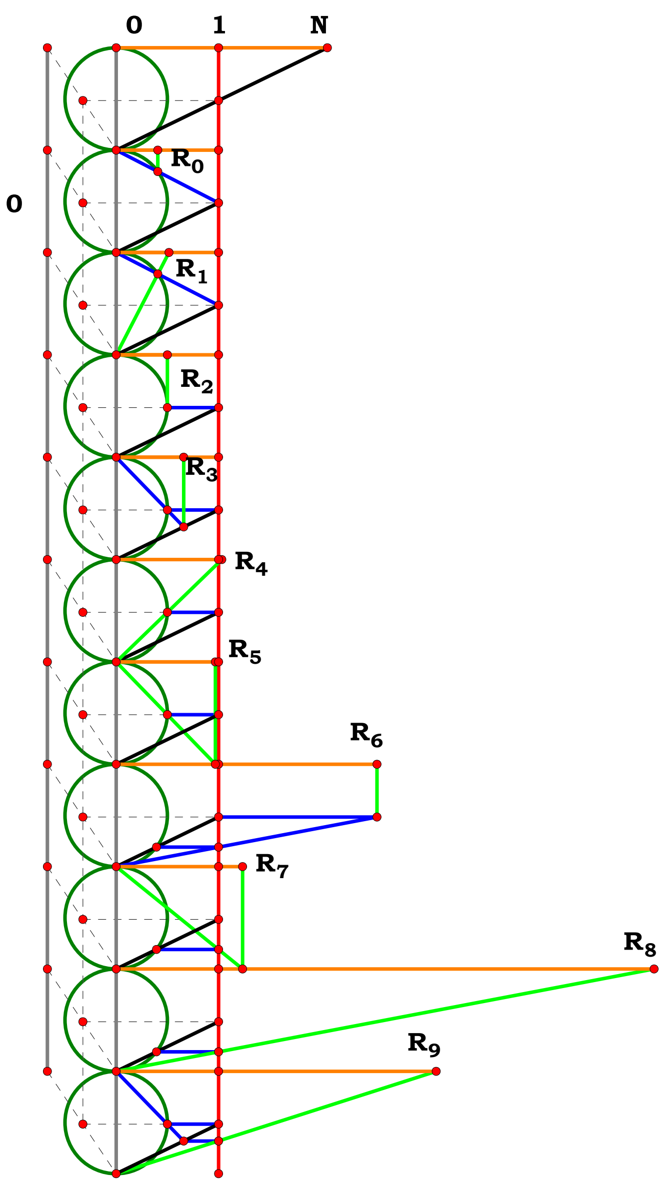
$$\frac{1}{\sqrt{N-1}}-R_5 = 0$$

$$\frac{N^2+1}{N}-R_6 = 0$$

$$\frac{N^2+1}{N^2}-R_7 = 0$$

$$(N^2+1)-R_8 = 0$$

$$(N\cdot\sqrt{N-1}+1)-R_9 = 0$$



2SMT3

$N = 3.70149$

$$\frac{N}{(N^2-N)+1} - R_0 = 0$$

$$\frac{1}{N^{\frac{1}{2}}} - R_1 = 0$$

$$\sqrt{\frac{N}{(N^2-N)+1}} - R_2 = 0$$

$$\frac{1}{N^{\frac{1}{4}}} - R_3 = 0$$

$$\frac{1}{N^{\frac{1}{8}}} - R_4 = 0$$

$$N^{\frac{1}{8}} - R_5 = 0$$

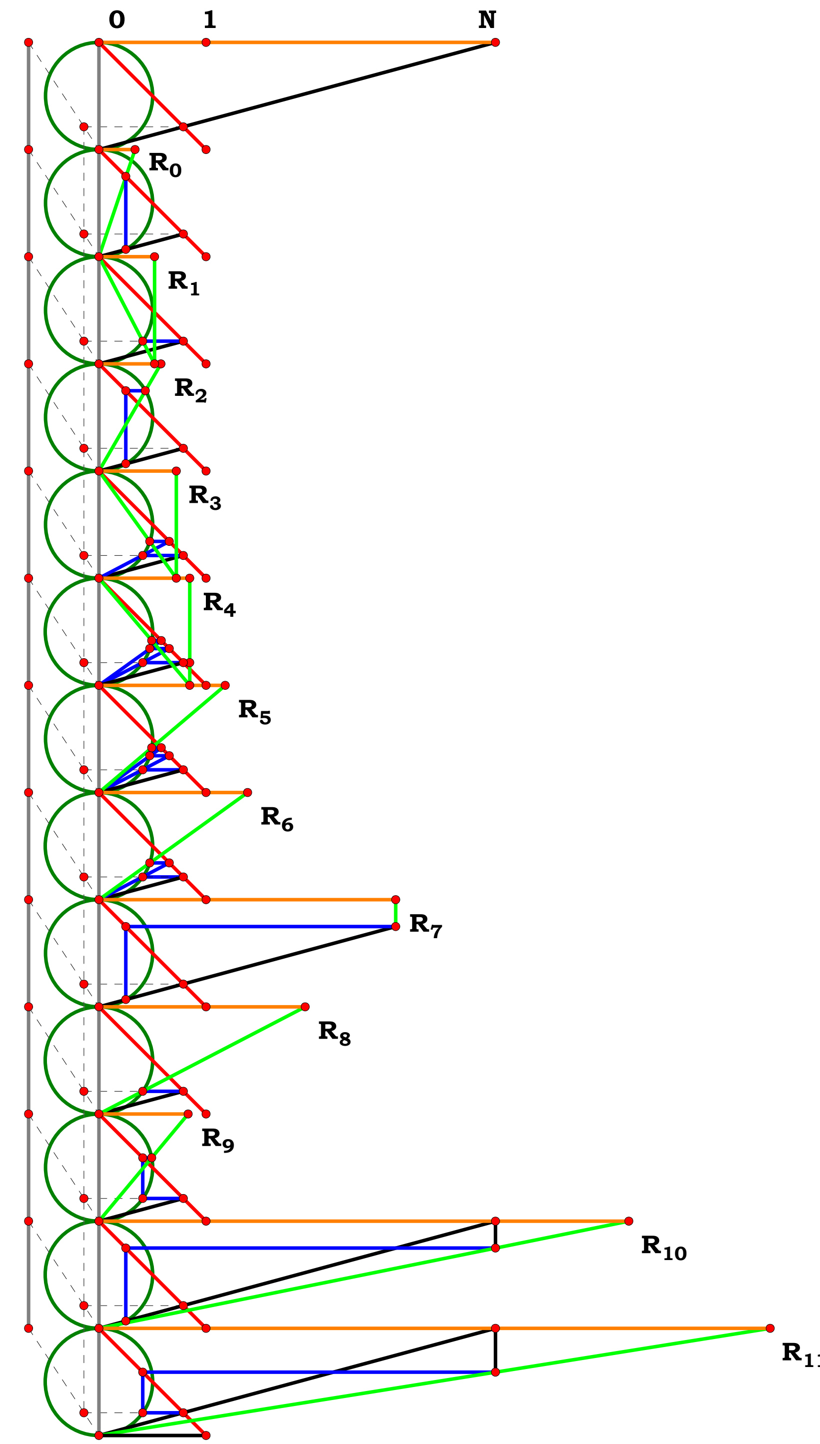
$$N^{\frac{1}{4}} - R_6 = 0$$

$$\frac{(N^3-N^2)+N}{N^2+1} - R_7 = 0$$

$$N^{\frac{1}{2}} - R_8 = 0$$

$$\sqrt{\frac{\sqrt{N}}{(N-\sqrt{N})+1}} - R_9 = 0$$

$$\frac{N^2+N}{(N-\sqrt{N})+1} - R_{11} = 0$$



2SMT4

$N = 2.05333$

$$\frac{(((N^4-N^3)+2\cdot N^2)-N)+1}{(((2\cdot N^4-2\cdot N^3)+5\cdot N^2)-2\cdot N)+2}-R_0 = 0$$

$$\frac{N^2-N\cdot\sqrt{N^2-4}}{3\cdot N-\sqrt{N^2-4}}-R_1 = 0$$

$$\frac{(N^2-N)+1}{N^2+1}-R_2 = 0$$

$$\frac{N-\sqrt{N^2-4}}{2}-R_3 = 0$$

$$\sqrt{\frac{(N^2-N)+1}{N}}-R_4 = 0$$

$$\frac{N+\sqrt{N^2-4}}{2}-R_5 = 0$$

$$\frac{N^2-N\cdot\sqrt{N^2-4}}{2}-R_6 = 0$$

$$\frac{N^2+1}{N}-R_7 = 0$$

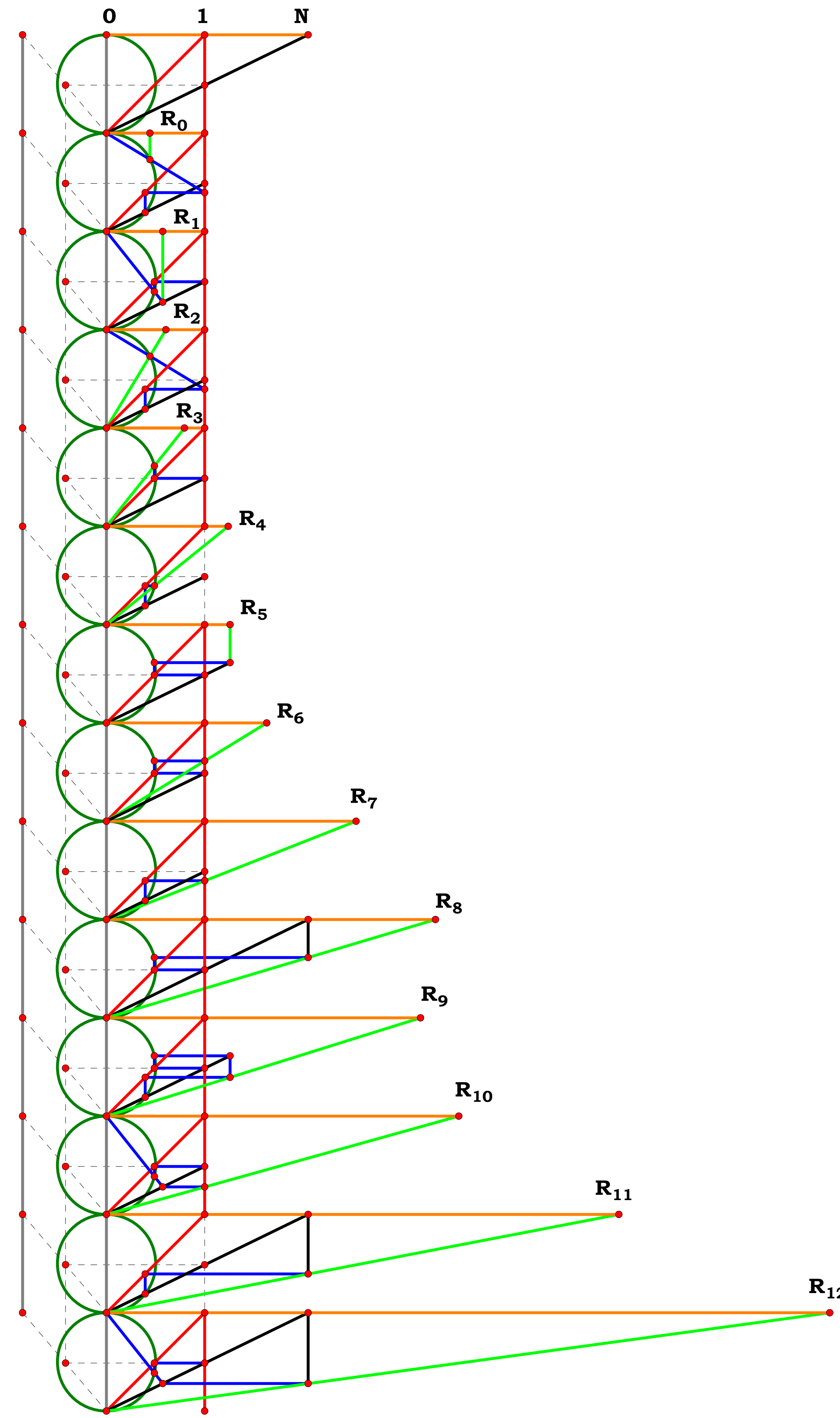
$$\frac{N^3-N^2\cdot\sqrt{N^2-4}}{2}-R_8 = 0$$

$$\frac{N^3+N^2\cdot\sqrt{N^2-4}+N+\sqrt{N^2-4}}{2\cdot N}-R_9 = 0$$

$$\frac{3\cdot N-\sqrt{N^2-4}}{N-\sqrt{N^2-4}}-R_{10} = 0$$

$$(N^2+1)-R_{11} = 0$$

$$\frac{3\cdot N^2-N\cdot\sqrt{N^2-4}}{N-\sqrt{N^2-4}}-R_{12} = 0$$



2SMT5

N = 2.01235

$$\frac{N^3+N}{N^4+3\cdot N^2+1}-R_0=0$$

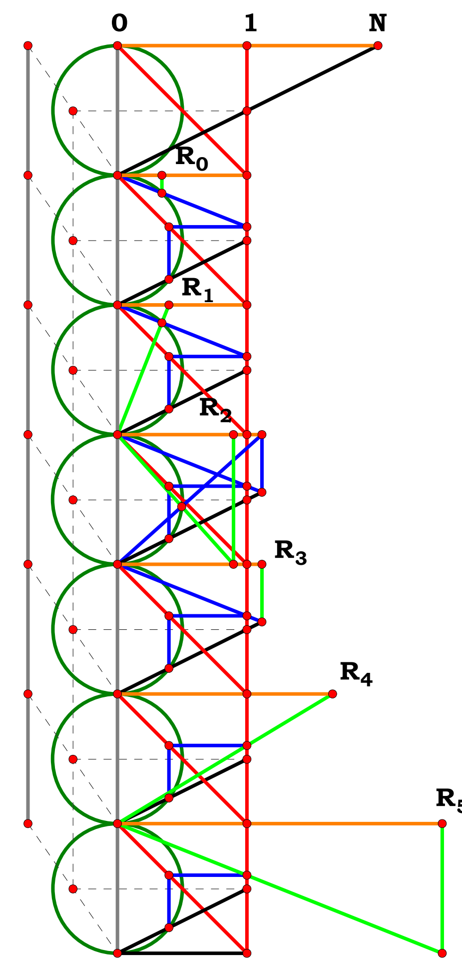
$$\frac{N}{N^2+1}-R_1=0$$

$$\frac{2\cdot N^2+1}{N^3+N}-R_2=0$$

$$\frac{N^3+N}{2\cdot N^2+1}-R_3=0$$

$$\frac{N^2+1}{(N^2-N)+1}-R_4=0$$

$$\frac{N^2+1}{N}-R_5=0$$



2SMT6

N = 1.98571

$$\frac{(N+1)-\sqrt{(N^2+2\cdot N)-3}}{2}-R_0=0$$

$$\sqrt{\sqrt{N}-1}-R_1=0$$

$$\frac{1}{N^{\frac{1}{4}}}-R_2=0$$

$$\frac{1}{\sqrt{N}}-R_3=0$$

$$\frac{N^3+N^2}{N^4+N^2+2\cdot N+1}-R_4=0$$

$$\sqrt{N}-R_5=0$$

$$\frac{2\cdot N}{(((N^2+2\cdot N)-N\cdot\sqrt{(N^2+2\cdot N)-3})+1)-\sqrt{(N^2+2\cdot N)-3}}-R_6=0$$

$$\frac{(((N^3+2\cdot N^2)-N^2\cdot\sqrt{(N^2+2\cdot N)-3})+N)-N\cdot\sqrt{(N^2+2\cdot N)-3}}{2}-R_7=0$$

$$\frac{N^2}{\sqrt{N}}-R_8=0$$

$$\frac{1}{\sqrt{\frac{N^2}{(N^2-N)+1}}}-R_9=0$$

$$\frac{N^2}{(N^2-N)+1}-R_{10}=0$$

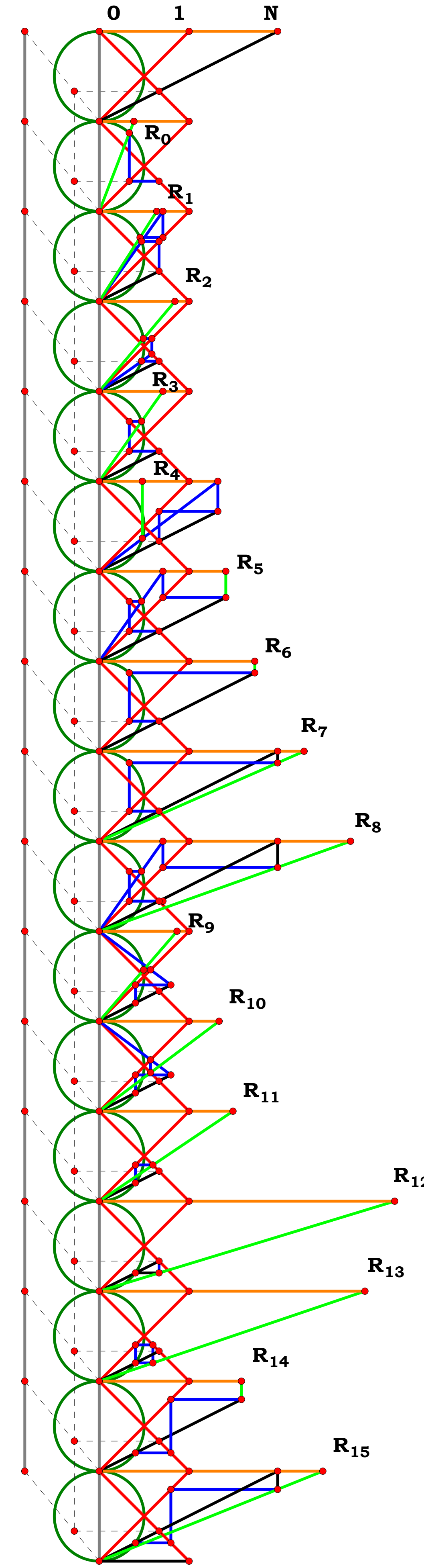
$$\frac{(N^2-N)+1}{N}-R_{11}=0$$

$$\frac{N^3+N}{N+1}-R_{12}=0$$

$$((N^2-N)+1)-R_{13}=0$$

$$\frac{N^3}{N^2+1}-R_{14}=0$$

$$\frac{N^2+1}{N}-R_{15}=0$$



2SMT7

N = 2.19753

$$\frac{1}{(N^2-2\cdot N)+2}-R_0=0$$

$$\frac{1}{(N^2-2\cdot N)+1}-R_1=0$$

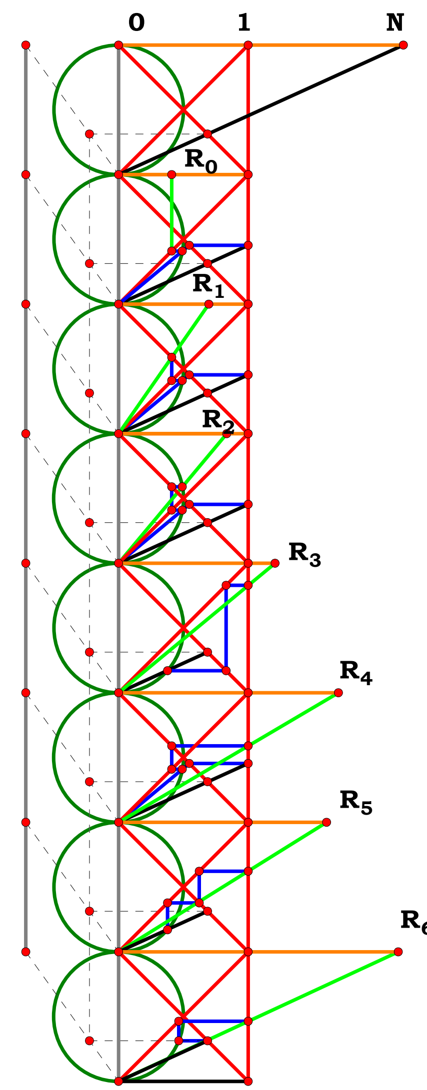
$$\frac{1}{N-1}-R_2=0$$

$$\frac{N^2+1}{N^2}-R_3=0$$

$$\frac{(N^2-2\cdot N)+2}{(N^2-2\cdot N)+1}-R_4=0$$

$$\frac{N^2+1}{(N^2-N)+1}-R_5=0$$

$$\frac{N+1}{\sqrt{N}}-R_6=0$$



2SMT8

$N = 1.84615$

$$\frac{2\cdot N-2}{(2\cdot N+\sqrt{4\cdot N-3})-1}-R_0 = 0$$

$$\frac{\sqrt{N^2-N}}{2\cdot N-1}-R_1 = 0$$

$$\frac{\sqrt{N^2-N}}{N}-R_2 = 0$$

$$\sqrt{N-1}-R_3 = 0$$

$$\frac{N}{(N^2-N)+1}-R_4 = 0$$

$$\sqrt{N-\sqrt{\frac{1}{N}}}-R_5 = 0$$

$$\sqrt{N^2-1}-R_6 = 0$$

$$N^2-1-R_7 = 0$$

$$\frac{1}{N-1}-R_8 = 0$$

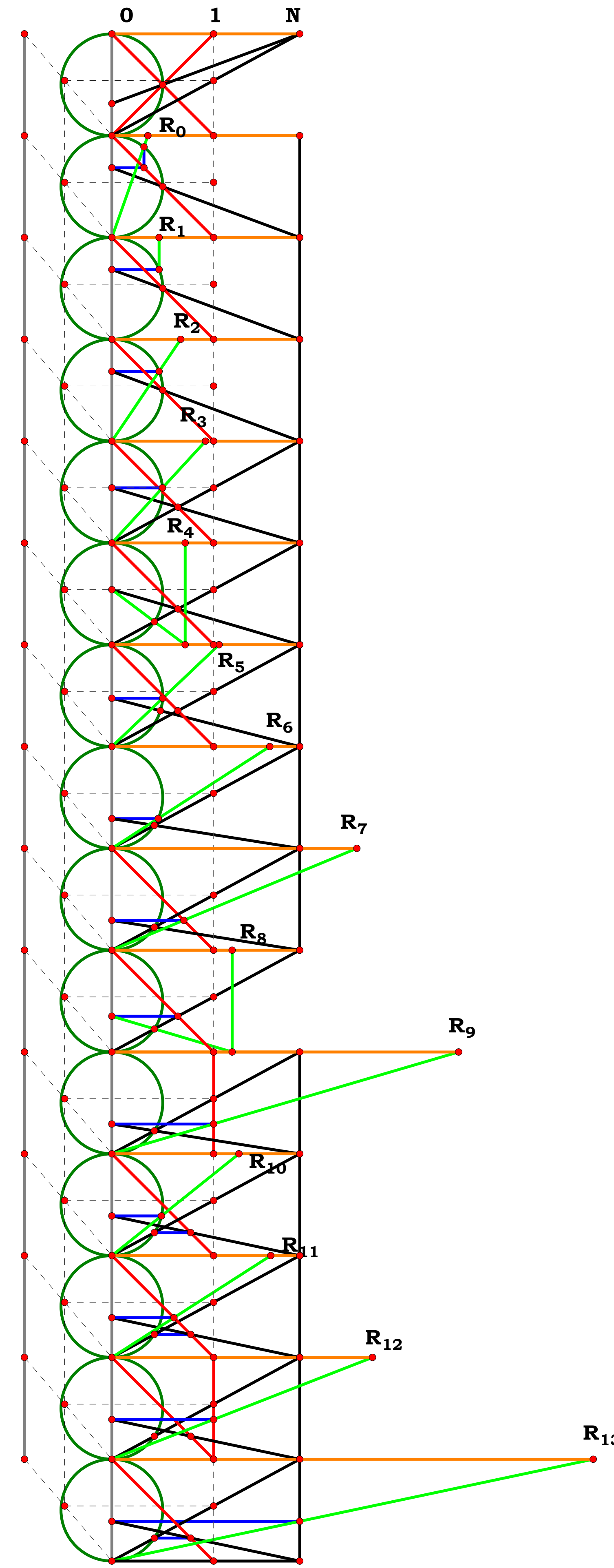
$$N^2-R_9 = 0$$

$$\sqrt{N^2-N}-R_{10} = 0$$

$$N^2-N-R_{11} = 0$$

$$((N^2-N)+1)-R_{12} = 0$$

$$((N^3-N^2)+N)-R_{13} = 0$$



3OBT1

$N = 2.23529$

$$\frac{1}{(N^2-N)+1}-R_0 = 0$$

$$\frac{1}{(N^2-N-\sqrt{N^2-N})+1}-R_1 = 0$$

$$\frac{1}{N}-R_2 = 0$$

$$\frac{1}{\sqrt{N^2-N}}-R_3 = 0$$

$$\frac{1}{\left(N^2-N-\sqrt{N^2-N}-\sqrt{N^2-N-\sqrt{N^2-N}}\right)+1}-R_4 = 0$$

$$\frac{N-1}{N}-R_5 = 0$$

$$\frac{N^2}{N^2+1}-R_6 = 0$$

$$\frac{N^4+2\cdot N^2+1}{N^4+2\cdot N^2+2}-R_7 = 0$$

$$\sqrt{N^2-N-\sqrt{N^2-N}}-R_8 = 0$$

$$\sqrt{N^2-N}-R_9 = 0$$

$$\frac{N^3+N}{(N^2-N)+1}-R_{10} = 0$$

$$(N^2+1)-R_{11} = 0$$

$$((N^2-N)+1)-R_{12} = 0$$

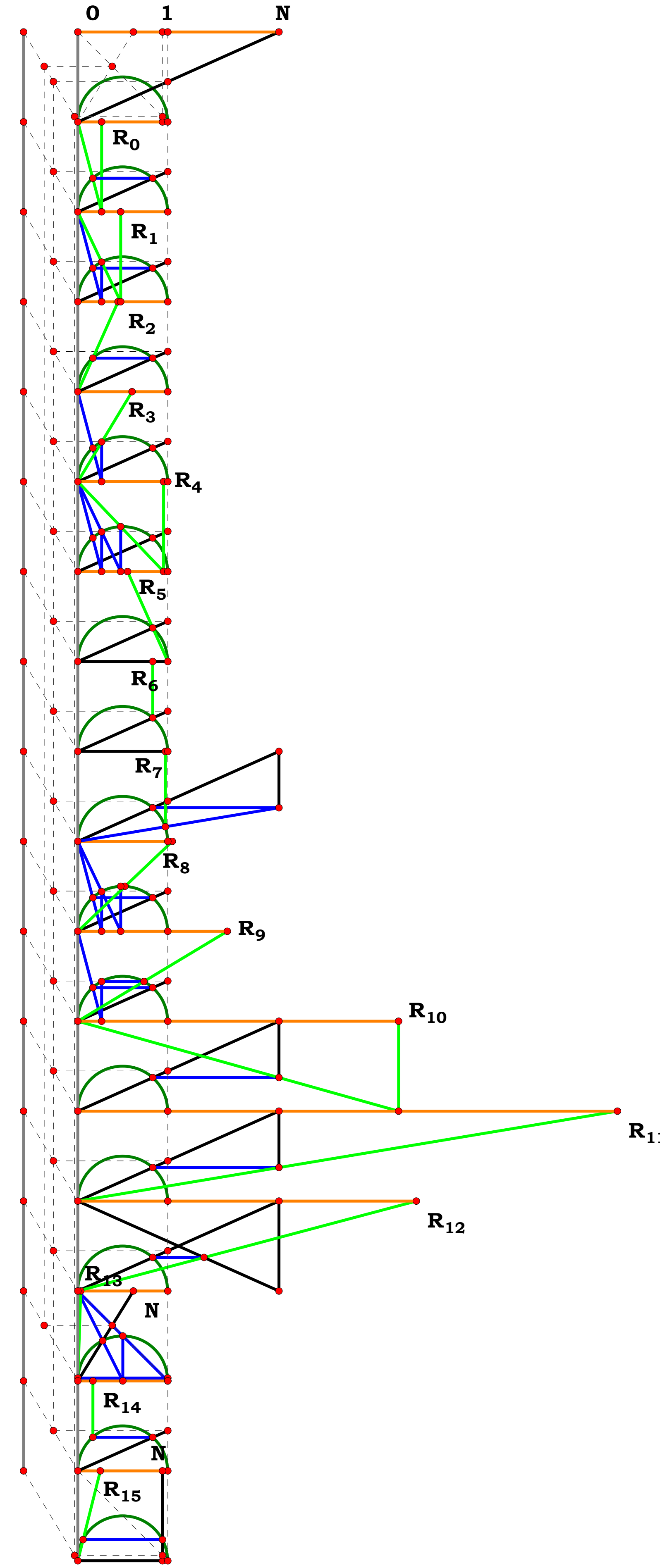
$N = 0.61765$

$$\frac{1-N-\sqrt{N^2-N^3}}{\sqrt{N^2-N^3-N^2\cdot\sqrt{N^2-N^3}}}-R_{13} = 0$$

$$\frac{1}{N^2+1}-R_{14} = 0$$

$N = 0.94118$

$$\frac{1-N}{\sqrt{N-N^2}}-R_{15} = 0$$



3OBT2

$N = 1.76250$

$$\frac{N^2}{N^4+3\cdot N^2+1}-R_0 = 0$$

$$\frac{1}{N^3+N+1}-R_1 = 0$$

$$\frac{N^3+N}{N^3+N+1}-R_2 = 0$$

$$\frac{N^4+2\cdot N^2+1}{N^4+3\cdot N^2+1}-R_3 = 0$$

$$\frac{N^4+3\cdot N^2+1}{(((N^4-N^3)+3\cdot N^2)-N)+1}-R_4 = 0$$

$$\frac{N^3+N+1}{(N^3+N+1)-\sqrt{N^3+N}}-R_5 = 0$$

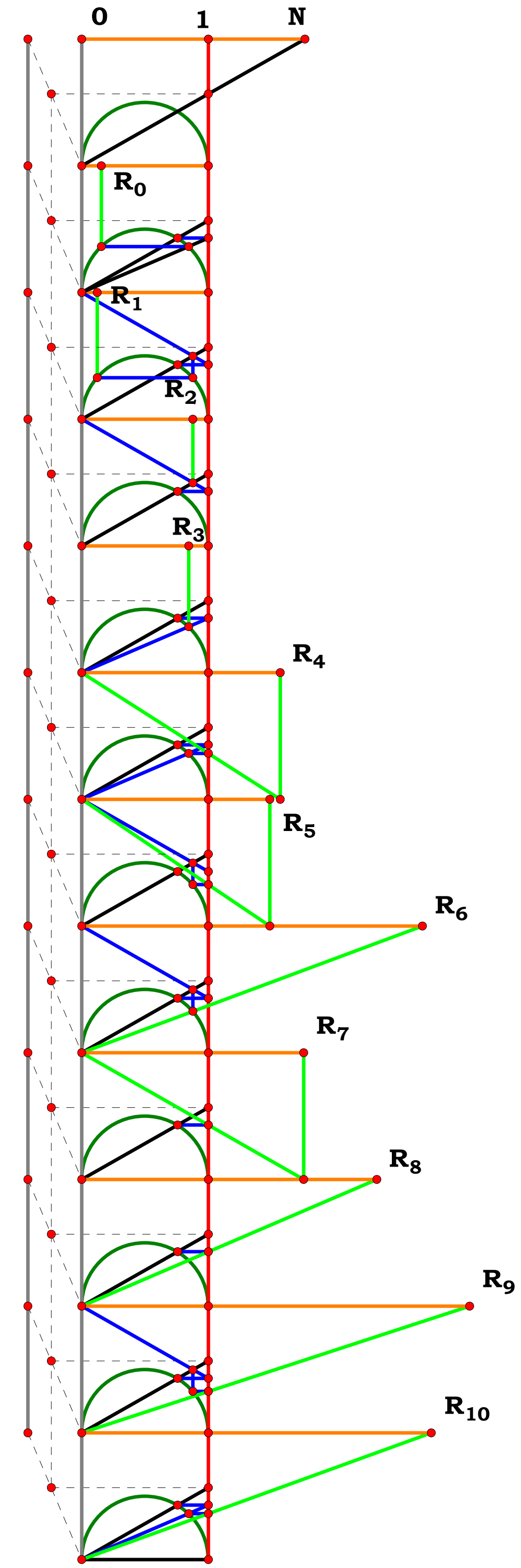
$$\sqrt{N^3+N}-R_6 = 0$$

$$\frac{N^2+1}{(N^2-N)+1}-R_7 = 0$$

$$\frac{N^2+1}{N}-R_8 = 0$$

$$\frac{N^3+N+1}{\sqrt{N^3+N}}-R_9 = 0$$

$$\frac{N^4+3\cdot N^2+1}{N^3+N}-R_{10} = 0$$



3OBT3

N = 2.40845

$$\frac{(N^2-N)+1}{(N^2-N)+1+\sqrt{(N^3-N^2)+N}}-R_0 = 0$$

$$\frac{(N^2-N)+1}{N^2+1}-R_1 = 0$$

$$\frac{(((N^4-2\cdot N^3)+3\cdot N^2)-2\cdot N)+1}{(((N^4-2\cdot N^3)+4\cdot N^2)-2\cdot N)+1}-R_2 = 0$$

$$\frac{(N^2-N)+1}{\sqrt{((N^2-N)+1)\cdot\sqrt{(N^3-N^2)+N}}}-R_3 = 0$$

$$\frac{(N^2-N)+1}{\sqrt{(N^3-N^2)+N}}-R_4 = 0$$

$$\sqrt{N}-R_5 = 0$$

$$\frac{(N^2-N)+1}{N}-R_6 = 0$$

$$\frac{N+1+\sqrt{(N^2+2\cdot N)-3}}{2}-R_7 = 0$$

$$\frac{N+3+\sqrt{(N^2+2\cdot N)-3}+\sqrt{((2\cdot N^2+8\cdot N+2\cdot N\cdot\sqrt{(N^2+2\cdot N)-3})-10)+6\cdot\sqrt{(N^2+2\cdot N)-3}}}{4}-R_8 = 0$$

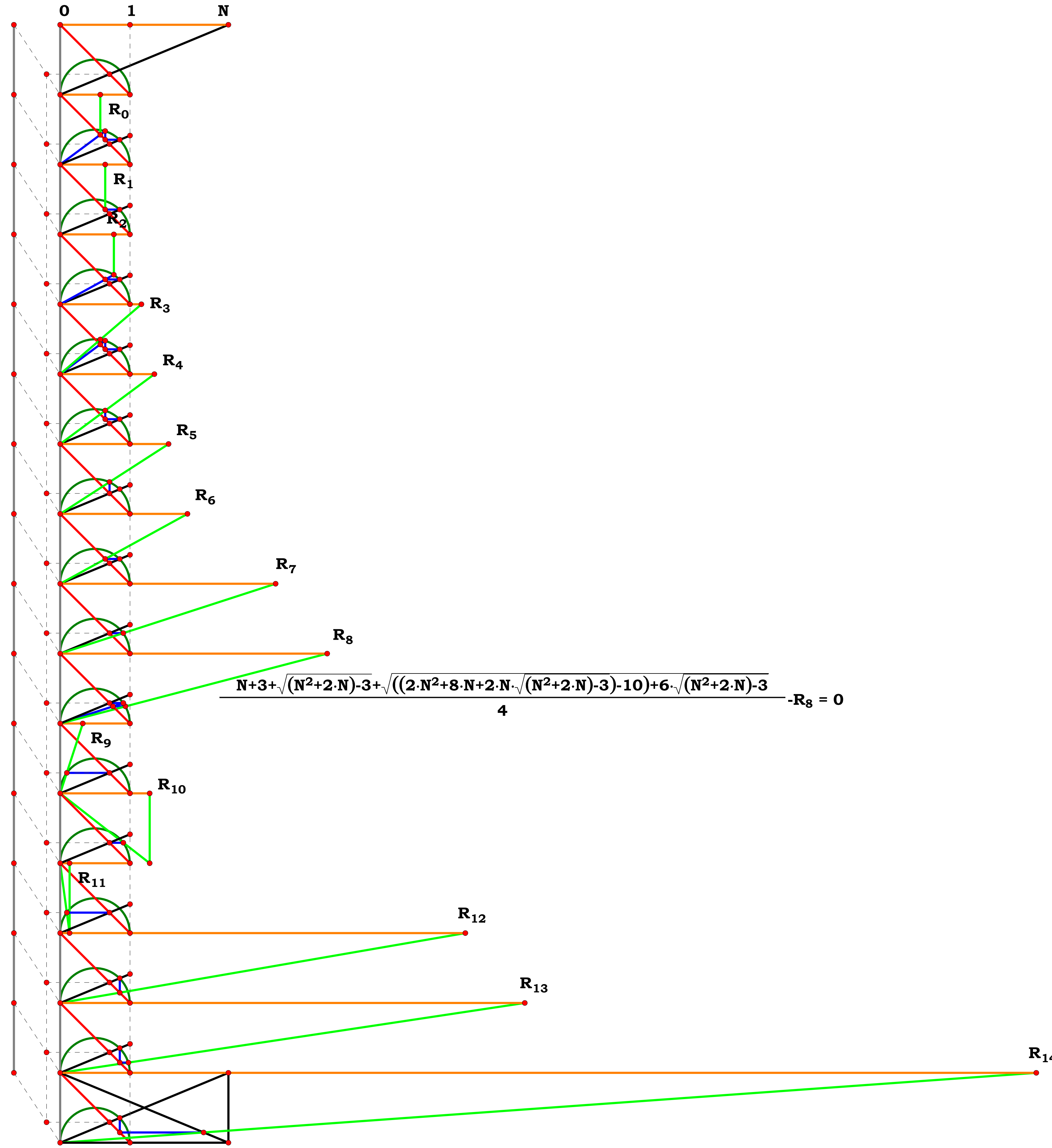
$$\frac{(N+1)-\sqrt{(N^2+2\cdot N)-3}}{2}-R_9 = 0$$

$$\frac{N+1+\sqrt{(N^2+2\cdot N)-3}}{2\cdot N}-R_{10} = 0$$

$$\frac{(N+1)-\sqrt{(N^2+2\cdot N)-3}}{2\cdot N}-R_{11} = 0$$

$$N^2-R_{12} = 0$$

$$N^3-R_{14} = 0$$



3OBT4

$N = 2.24691$

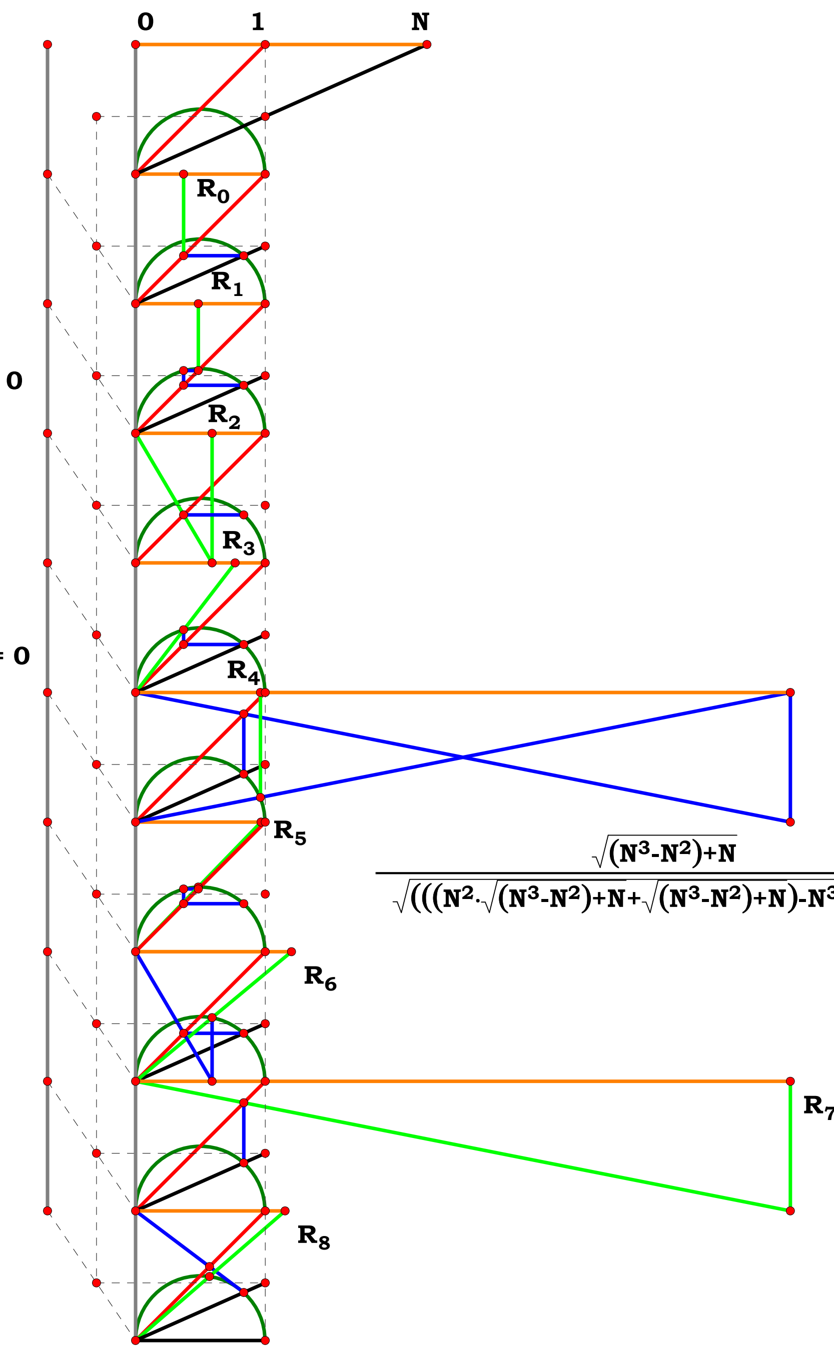
$$\frac{N}{N^2+1}-R_0 = 0$$

$$\frac{\sqrt{(N^3-N^2)+N}}{N^2+1}-R_1 = 0$$

$$\frac{N}{(N^2-N)+1}-R_2 = 0$$

$$\sqrt{\frac{N}{(N^2-N)+1}}-R_3 = 0$$

$$\frac{N^4}{N^4+1}-R_4 = 0$$

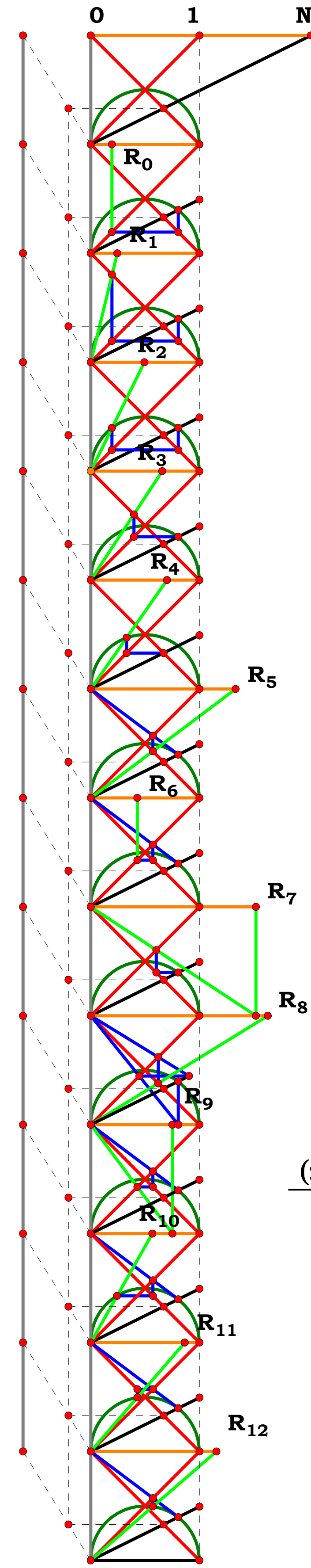


$$\frac{\sqrt{(N^3-N^2)+N}}{\sqrt{(((N^2\cdot\sqrt{(N^3-N^2)+N}+N+\sqrt{(N^3-N^2)+N})-N^3)+N^2)-N}}-R_5 = 0$$

$$\left|\frac{\sqrt{N}}{N-1}\right|-R_6 = 0$$

$$N^2-R_7 = 0$$

20BT5



N = 2.02597

$$\frac{1}{N^2+1}-R_0 = 0$$

$$\frac{1}{N^2}-R_1 = 0$$

$$\frac{1}{N}-R_2 = 0$$

$$\frac{N}{(N^2-N)+1}-R_3 = 0$$

$$\frac{1}{\sqrt{N}}-R_4 = 0$$

$$\frac{N^2}{(N^2-N)+1}-R_5 = 0$$

$$\frac{(N^2-N)+1}{(2\cdot N^2-N)+1}-R_6 = 0$$

$$\frac{(N^2-N)+1}{N}-R_7 = 0$$

$$\frac{N^3}{N^2+1}-R_8 = 0$$

$$\frac{(N^2-N)+1}{N^2}-R_9 = 0$$

$$\frac{(2\cdot N^2+1)-N-\sqrt{((4\cdot N^3-7\cdot N^2)+6\cdot N)-3}}{(2\cdot N^2-2\cdot N)+2}-R_{10} = 0$$

$$\frac{(N^2-N)+1}{\sqrt{(N^4-N^3)+N^2}}-R_{11} = 0$$

$$\frac{N}{\sqrt{(N^2-N)+1}}-R_{12} = 0$$

3OBT6

$N = 2.00000$

$$\frac{N+1}{N+1+\sqrt{N}}-R_0 = 0$$

$$\sqrt{N-1}-R_1 = 0$$

$$\frac{N^2+2\cdot N+1}{N^2+3\cdot N+1}-R_2 = 0$$

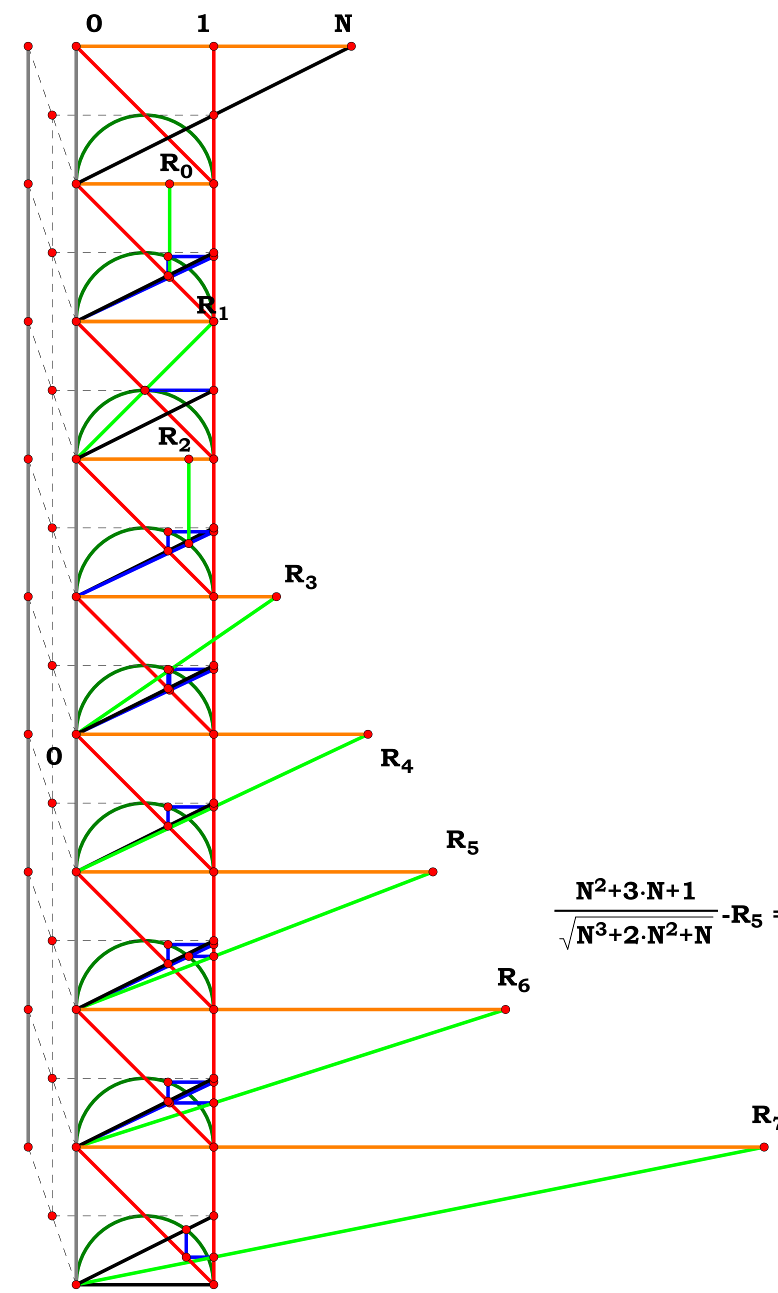
$$\frac{N+1}{\sqrt{N\cdot\sqrt{N}+\sqrt{N}}}-R_3 = 0$$

$$\frac{N+1}{\sqrt{N}}-R_4 = 0$$

$$\frac{N^2+3\cdot N+1}{\sqrt{N^3}+\sqrt{N}}-R_5 = 0$$

$$\frac{N+1+\sqrt{N}}{\sqrt{N}}-R_6 = 0$$

$$(N^2+1)-R_7 = 0$$



$$\frac{N^2+3\cdot N+1}{\sqrt{N^3+2\cdot N^2+N}}-R_5 = 0.00000$$

2OBT7

$N = 1.30233$

$N-\sqrt{N-1}-R_0 = 0$

$\frac{1}{\sqrt{N-1}}-R_1 = 0$

$\frac{N+1}{N}-R_2 = 0$

$\frac{N}{\sqrt{N-1}}-R_3 = 0$

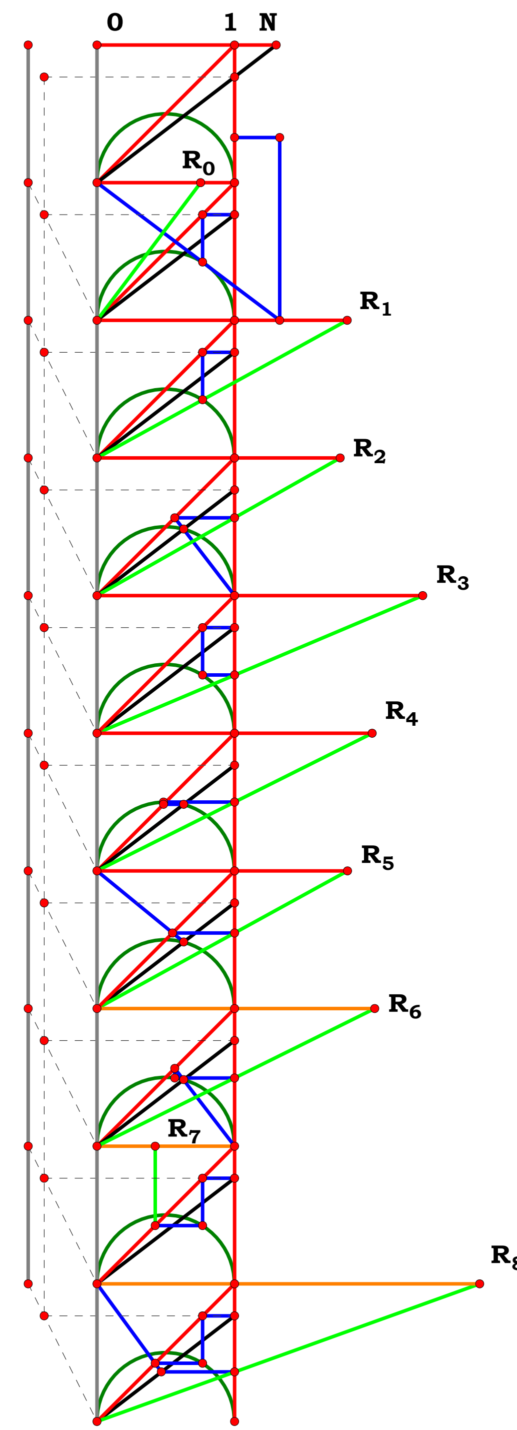
$\frac{N^2+1}{\sqrt{(N^3-N^2)+N}}-R_4 = 0$

$\frac{(2\cdot N^2-N)+1}{N^2}-R_5 = 0$

$\frac{N+1}{\sqrt{N}}-R_6 = 0$

$\frac{\sqrt{N-1}}{N}-R_7 = 0$

$\frac{(N^2-N\cdot\sqrt{N-1})+\sqrt{N-1}}{\sqrt{N-1}}-R_8 = 0$

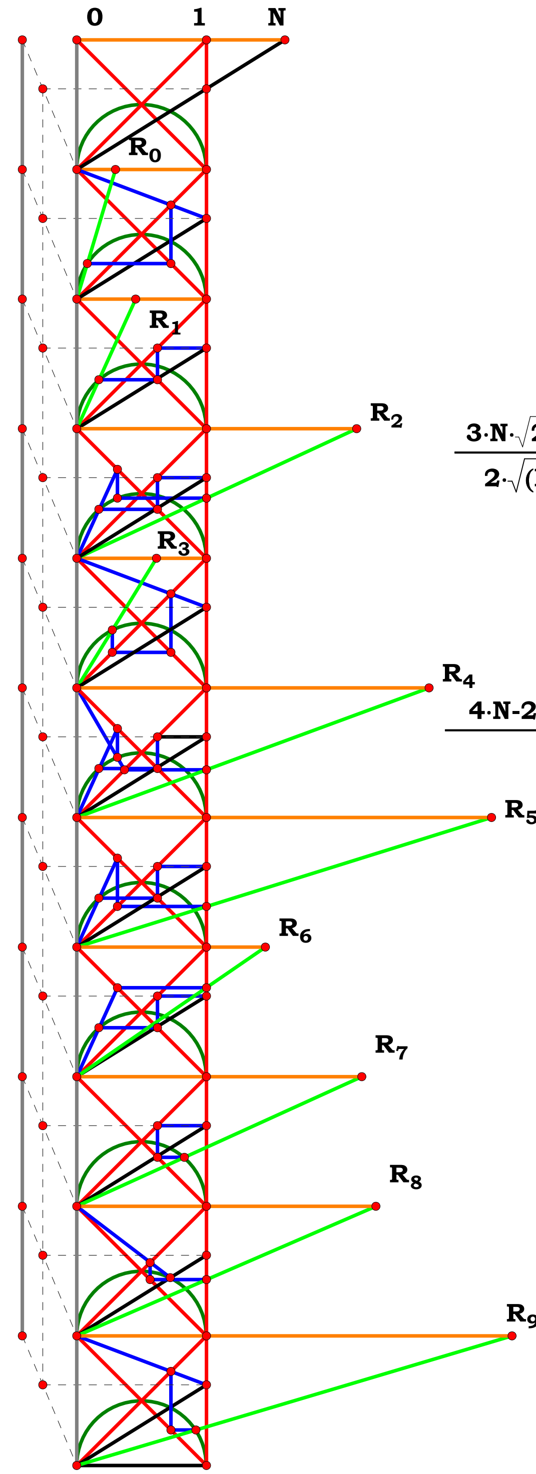


2OBT8

N = 1.60494

$$\frac{2\cdot N-1-\sqrt{4\cdot N-3}}{2\cdot N-2}-R_0 = 0$$

$$\frac{N-\sqrt{8\cdot N-3\cdot N^2-4}}{2\cdot N-2}-R_1 = 0$$



$$\frac{3\cdot N\cdot\sqrt{2}-\sqrt{2}\cdot\sqrt{8\cdot N-4-3\cdot N^2}-2\cdot\sqrt{2}}{2\cdot\sqrt{(N-1)\cdot\left(N-\sqrt{8\cdot N-4-3\cdot N^2}\right)}}-R_2 = 0$$

$$\frac{N-1}{\sqrt{N^2-N}}-R_3 = 0$$

$$\frac{4\cdot N-2-2\cdot\sqrt{8\cdot N-3\cdot N^2-4}-\sqrt{2}\cdot\sqrt{\left(N^2-N-\sqrt{8\cdot N-3\cdot N^2-4}\cdot N\right)+\sqrt{8\cdot N-3\cdot N^2-4}}}{N-\sqrt{8\cdot N-3\cdot N^2-4}}-R_4 = 0$$

$$\frac{3\cdot N-2-\sqrt{8\cdot N-3\cdot N^2-4}}{N-\sqrt{8\cdot N-3\cdot N^2-4}}-R_5 = 0$$

$$\frac{3\cdot N-2-\sqrt{8\cdot N-3\cdot N^2-4}}{2\cdot N-2}-R_6 = 0$$

$$\frac{N+\sqrt{8\cdot N-3\cdot N^2-4}}{2\cdot N-2}-R_7 = 0$$

$$\frac{(2\cdot N^2-N)+1}{(N^2-N)+1}-R_8 = 0$$

$$\frac{2\cdot N-2}{2\cdot N-1-\sqrt{4\cdot N-3}}-R_9 = 0$$

03BT9

N = 0.61628

$$\frac{(N^2+2\cdot N)-N\cdot\sqrt{4\cdot N-3\cdot N^2}}{2\cdot N^2+2}-R_0=0$$

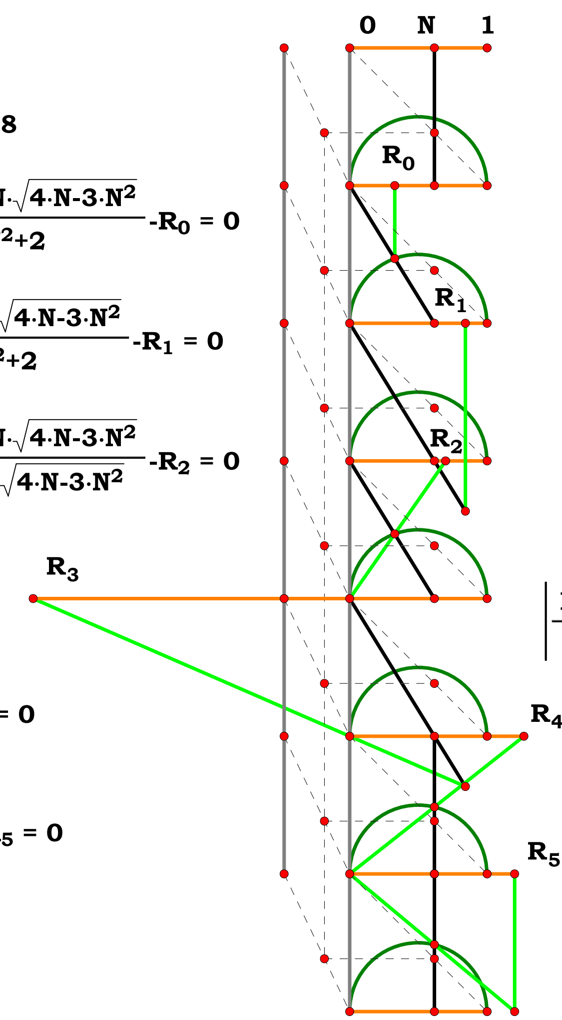
$$\frac{N^2+2\cdot N+N\cdot\sqrt{4\cdot N-3\cdot N^2}}{2\cdot N^2+2}-R_1=0$$

$$\frac{(N^2+2\cdot N)-N\cdot\sqrt{4\cdot N-3\cdot N^2}}{(2\cdot N^2-N)+\sqrt{4\cdot N-3\cdot N^2}}-R_2=0$$

$$\left|\frac{N^2+2\cdot N+N\cdot\sqrt{4\cdot N-3\cdot N^2}}{2\cdot N^2-N-\sqrt{4\cdot N-3\cdot N^2}}\right|-R_3=0$$

$$\frac{N}{\sqrt{N-N^2}}-R_4=0$$

$$\frac{N}{1-\sqrt{N-N^2}}-R_5=0$$



3OBT10A

N = 0.68605

$$\frac{(N^2+1)-\sqrt{(2\cdot N^2+1)-3\cdot N^4}}{2\cdot N^2+2}-R_0 = 0$$

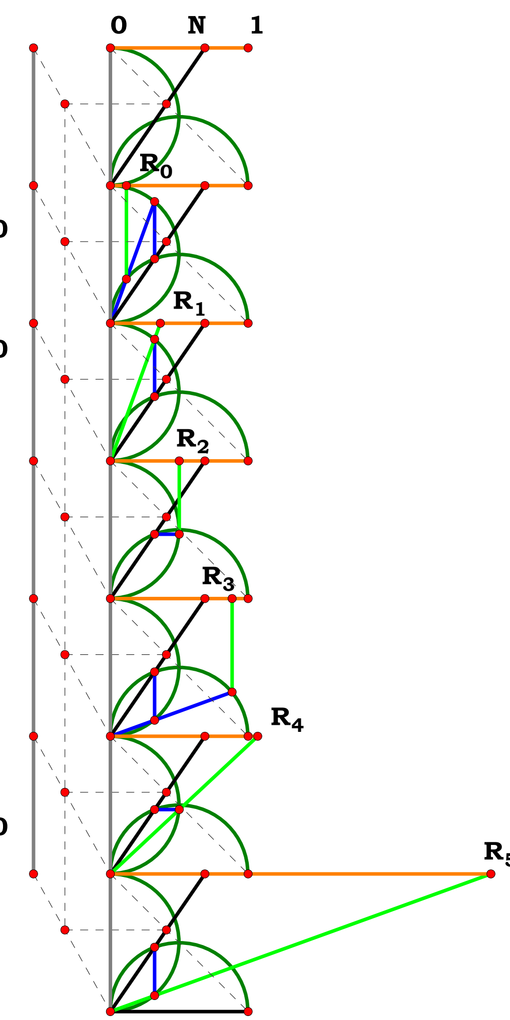
$$\frac{(N^2+1)-\sqrt{(2\cdot N^2+1)-3\cdot N^4}}{2\cdot N^2}-R_1 = 0$$

$$\frac{\sqrt{(N^3-N^2)+N}}{N^2+1}-R_2 = 0$$

$$\frac{N^2+\sqrt{(2\cdot N^2-3\cdot N^4)+1}+1}{2\cdot N^2+2}-R_3 = 0$$

$$\sqrt{\frac{(N^2-N)+1}{N}}-R_4 = 0$$

$$\frac{2\cdot N^2}{(N^2+1)-\sqrt{(2\cdot N^2+1)-3\cdot N^4}}-R_5 = 0$$



3OBT10B

N = 1.61628

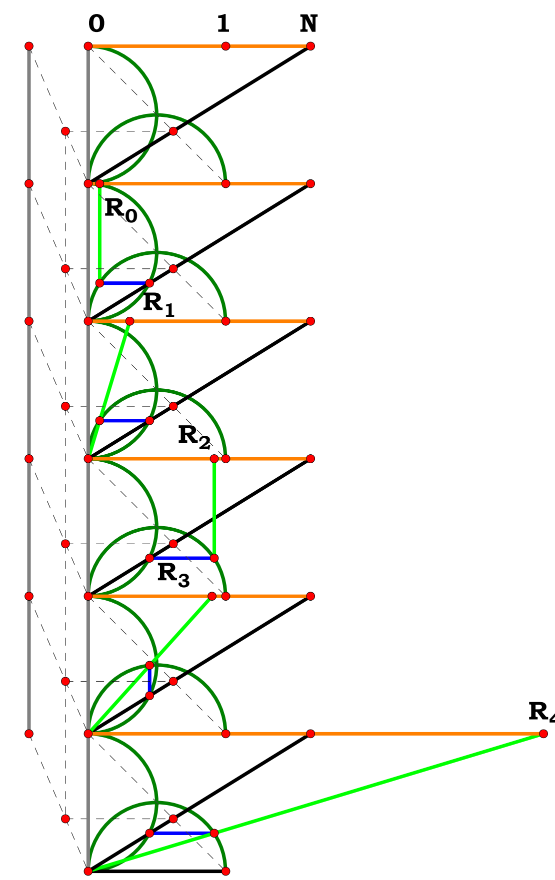
$$\frac{(N^2+1)-\sqrt{(N^4+2\cdot N^2)-3}}{2\cdot N^2+2}-R_0 = 0$$

$$\frac{(N^2+1)-\sqrt{(N^4+2\cdot N^2)-3}}{2}-R_1 = 0$$

$$\frac{N^2+1+\sqrt{(N^4+2\cdot N^2)-3}}{2\cdot N^2+2}-R_2 = 0$$

$$\sqrt{\frac{N}{(N^2-N)+1}}-R_3 = 0$$

$$\frac{N^2+1+\sqrt{(N^4+2\cdot N^2)-3}}{2}-R_4 = 0$$



3OBT10C

N = 0.36047

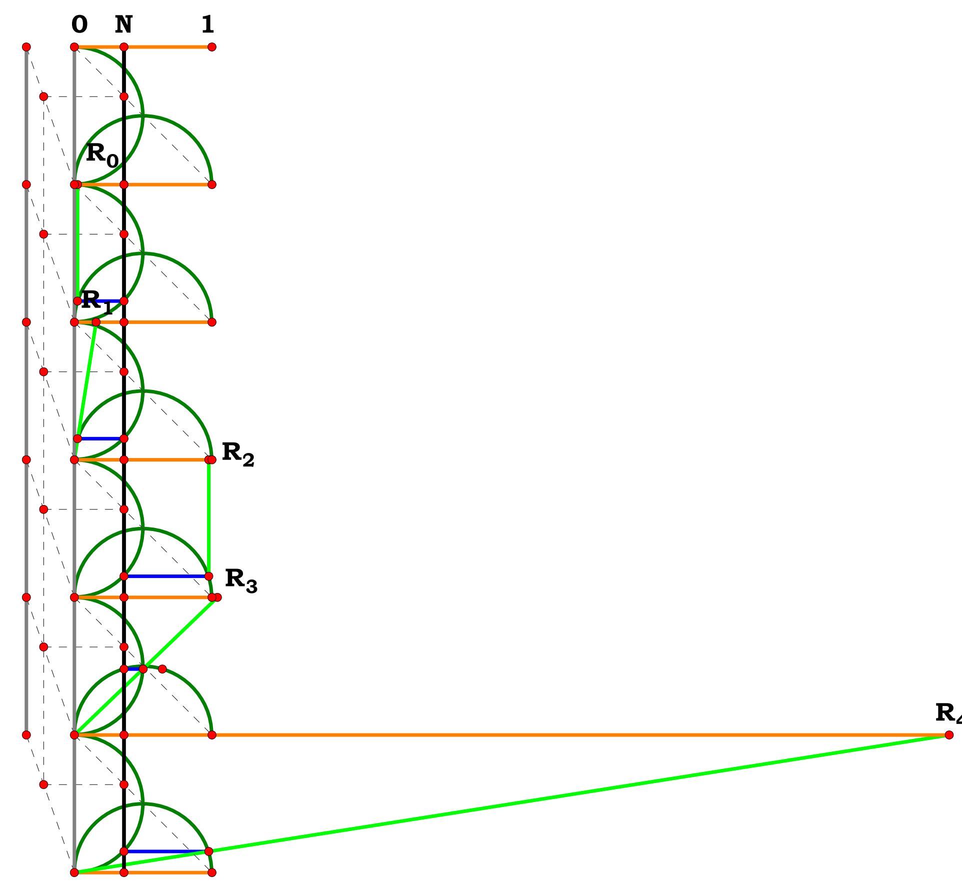
$$\frac{1-\sqrt{(4\cdot N^2-1)+2\cdot\sqrt{1-4\cdot N^2}}}{2}-R_0=0$$

$$\frac{\sqrt{(4\cdot N^2-1)+2\cdot\sqrt{1-4\cdot N^2}}-1}{\sqrt{1-4\cdot N^2}-1}-R_1=0$$

$$\frac{1+\sqrt{(4\cdot N^2-1)+2\cdot\sqrt{1-4\cdot N^2}}}{2}-R_2=0$$

$$\frac{\sqrt{(N^2-N)+\sqrt{N-N^2}}}{\sqrt{N-N^2}}-R_3=0$$

$$\frac{1+\sqrt{(4\cdot N^2-1)+2\cdot\sqrt{1-4\cdot N^2}}}{1-\sqrt{1-4\cdot N^2}}-R_4=0$$



Just Trees.

4RST1

4RST1AB1

N = 1.29070

$$\left|\frac{(2\cdot N^5+2\cdot N^3)-(N^6+N^4)}{((N^6+3\cdot N^4+2\cdot N^3)-2\cdot N)+1}\right|-R_0 = 0$$

$$\left|\frac{2\cdot N-N^2}{N^2+1}\right|-R_1 = 0$$

$$\left|\frac{(4\cdot N^4+N^3)-2\cdot N^2-N^6}{((N^6+2\cdot N^5)-2\cdot N^3)+3\cdot N^2+1}\right|-R_2 = 0$$

$$\left|\frac{2\cdot N-N^2}{(N^2+N)-1}\right|-R_3 = 0$$

$$\left|\frac{2\cdot N-N^2}{(N^2+2\cdot N)-3}\right|-R_4 = 0$$

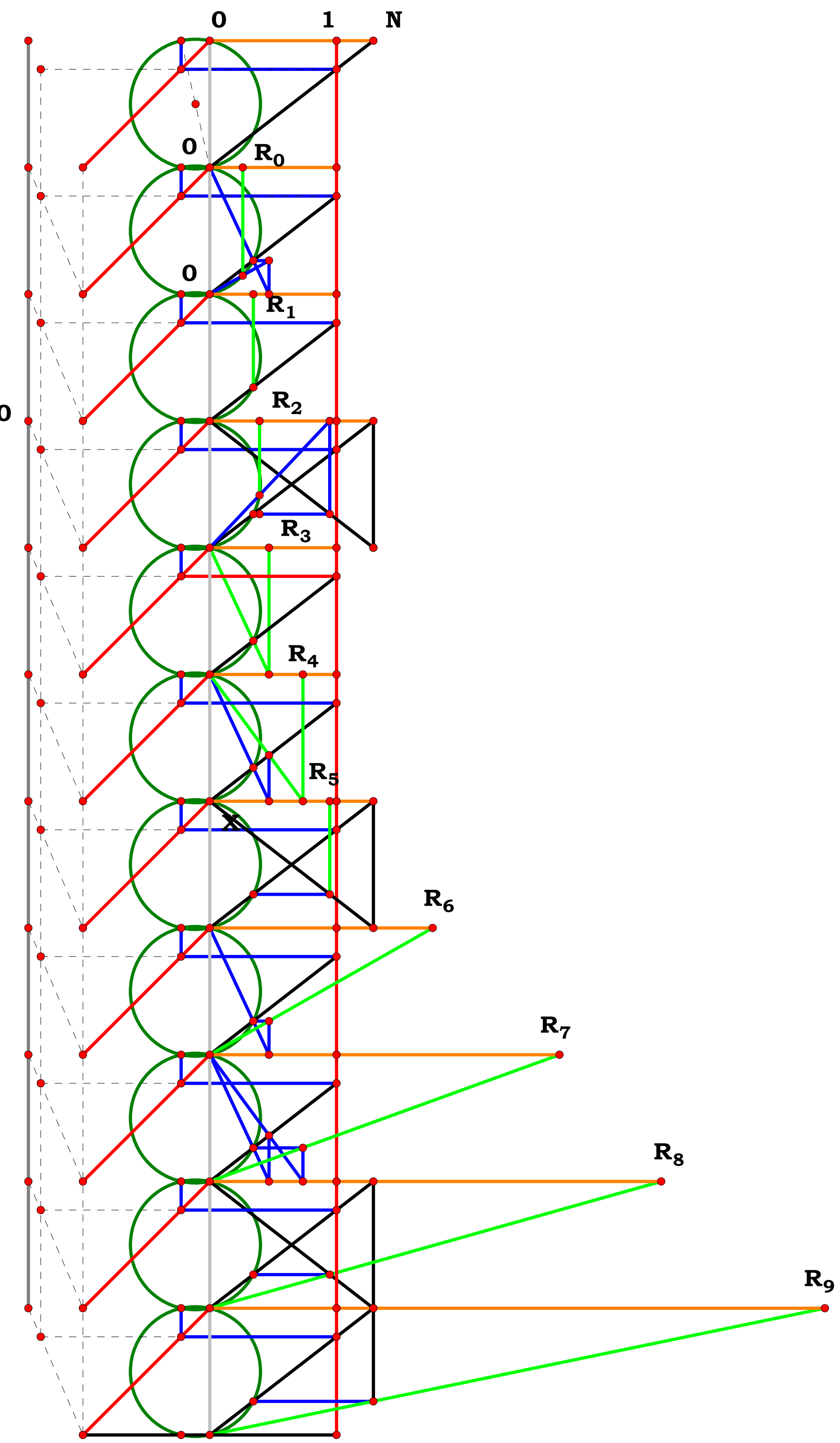
$$\left|\frac{(N^3+N^2)-N}{N^2+1}\right|-R_5 = 0$$

$$\left|\frac{N^3+N}{(N^2+N)-1}\right|-R_6 = 0$$

$$\left|\frac{N^3+N}{(N^2+2\cdot N)-3}\right|-R_7 = 0$$

$$\left|\frac{(N^3+N^2)-N}{2-N}\right|-R_8 = 0$$

$$\left|\frac{N^3+N}{N-2}\right|-R_9 = 0$$



4RST1AB3

N = 0.87209

$$\left|\frac{(N^4+N^3+2\cdot N^2+N)-N^6}{N^6+2\cdot N^5+4\cdot N^4+8\cdot N^3+7\cdot N^2+2\cdot N+1}\right|-R_0 = 0$$

$$\left|\frac{N^3-N^2-N}{1-N^3-N^2-N-2}\right|-R_1 = 0$$

$$\left|\frac{(6\cdot N^6+6\cdot N^5+5\cdot N^4+2\cdot N^3)-N^9-3\cdot N^8}{N^9+5\cdot N^8+9\cdot N^7+7\cdot N^6+5\cdot N^5+7\cdot N^4+7\cdot N^3+5\cdot N^2+3\cdot N+1}\right|-R_2 = 0$$

$$\left|\frac{(N-N^2)+1}{N^2+2\cdot N}\right|-R_3 = 0$$

$$\left|\frac{(N^2-N^3)+N}{(N^3+3\cdot N^2)-N-1}\right|-R_4 = 0$$

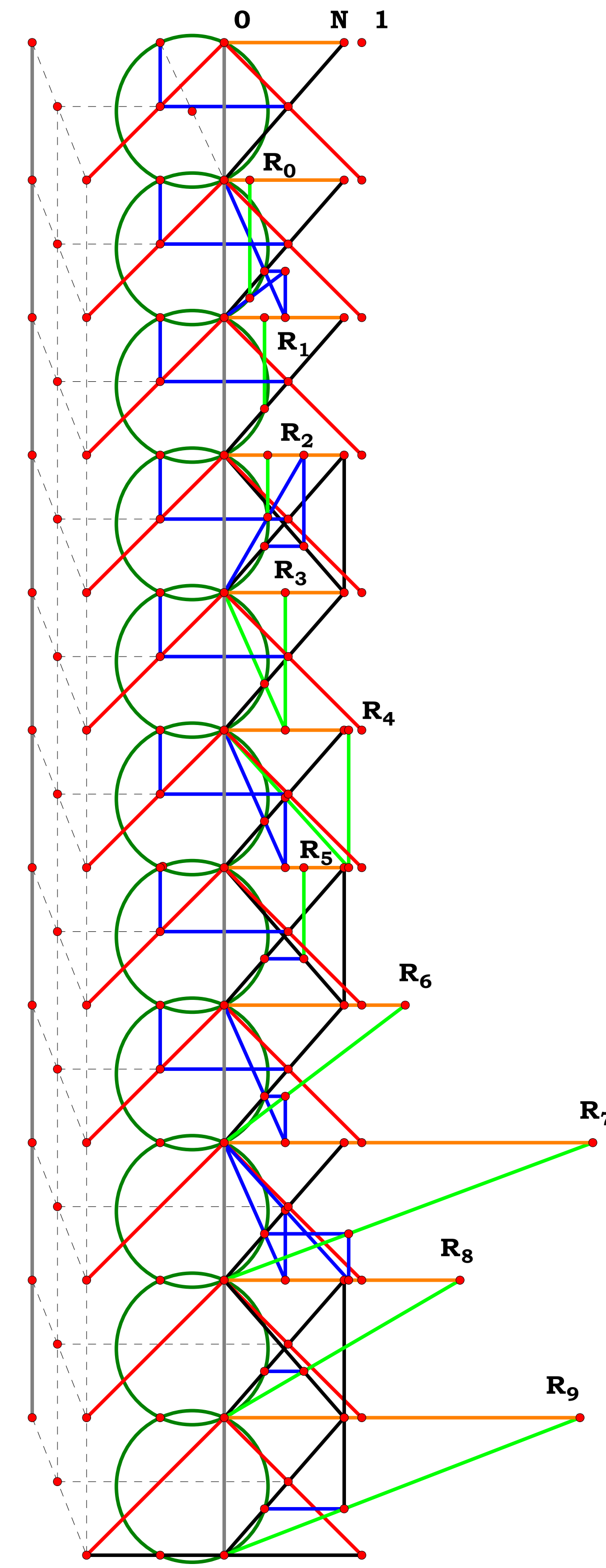
$$\frac{N^4+2\cdot N^3}{N^3+N^2+N+1}-R_5 = 0$$

$$\frac{N^3+N^2+N+1}{N^2+2\cdot N}-R_6 = 0$$

$$\left|\frac{N^4+N^3+N^2+N}{(N^3+3\cdot N^2)-N-1}\right|-R_7 = 0$$

$$\left|\frac{N^4+2\cdot N^3}{(N-N^2)+1}\right|-R_8 = 0$$

$$\left|\frac{N^4+N^3+N^2+N}{(N-N^2)+1}\right|-R_9 = 0$$



4RST1AB4

N = 0.81395

$$\frac{N^4+N^3+N^2+N}{N^6+2\cdot N^5+4\cdot N^4+6\cdot N^3+6\cdot N^2+4\cdot N+2}-R_0 = 0$$

$$\frac{N}{N^3+N^2+N+1}-R_1 = 0$$

$$\frac{N^7+2\cdot N^6+4\cdot N^5+4\cdot N^4+3\cdot N^3+N^2}{N^9+3\cdot N^8+6\cdot N^7+8\cdot N^6+8\cdot N^5+8\cdot N^4+7\cdot N^3+5\cdot N^2+3\cdot N+1}-R_2 = 0$$

$$\frac{1}{N^2+N+1}-R_3 = 0$$

$$\left|\frac{N}{(N^3+N^2+N)-1}\right|-R_4 = 0$$

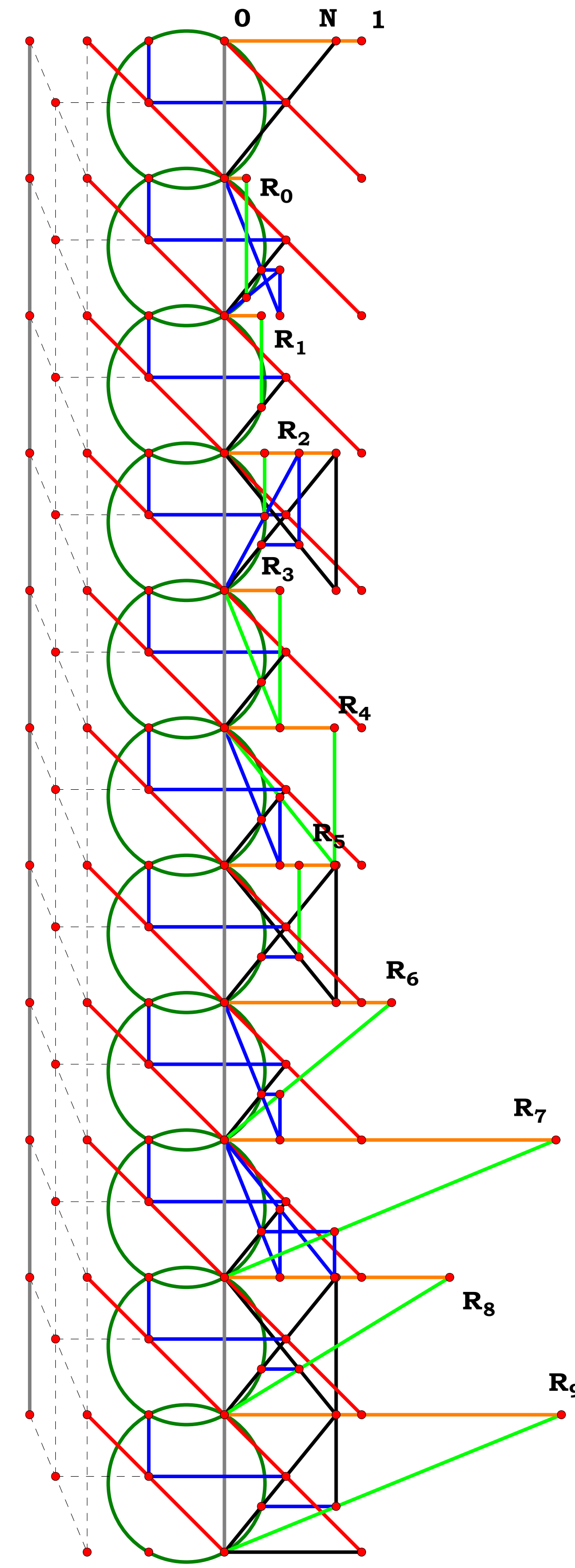
$$\frac{N^4+N^3+N^2}{N^3+N^2+N+1}-R_5 = 0$$

$$\frac{N^3+N^2+N+1}{N^2+N+1}-R_6 = 0$$

$$\left|\frac{N^4+N^3+N^2+N}{(N^3+N^2+N)-1}\right|-R_7 = 0$$

$$(N^4+N^3+N^2)-R_8 = 0$$

$$(N^4+N^3+N^2+N)-R_9 = 0$$



4RST1AB5

N = 0.71765

$$\left|\frac{N-N^5}{N^4+6\cdot N^2+1}\right|-R_0 = 0$$

$$\left|\frac{N-N^3}{N^2+1}\right|-R_1 = 0$$

$$\left|\frac{(2\cdot N^5+2\cdot N^3)-4\cdot N^7}{4\cdot N^6+N^4+2\cdot N^2+1}\right|-R_2 = 0$$

$$\left|\frac{1-N^2}{2\cdot N}\right|-R_3 = 0$$

$$\left|\frac{N-N^3}{3\cdot N^2-1}\right|-R_4 = 0$$

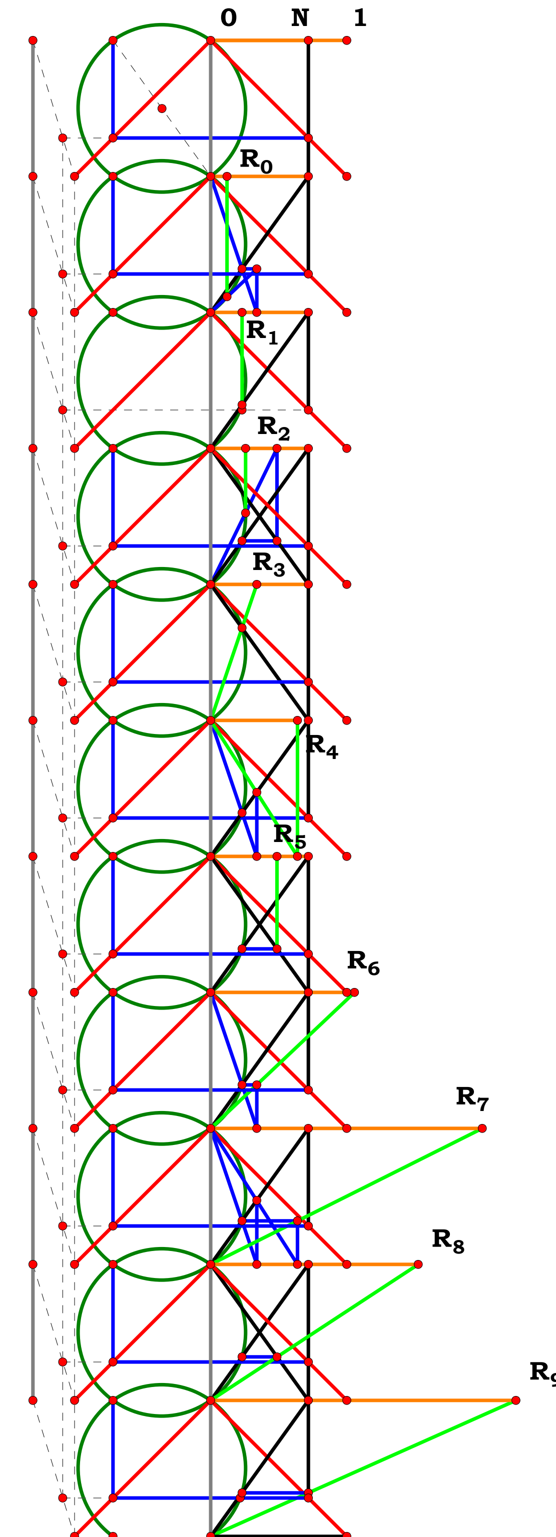
$$\frac{2\cdot N^3}{N^2+1}-R_5 = 0$$

$$\frac{N^2+1}{2\cdot N}-R_6 = 0$$

$$\left|\frac{N^3+N}{3\cdot N^2-1}\right|-R_7 = 0$$

$$\left|\frac{2\cdot N^3}{1-N^2}\right|-R_8 = 0$$

$$\left|\frac{N^3+N}{1-N^2}\right|-R_9 = 0$$



4RST1AB6

N = 0.50000

$$\frac{(N^5+2\cdot N^3+N)-(N^4+N^2)}{N^4+2\cdot N^2+2}-R_0 = 0$$

$$\frac{(N^3+N)-N^2}{N^2+1}-R_1 = 0$$

$$\frac{N^5+N^2}{2\cdot N^4+2\cdot N^2+1}-R_2 = 0$$

$$((N^2-N)+1)-R_3 = 0$$

$$\left|\frac{N^2-(N^3+N)}{(N^2-2\cdot N)+1}\right|-R_4 = 0$$

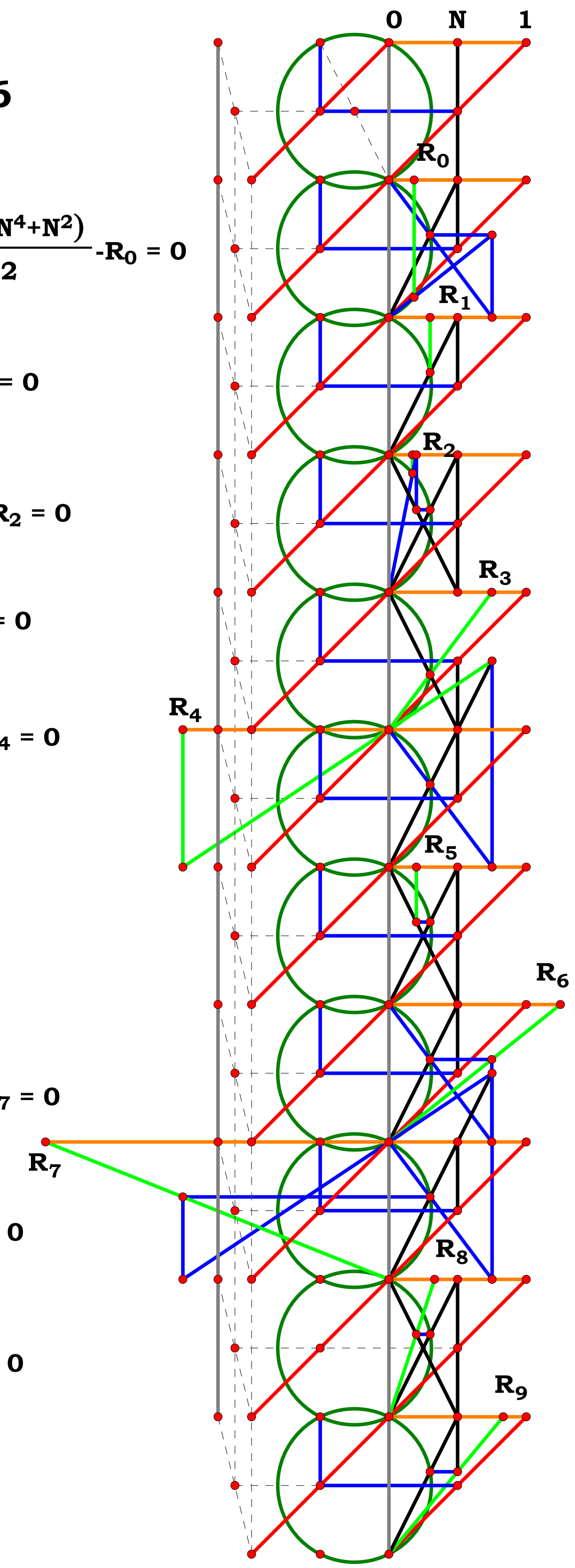
$$\frac{N^2}{N^2+1}-R_5 = 0$$

$$(N^2+1)-R_6 = 0$$

$$\left|\frac{N^3+N}{2\cdot N-(N^2+1)}\right|-R_7 = 0$$

$$\frac{N^2}{(N^2-N)+1}-R_8 = 0$$

$$\frac{N^3+N}{(N^2-N)+1}-R_9 = 0$$



4RST1BB1

N = 1.26744

$$\left|\frac{((7\cdot N^3-9\cdot N^2)+6\cdot N)-2\cdot N^4}{(2\cdot N^4-4\cdot N^3)+6\cdot N^2+1}\right|-R_0 = 0$$

$$\left|\frac{(N^7+N^6+N^4)-N^3}{((((((N^8+4\cdot N^7+N^6)-6\cdot N^5)+8\cdot N^4)-8\cdot N^3)+6\cdot N^2)-2\cdot N)+1}\right|-R_2 = 0$$

$$\left|\frac{2-N}{N}\right|-R_3 = 0$$

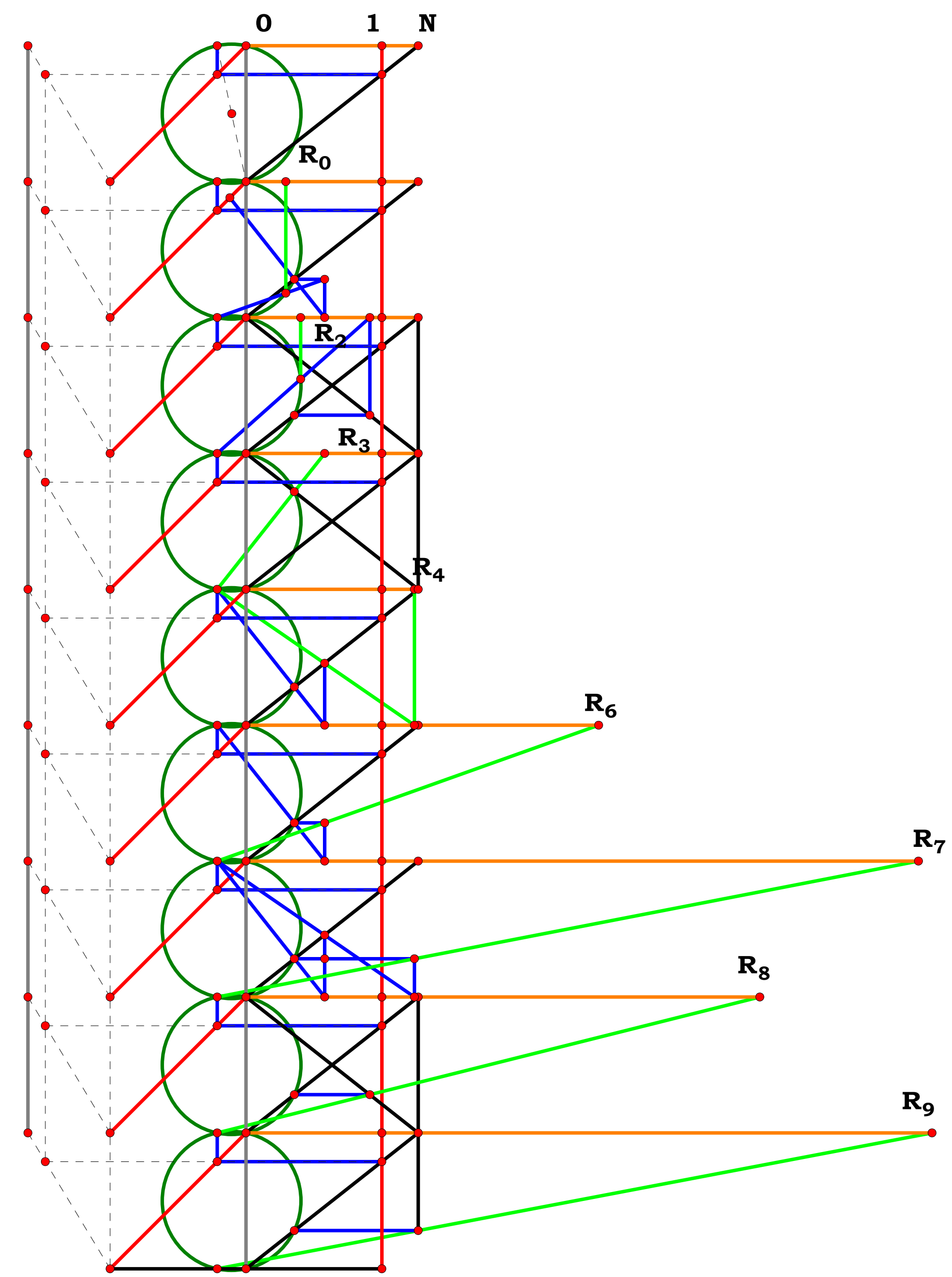
$$\left|\frac{((N^2-N^3)+3\cdot N)-2}{(N^3+N^2)-2\cdot N}\right|-R_4 = 0$$

$$\left|\frac{3\cdot N-2\cdot N^2-3}{N^2-2\cdot N}\right|-R_6 = 0$$

$$\left|\frac{(((2\cdot N^3-2\cdot N^4)+2\cdot N^2)-8\cdot N)+4}{(N^4-N^3-4\cdot N^2)+4\cdot N}\right|-R_7 = 0$$

$$\left|\frac{((N^4+2\cdot N^3)-N^2-2\cdot N)+1}{2\cdot N-N^2}\right|-R_8 = 0$$

$$\left|\frac{((N^4+N^3+N^2)-2\cdot N)+1}{2\cdot N-N^2}\right|-R_9 = 0$$



4RST1BB3

N = 1.11628

$$\left|\frac{(N^6+2\cdot N^5+2\cdot N^3+3\cdot N^2+N)-2\cdot N^7}{2\cdot N^7+2\cdot N^6+2\cdot N^5+8\cdot N^4+10\cdot N^3+6\cdot N^2+3\cdot N+1}\right|-R_0=0$$

$$\frac{N^7+3\cdot N^6+3\cdot N^5+3\cdot N^4+2\cdot N^3}{N^8+6\cdot N^7+10\cdot N^6+4\cdot N^5+9\cdot N^4+4\cdot N^3+4\cdot N^2+2\cdot N+1}-R_2=0$$

$$\left|\frac{(N+1)-N^2}{N^2+N}\right|-R_3=0$$

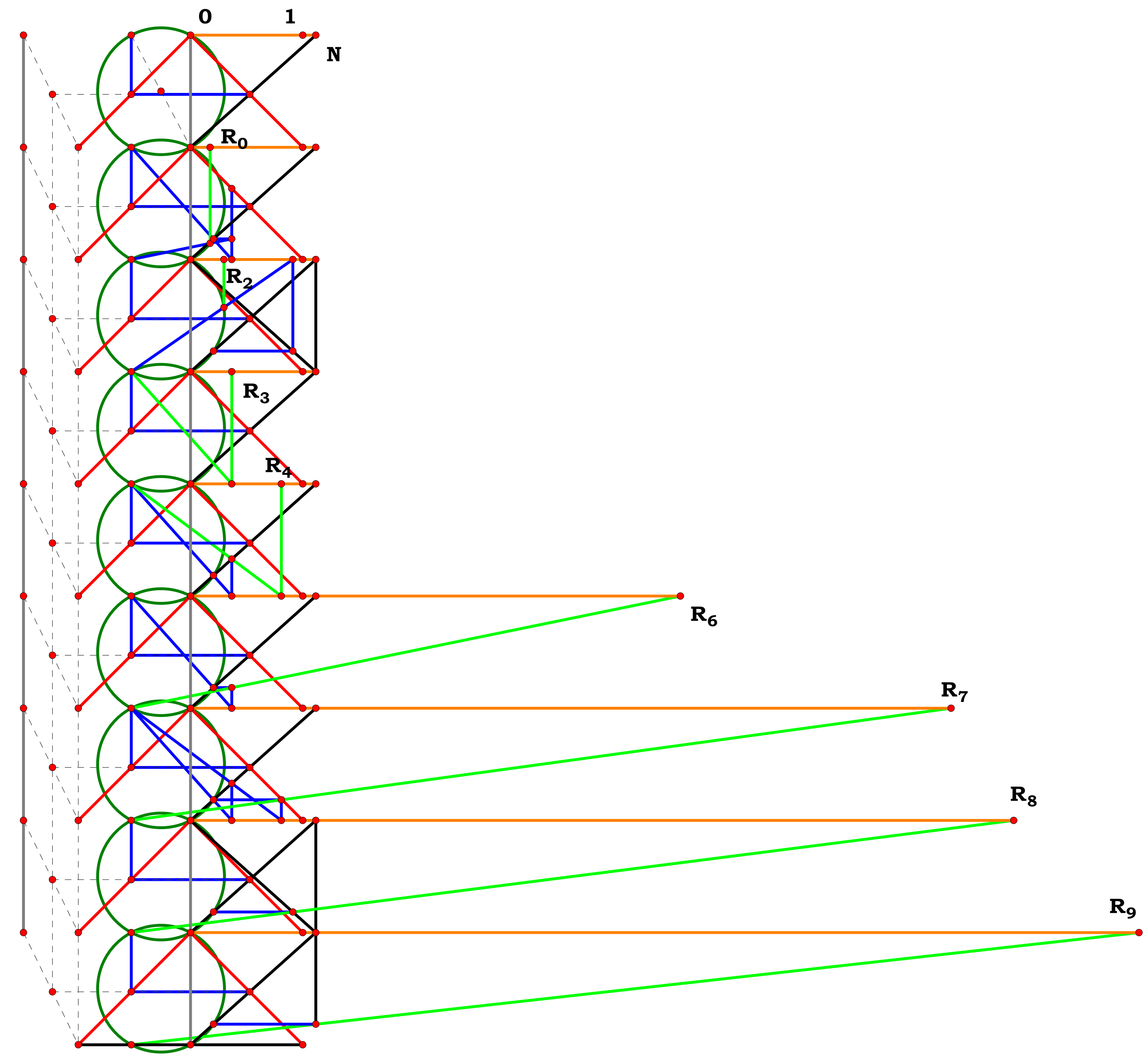
$$\left|\frac{(3\cdot N^2-N^4-N^3)+2\cdot N}{(N^4+3\cdot N^3+N^2)-2\cdot N-1}\right|-R_4=0$$

$$\left|\frac{2\cdot N^4+N^3+N^2+2\cdot N+1}{(2\cdot N^2-N^4)+N}\right|-R_6=0$$

$$\left|\frac{2\cdot N^6+4\cdot N^5+2\cdot N^3+5\cdot N^2+2\cdot N}{((3\cdot N^4-N^6-2\cdot N^5)+6\cdot N^3)-3\cdot N-1}\right|-R_7=0$$

$$\left|\frac{N^5+4\cdot N^4+4\cdot N^3}{(2\cdot N-N^3)+1}\right|-R_8=0$$

$$\left|\frac{N^5+3\cdot N^4+4\cdot N^3+2\cdot N^2+N}{(2\cdot N-N^3)+1}\right|-R_9=0$$



4RST1BB4

N = 1.12791

$$\frac{N^5+2\cdot N^4+2\cdot N^3+N^2+N}{N^7+3\cdot N^6+5\cdot N^5+7\cdot N^4+8\cdot N^3+6\cdot N^2+3\cdot N+1}-R_0 = 0$$

$$\frac{N^7+2\cdot N^6+3\cdot N^5+3\cdot N^4+2\cdot N^3+N^2}{N^8+2\cdot N^7+6\cdot N^6+6\cdot N^5+9\cdot N^4+6\cdot N^3+7\cdot N^2+2\cdot N+2}-R_2 = 0$$

$$\frac{1}{N^2+N}-R_3 = 0$$

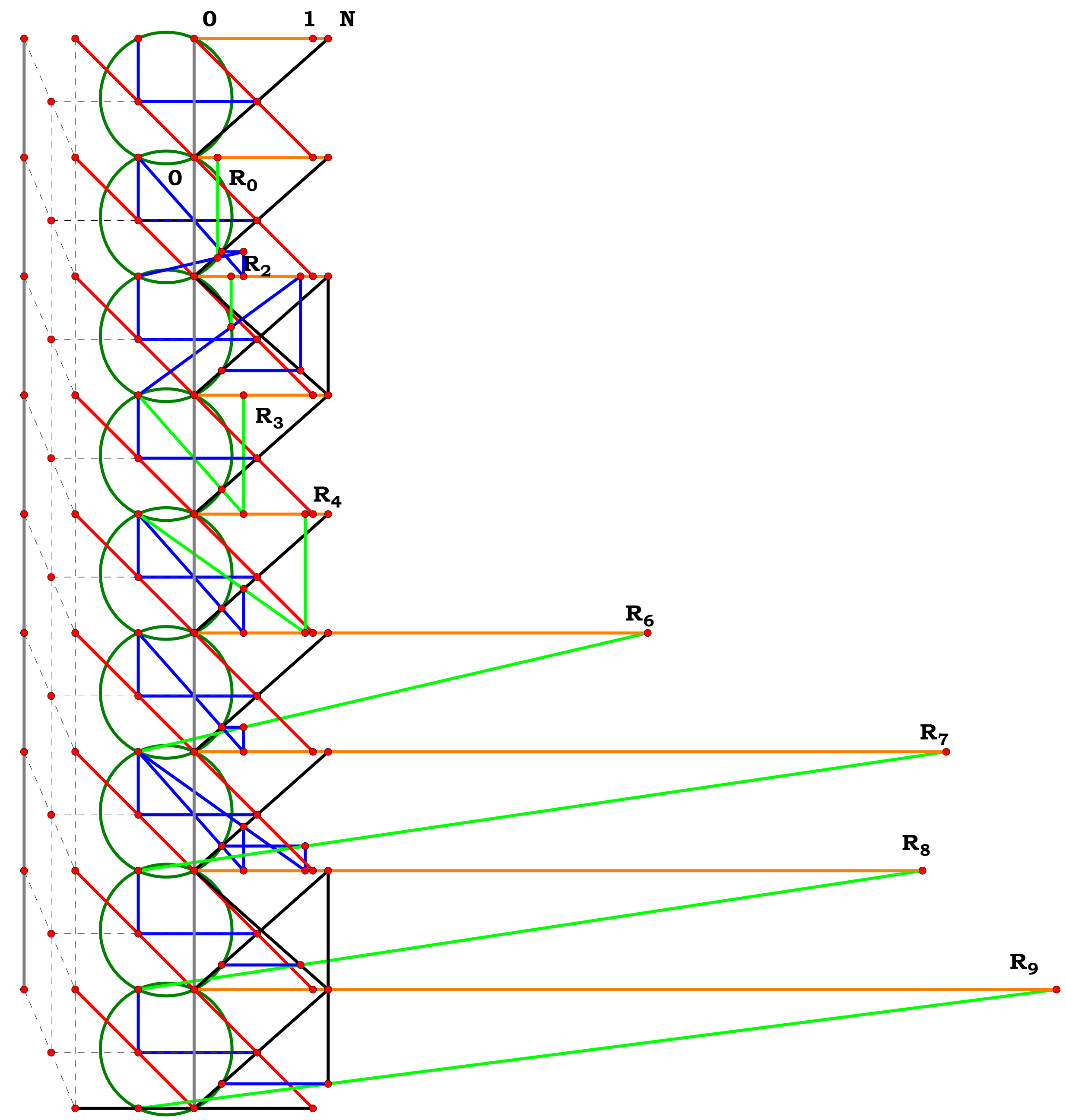
$$\left|\frac{N^2+N+1}{(N^4+2\cdot N^3+N^2)-N-1}\right|-R_4 = 0$$

$$\frac{N^4+2\cdot N^3+2\cdot N^2+N+1}{N^2+N}-R_6 = 0$$

$$\left|\frac{N^6+3\cdot N^5+4\cdot N^4+3\cdot N^3+2\cdot N^2+N+1}{(N^4+2\cdot N^3+N^2)-N-1}\right|-R_7 = 0$$

$$\frac{N^5+2\cdot N^4+3\cdot N^3+2\cdot N^2+N}{N+1}-R_8 = 0$$

$$\frac{N^5+2\cdot N^4+3\cdot N^3+3\cdot N^2+2\cdot N}{N+1}-R_9 = 0$$



4RST1BB5

N = 0.84706

$$\left|\frac{(N^5-N^7-N^3)+N}{(N^6-N^4)+3\cdot N^2+1}\right|-R_0 = 0$$

$$\frac{2\cdot N^5+2\cdot N^3}{9\cdot N^6+7\cdot N^4+3\cdot N^2+1}-R_2 = 0$$

$$\frac{1-N^2}{N}-R_3 = 0$$

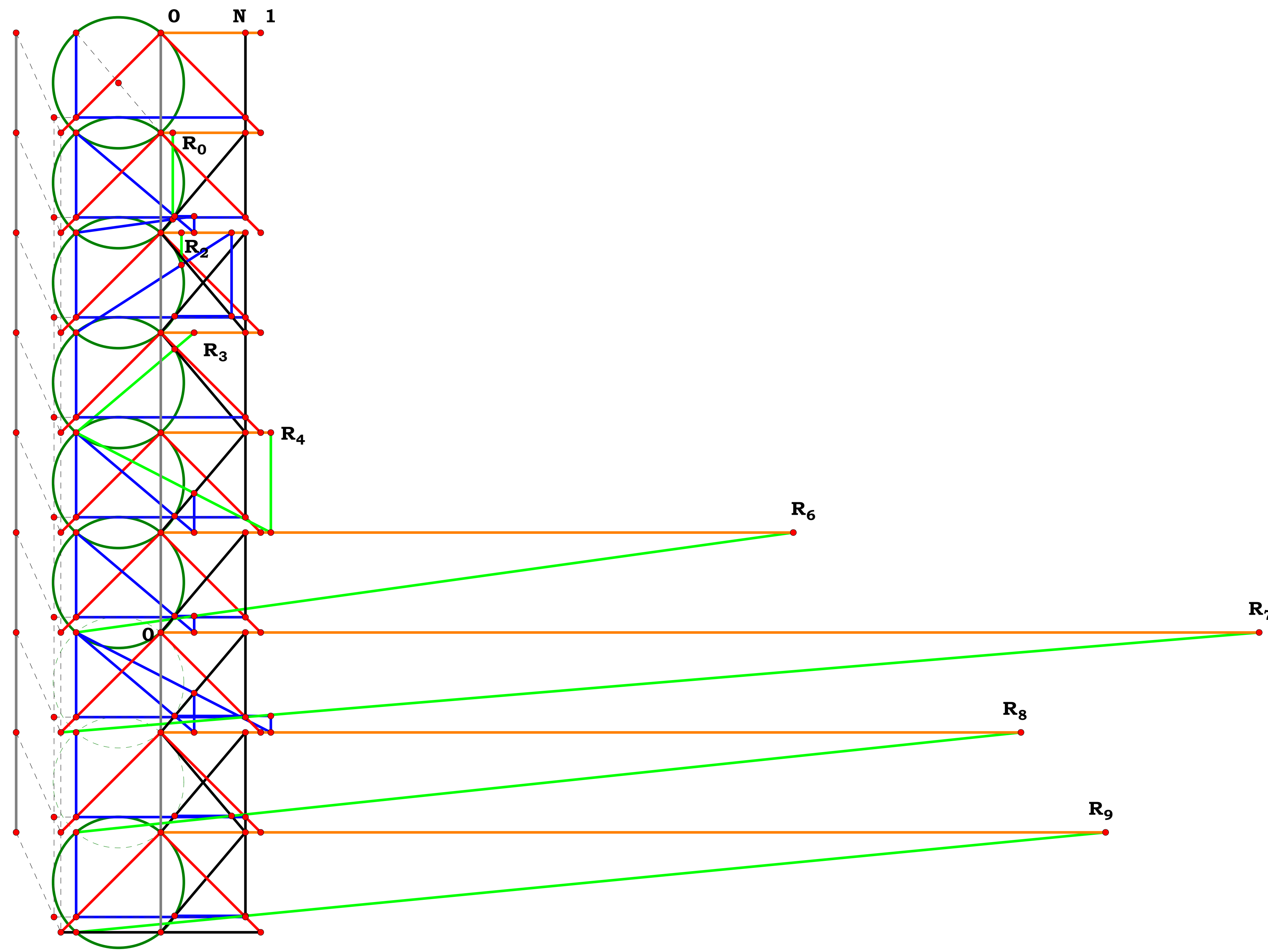
$$\left|\frac{2\cdot N-2\cdot N^3}{2\cdot N^2-1}\right|-R_4 = 0$$

$$\left|\frac{N^4+1}{N-N^3}\right|-R_6 = 0$$

$$\left|\frac{2\cdot N^3-2\cdot N^5-2\cdot N}{(2\cdot N^4-3\cdot N^2)+1}\right|-R_7 = 0$$

$$\left|\frac{4\cdot N^3}{1-N^2}\right|-R_8 = 0$$

$$\left|\frac{3\cdot N^3+N}{1-N^2}\right|-R_9 = 0$$



4RST1BB6

N = 0.46512

$$\left|\frac{(((((N^7-3\cdot N^6)+6\cdot N^5)-6\cdot N^4)+5\cdot N^3)-2\cdot N^2)+N}{(((N^6-2\cdot N^5)+4\cdot N^4)-2\cdot N^3)+3\cdot N^2+1}\right|-R_0=0$$

$$\frac{N^4+N^2}{(((((N^6-4\cdot N^5)+7\cdot N^4)-6\cdot N^3)+7\cdot N^2)-2\cdot N)+2}-R_2=0$$

$$\frac{(N^2-N)+1}{N}-R_3=0$$

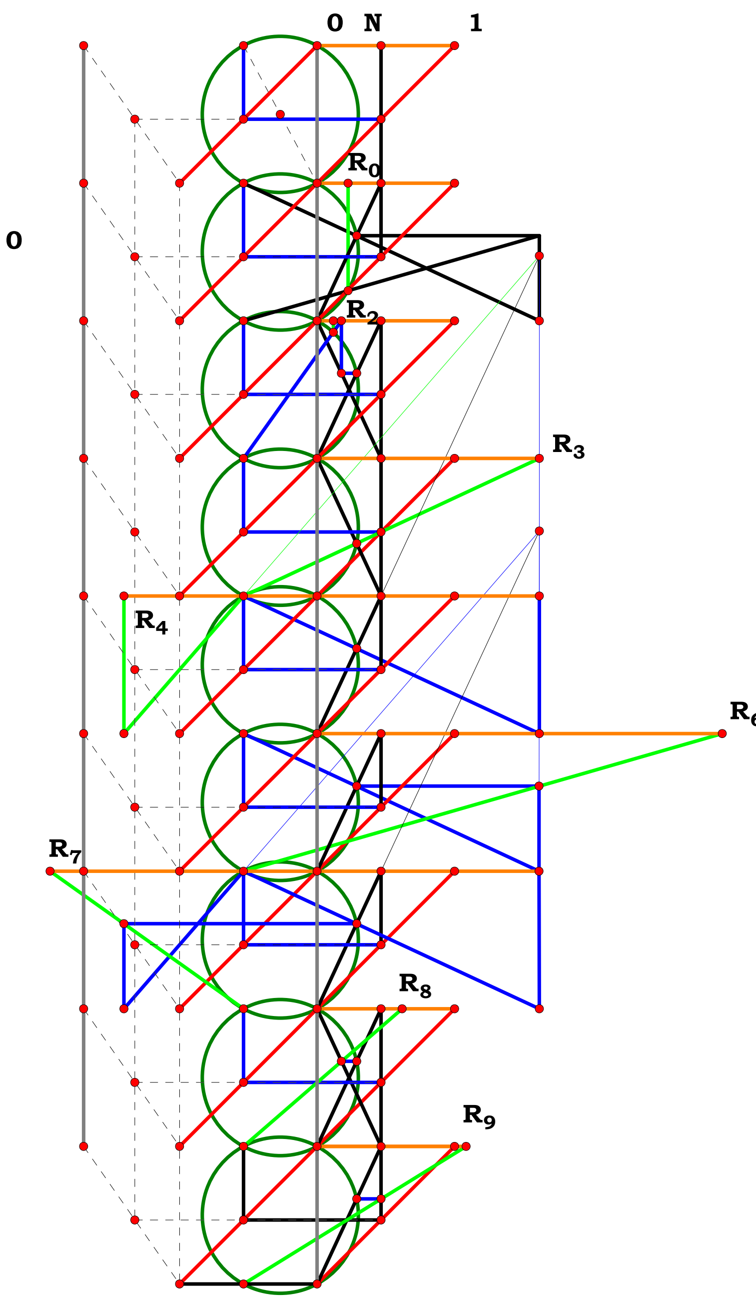
$$\frac{(N^2-N)+1}{1-N}-R_4=0$$

$$\frac{(((N^4-2\cdot N^3)+3\cdot N^2)-N)+1}{(N^3-N^2)+N}-R_6=0$$

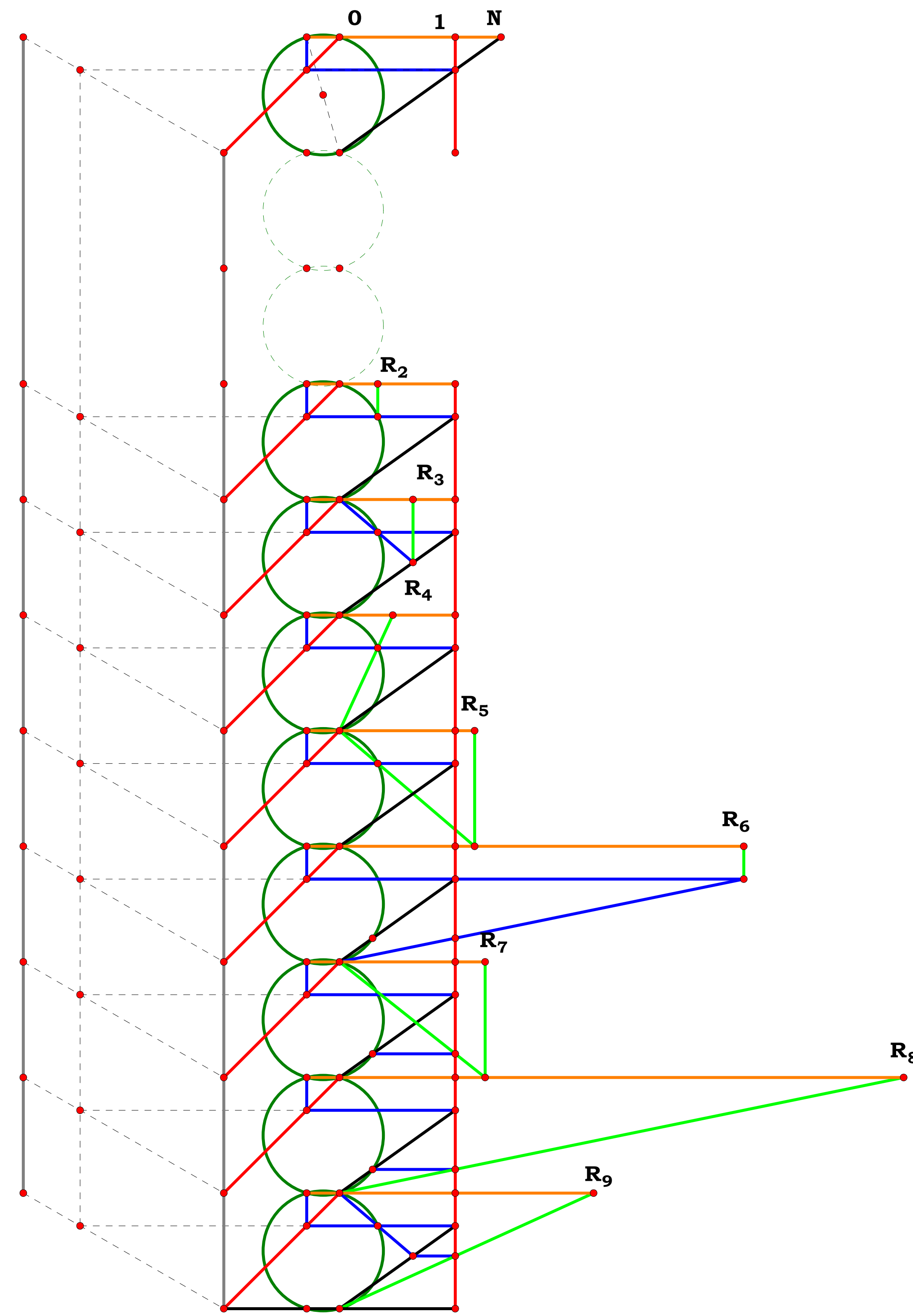
$$\left|\frac{(((N^4-2\cdot N^3)+4\cdot N^2)-2\cdot N)+1}{((N^3-2\cdot N^2)+2\cdot N)-1}\right|-R_7=0$$

$$\frac{N}{(N^2-N)+1}-R_8=0$$

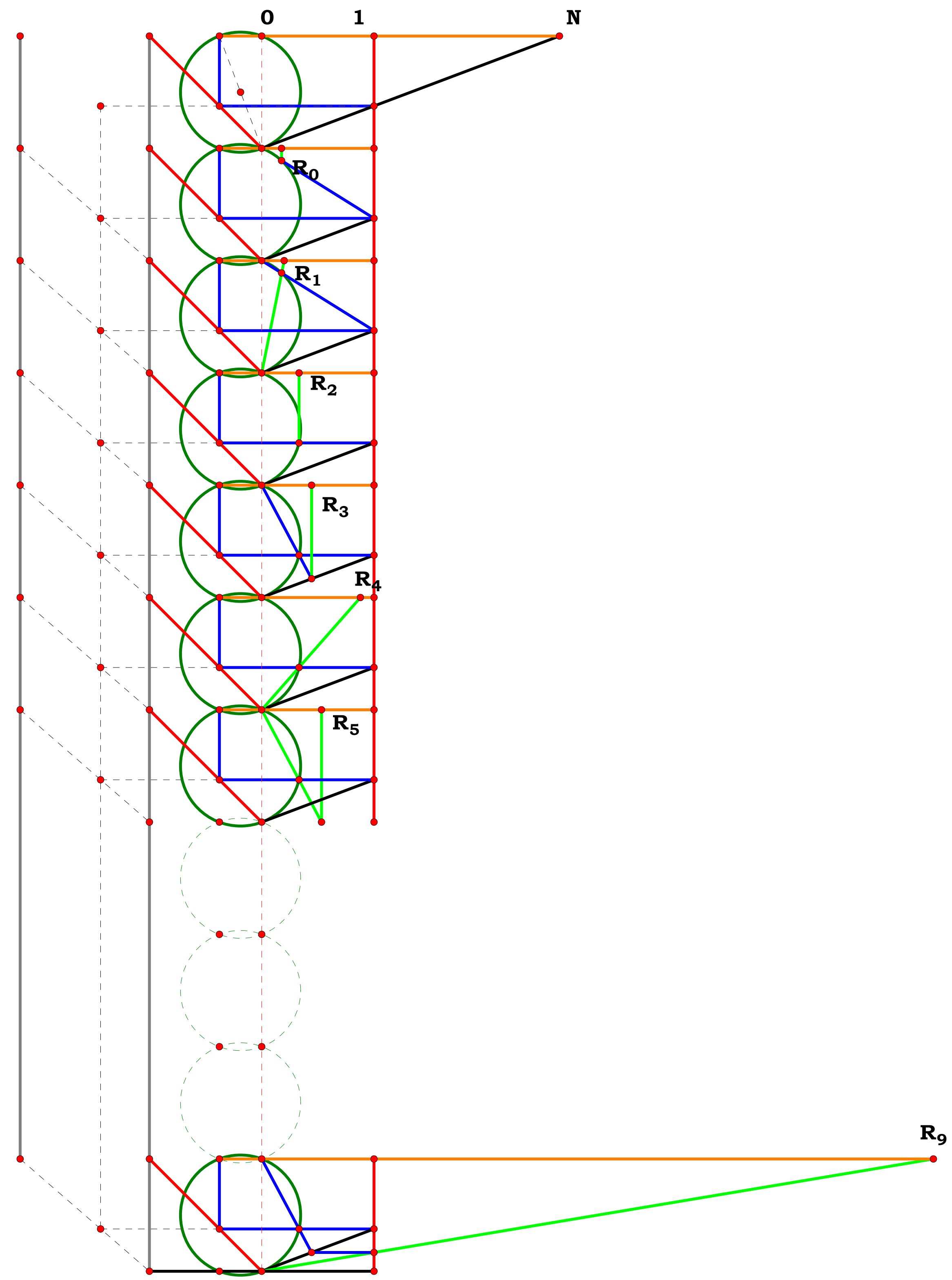
$$\frac{(N^3-N^2)+2\cdot N}{(N^2-N)+1}-R_9=0$$



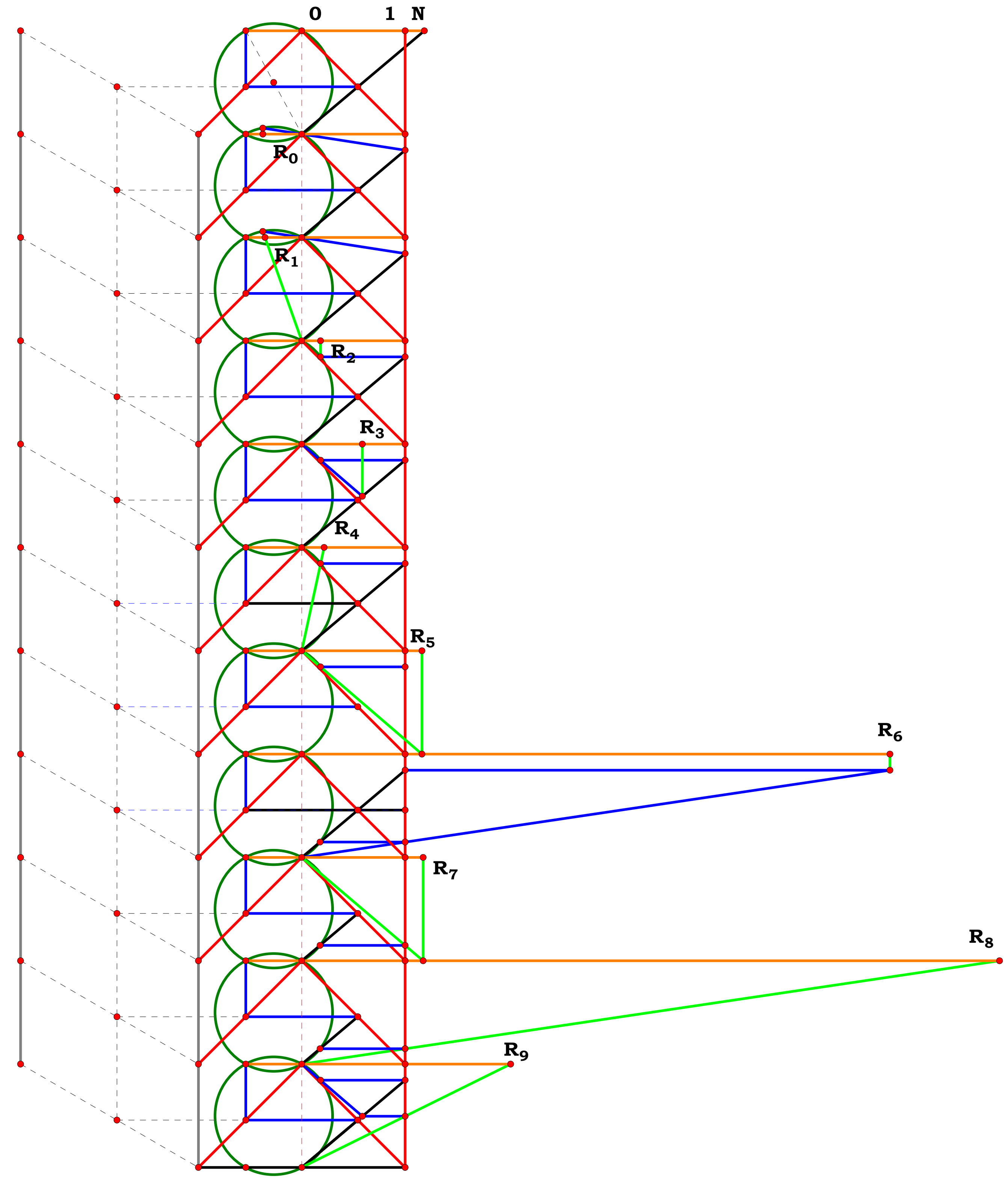
4RST2AB1



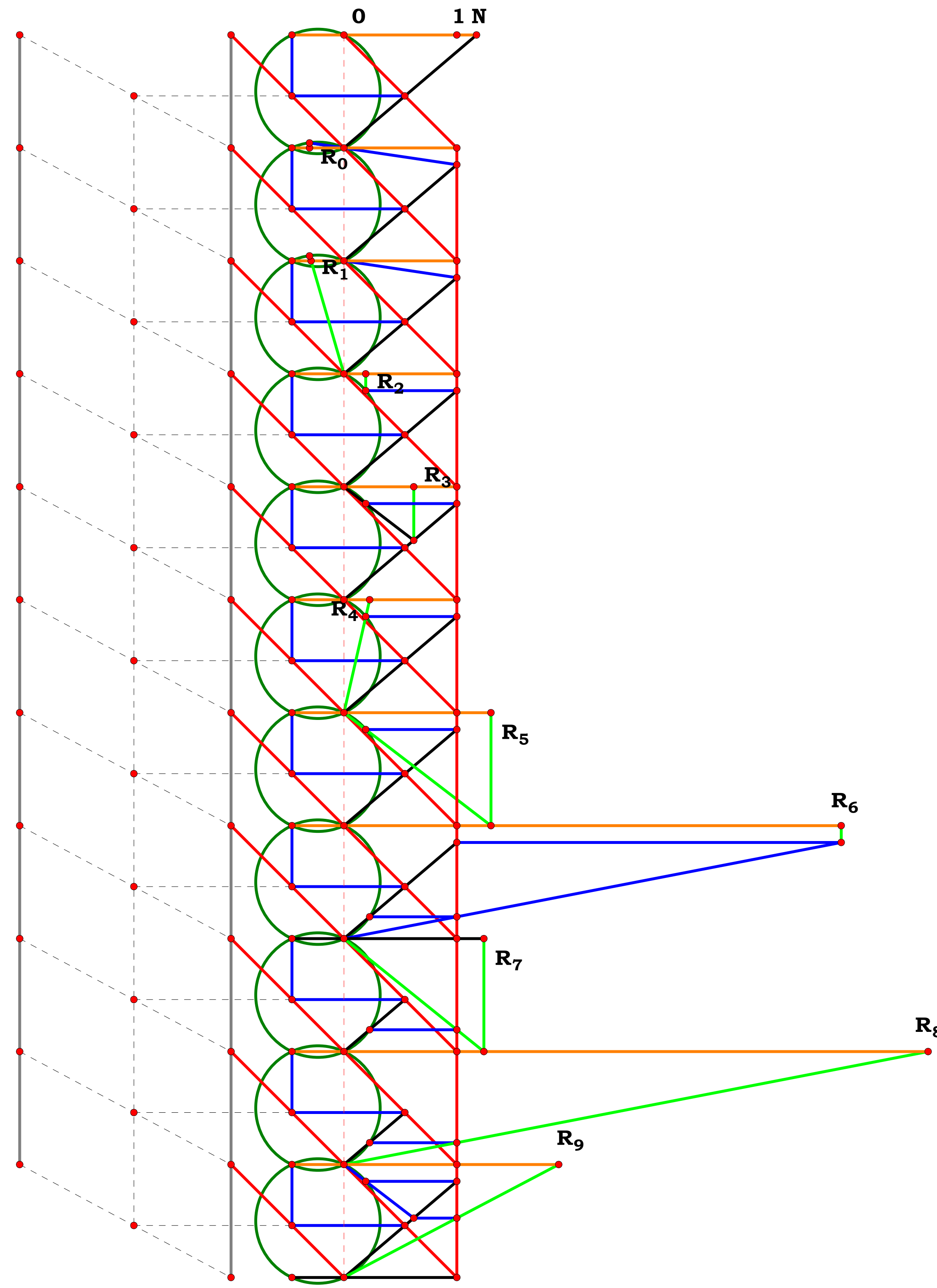
4RST2AB2



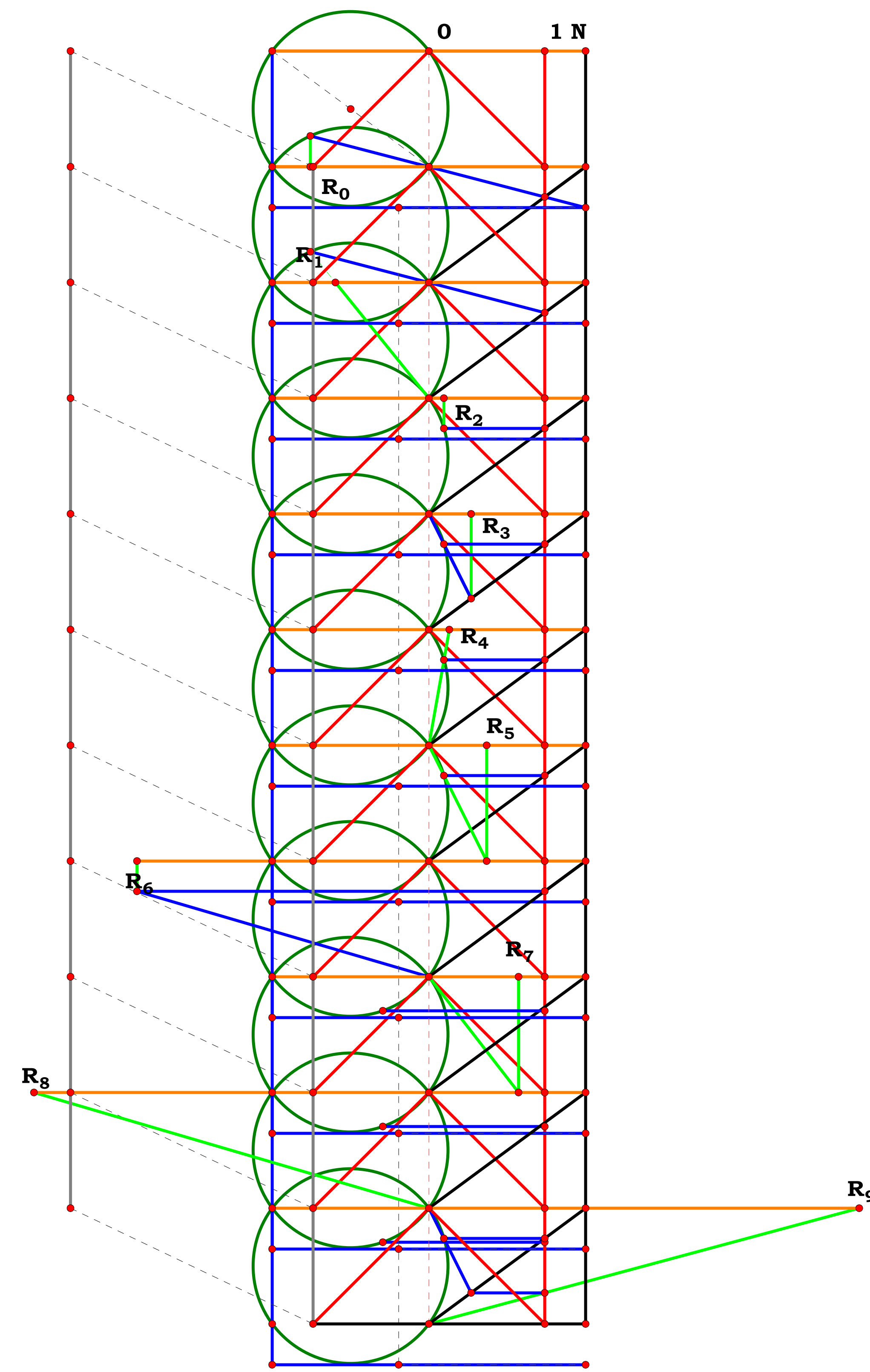
4RST2AB3



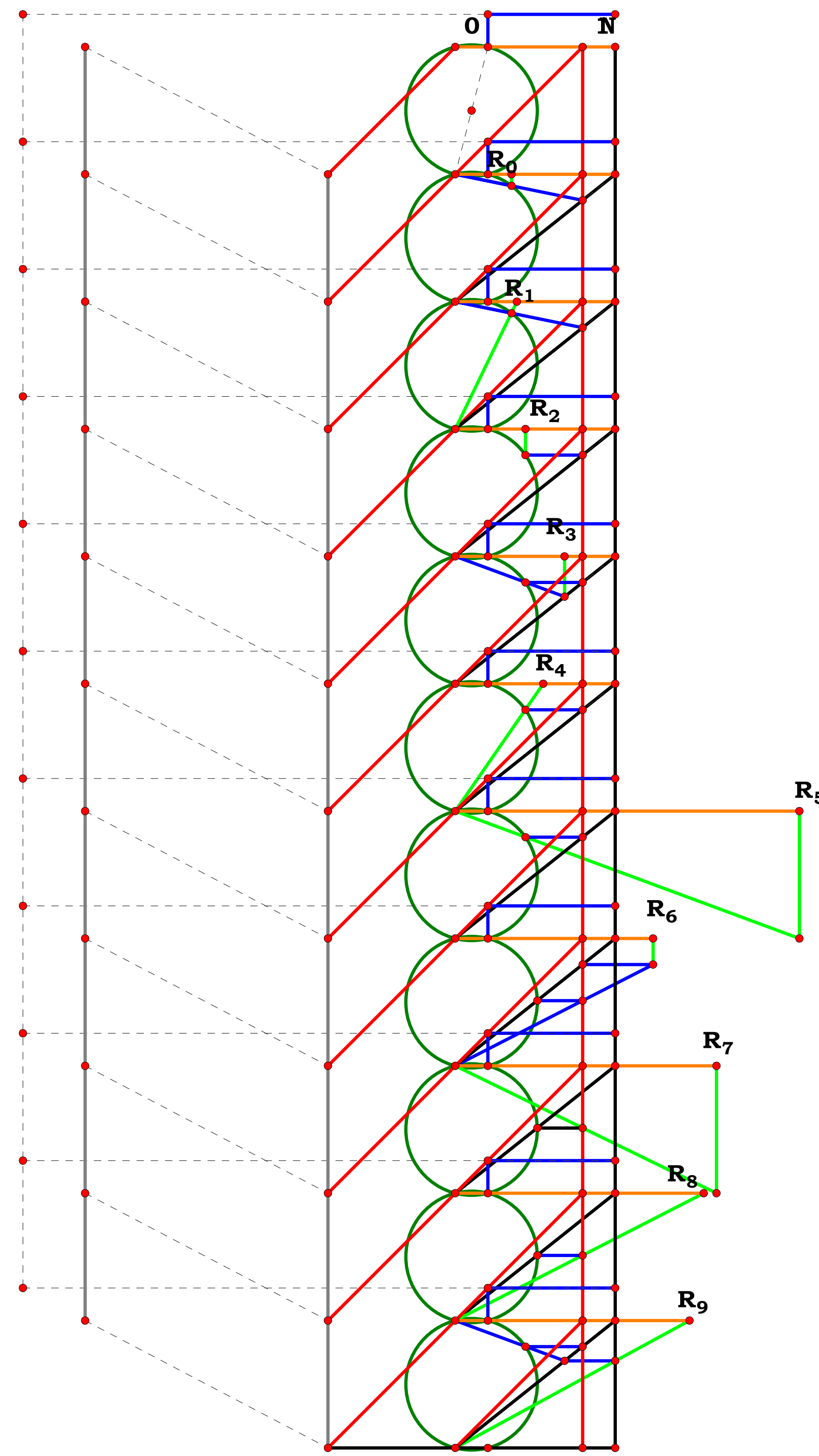
4RST2AB4

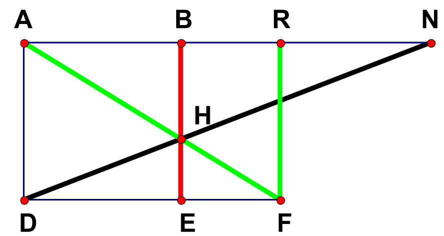


4RST2AB5



4RST2AB6





$\mathbf{AB} := 1$

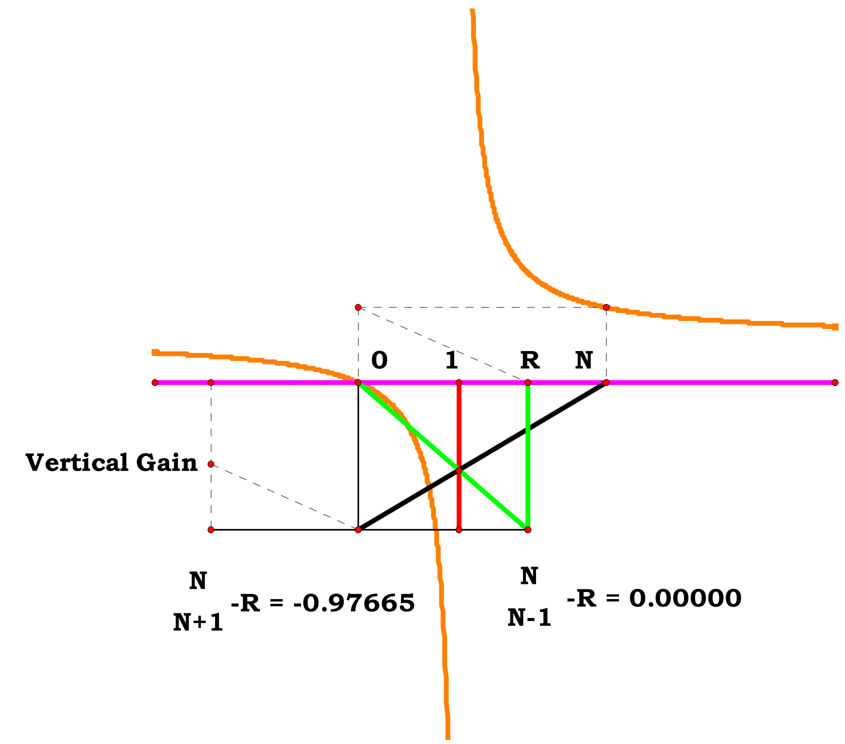
$\mathbf{AN} := 3 \cdot \mathbf{AB}$

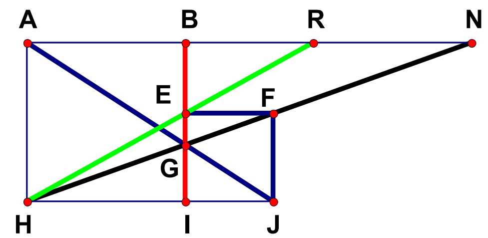
$\mathbf{AD} := \mathbf{AB}$

$\mathbf{DE} := \mathbf{AB}$

$$\mathbf{EH} := \frac{\mathbf{AD}^2}{\mathbf{AN}} \qquad \mathbf{EF} := \frac{\mathbf{DE} \cdot \mathbf{EH}}{\mathbf{AD} - \mathbf{EH}} \qquad \mathbf{DF} := \mathbf{DE} + \mathbf{EF} \qquad \mathbf{AR} := \mathbf{DF} \qquad \frac{\mathbf{AN}}{\mathbf{AN} - 1} - \mathbf{AR} = 0$$

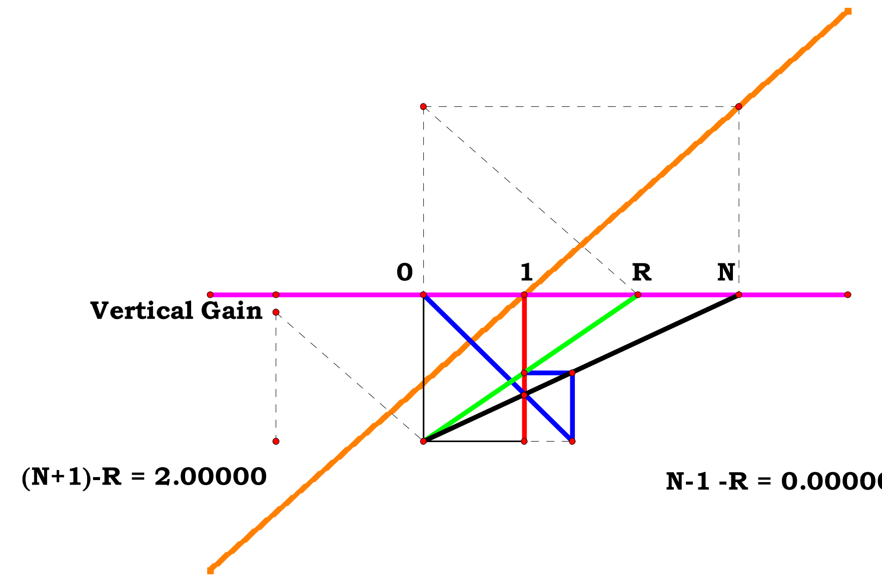
$$\mathbf{EH} - \frac{1}{\mathbf{AN}} = 0 \qquad \mathbf{EF} - \frac{1}{\mathbf{AN} - 1} = 0 \qquad \mathbf{DF} - \frac{\mathbf{AN}}{\mathbf{AN} - 1} = 0$$

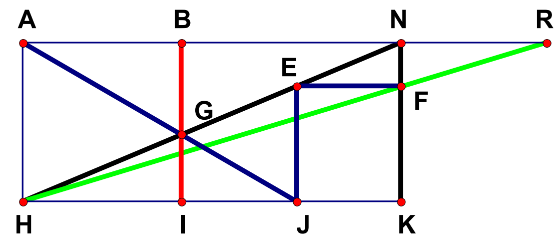




$$AB := 1$$
$$AH := AB$$
$$HI := AB$$
$$AN := 3$$
$$HJ := \frac{AN}{AN - 1}$$

$$FJ := \frac{AH \cdot HJ}{AN} \quad EI := FJ \quad AR := \frac{HI^2}{EI} \quad RN := AN - AR \quad (AN - 1) - AR = 0$$





$AB := 1$

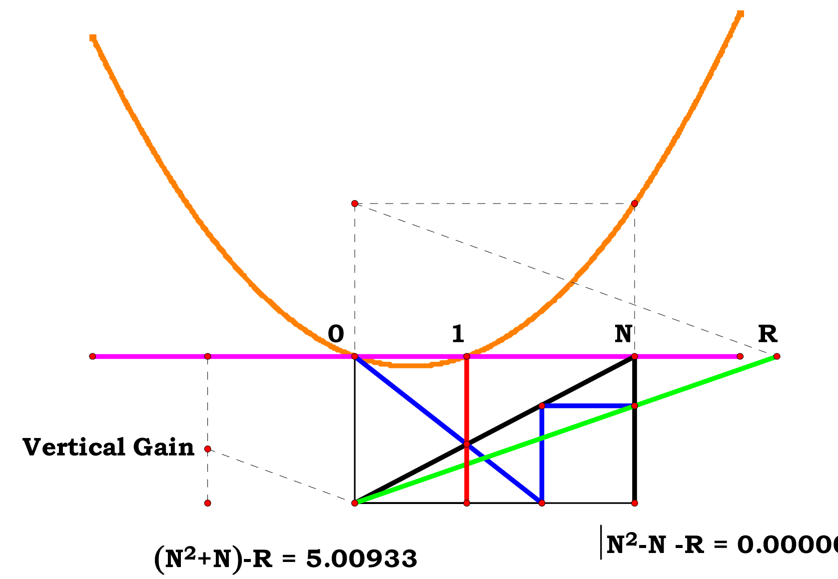
$AH := AB$

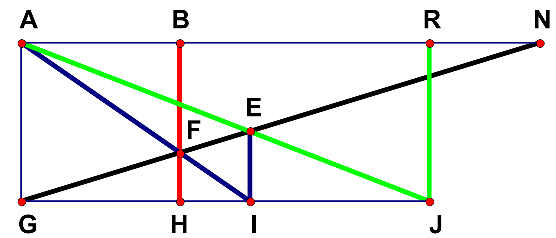
$HI := AB$

$AN := 3$

$$HJ := \frac{AN}{AN - 1} \quad EJ := \frac{AH \cdot HJ}{AN} \quad AR := \frac{AN \cdot AB}{EJ} \quad AN^2 - AN - AR = 0$$

$$EJ - \frac{1}{AN - 1} = 0$$





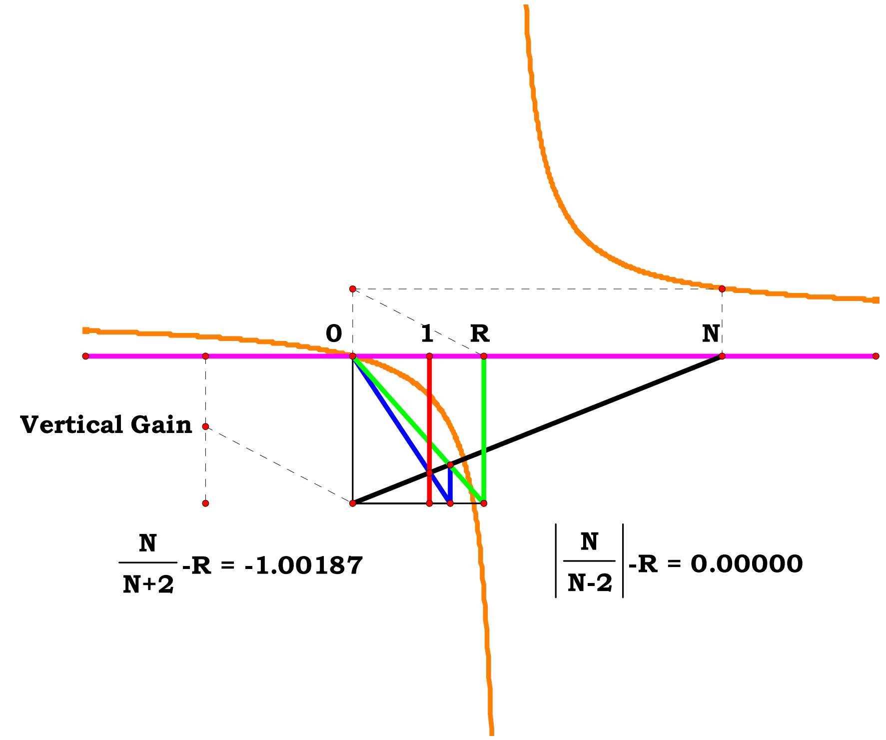
$AB := 1$

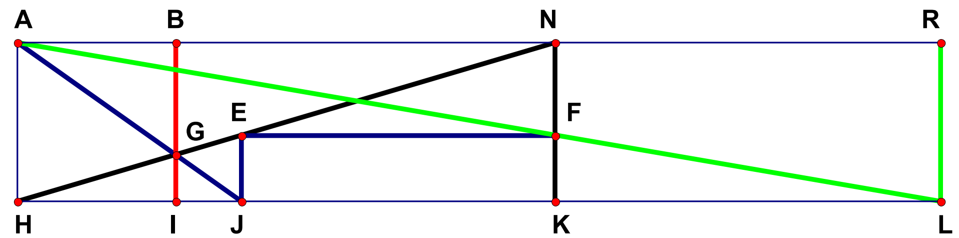
$AG := AB$

$GH := AB$

$AN := 5$

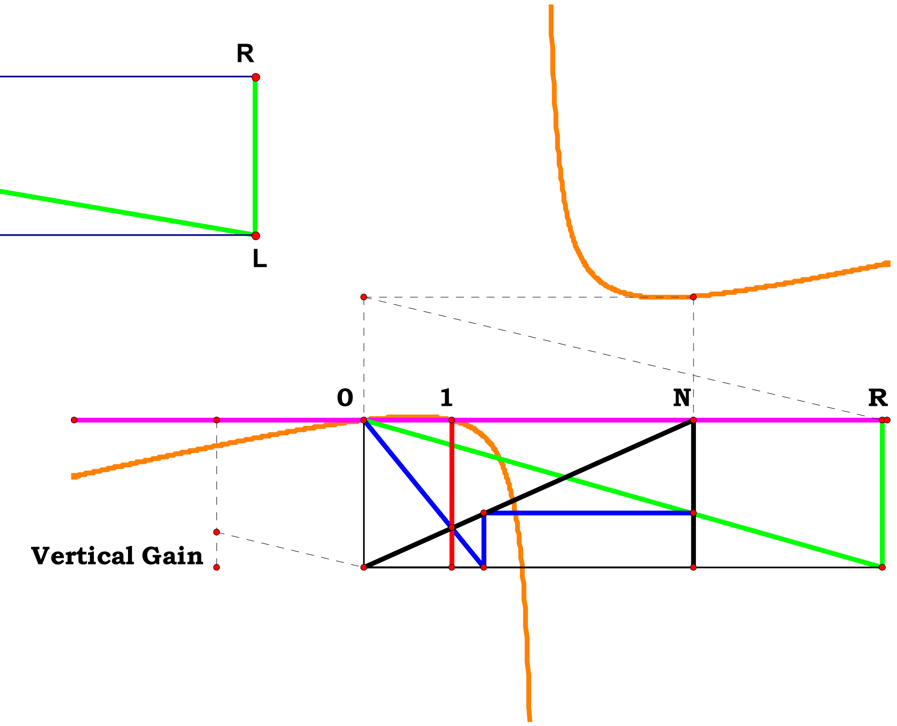
$$EI := \frac{1}{AN - 1} \qquad GI := \frac{AN}{AN - 1} \qquad GJ := \frac{GI \cdot AG}{AG - EI} \qquad AR := GJ \qquad AR - \frac{AN}{AN - 2} = 0$$

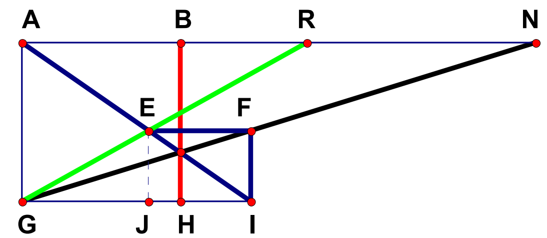




$\mathbf{AB} := 1 \quad \mathbf{AH} := \mathbf{AB} \quad \mathbf{HI} := \mathbf{AB} \quad \mathbf{AN} := 5$

$\mathbf{EJ} := \dfrac{1}{\mathbf{AN} - 1} \quad \mathbf{HL} := \dfrac{\mathbf{AN} \cdot \mathbf{AB}}{\mathbf{AB} - \mathbf{EJ}} \quad \mathbf{AR} := \mathbf{HL} \quad \mathbf{AR} - \dfrac{\mathbf{AN}^2 - \mathbf{AN}}{\mathbf{AN} - 2} = 0$





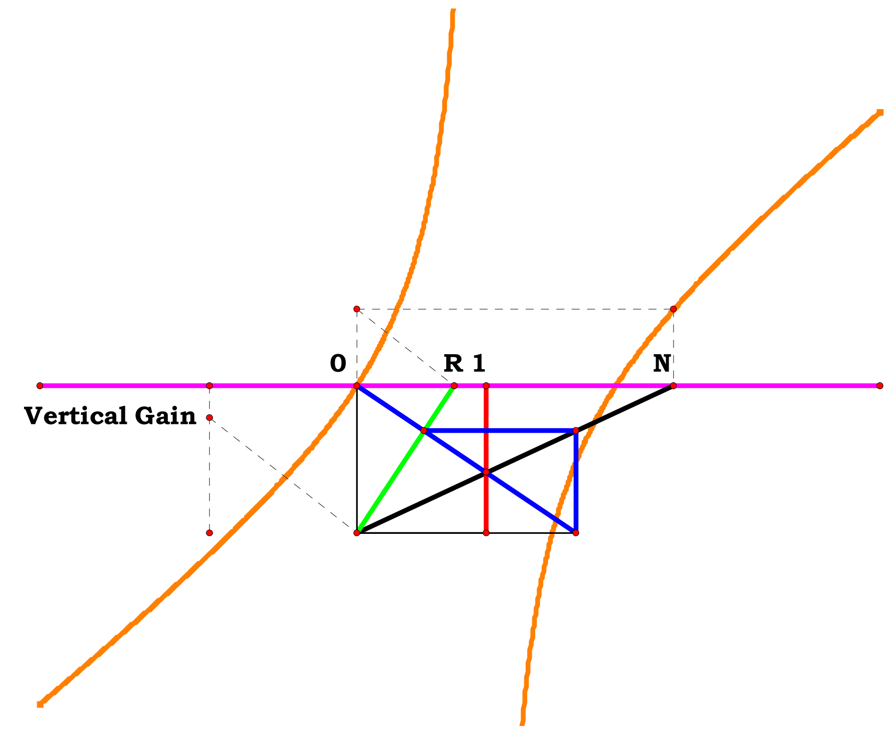
$$AB := 1$$

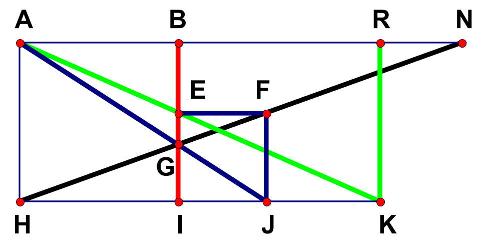
$$AG := AB$$

$$AN := 3$$

$$GI := \frac{AN}{AN - 1} \quad FI := \frac{1}{AN - 1} \quad IJ := \frac{GI \cdot FI}{AG} \quad GJ := GI - IJ \quad EJ := FI \quad AR := \frac{GJ \cdot AB}{EJ}$$

$$AR - \frac{AN^2 - 2AN}{AN - 1} = 0$$





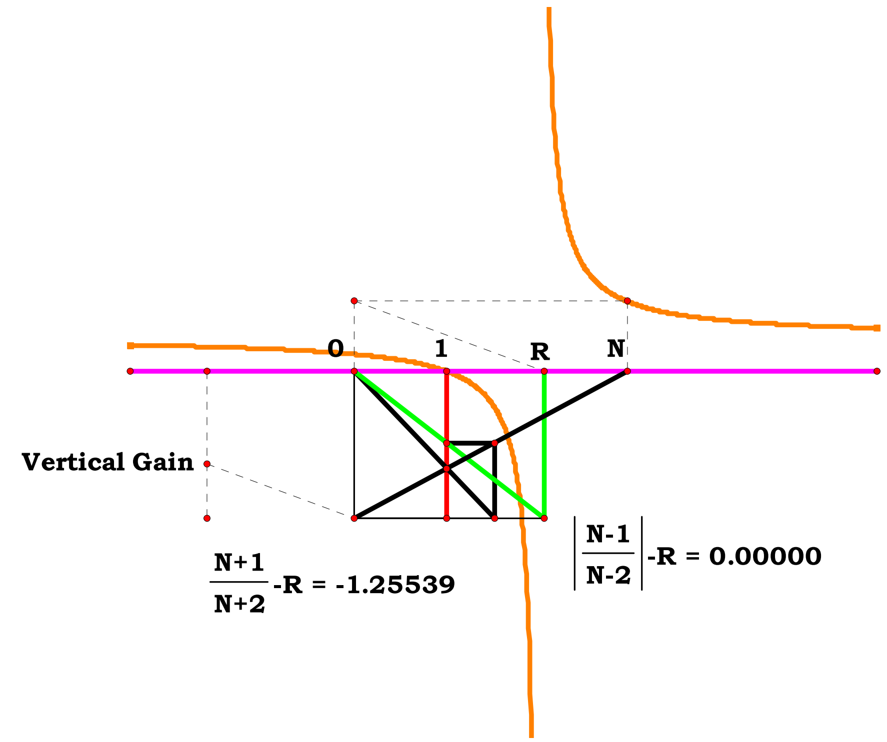
$AB := 1$

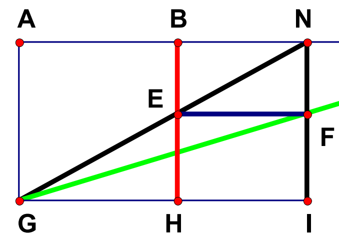
$AH := AB$

$HI := AB$

$AN := 4$

$FJ := \dfrac{1}{AN - 1} \qquad EI := FJ \qquad HK := \dfrac{HI^2}{HI - EI} \qquad AR := HK \qquad AR - \dfrac{AN - 1}{AN - 2} = 0$





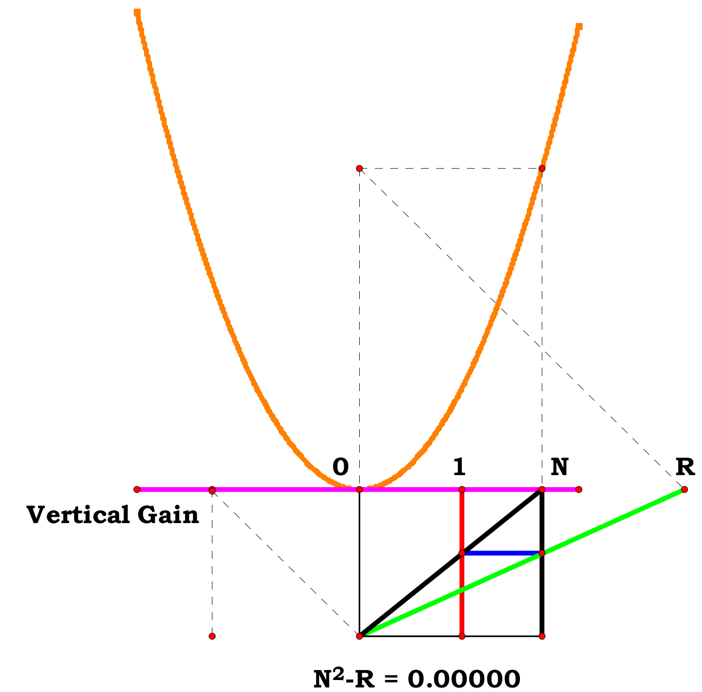
$\mathbf{AB} := 1$

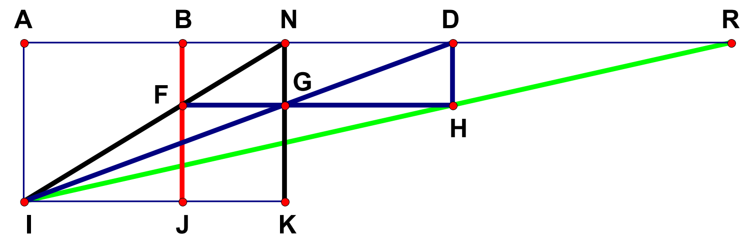
$\mathbf{AG} := \mathbf{AB}$

$\mathbf{AN} := 3$

$\mathbf{BN} := \mathbf{AN} - \mathbf{AB}$ $\quad \mathbf{BE} := \dfrac{\mathbf{AG} \cdot \mathbf{BN}}{\mathbf{AN}}$ $\quad \mathbf{NF} := \mathbf{BE}$ $\quad \mathbf{FI} := \mathbf{AG} - \mathbf{NF}$ $\quad \mathbf{AR} := \dfrac{\mathbf{AN} \cdot \mathbf{AG}}{\mathbf{FI}}$ $\quad \mathbf{AR} - \mathbf{AN}^2 = 0$

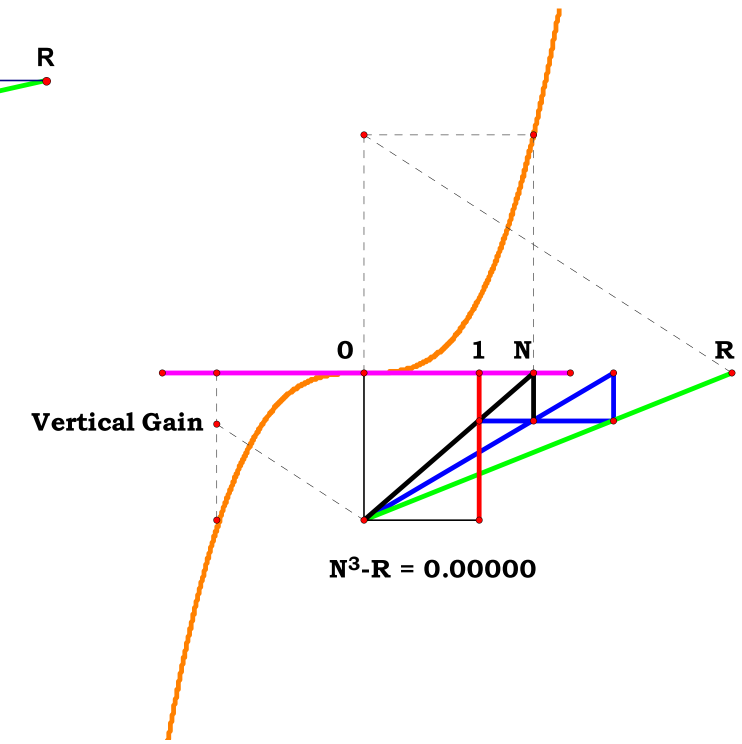
$\mathbf{BE} - \dfrac{\mathbf{AN} - 1}{\mathbf{AN}} = 0$

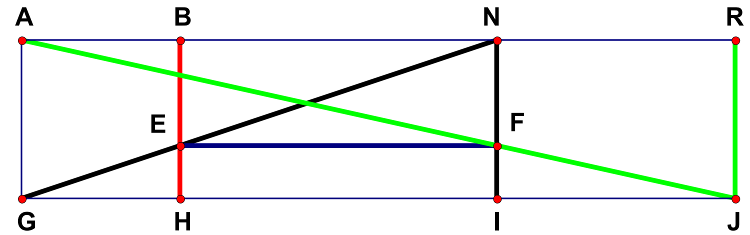




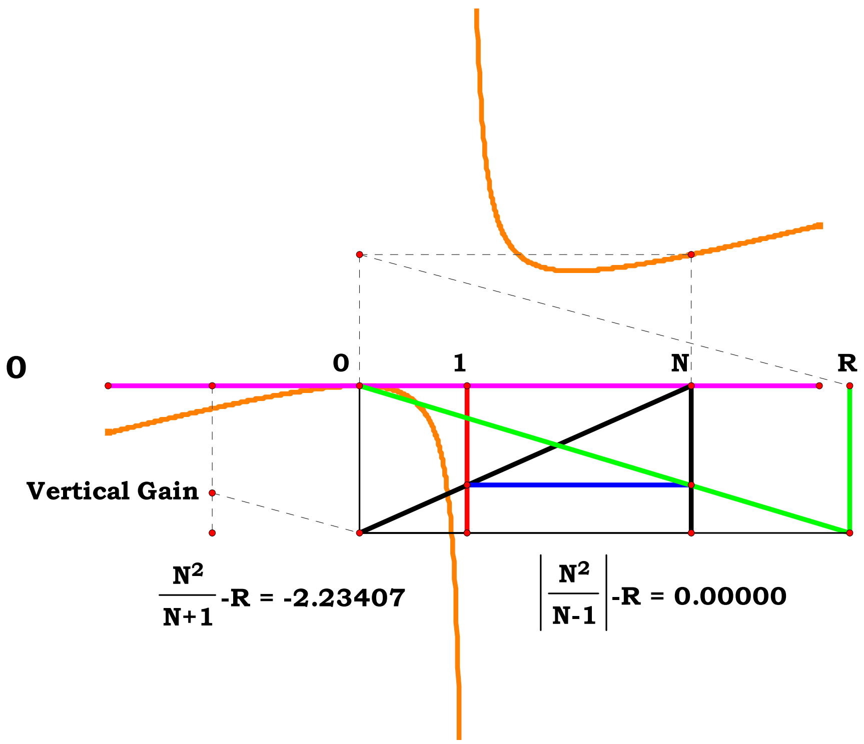
$AB := 1 \quad AI := AB \quad AN := 3 \quad AD := AN^2 \quad BF := \frac{AN - 1}{AN} \quad DH := BF$

$AR := \frac{AD \cdot AI}{AI - DH} \quad AR - AN^3 = 0$



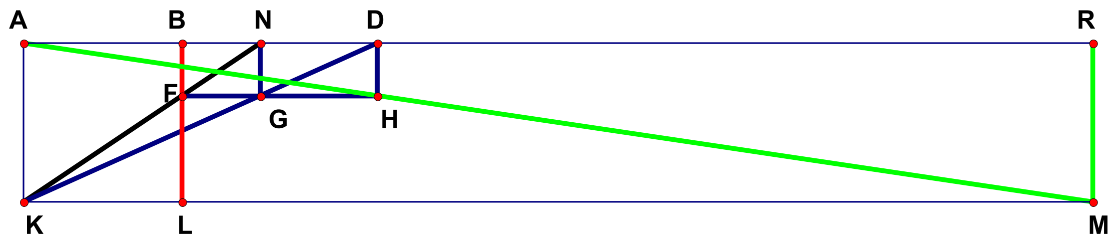


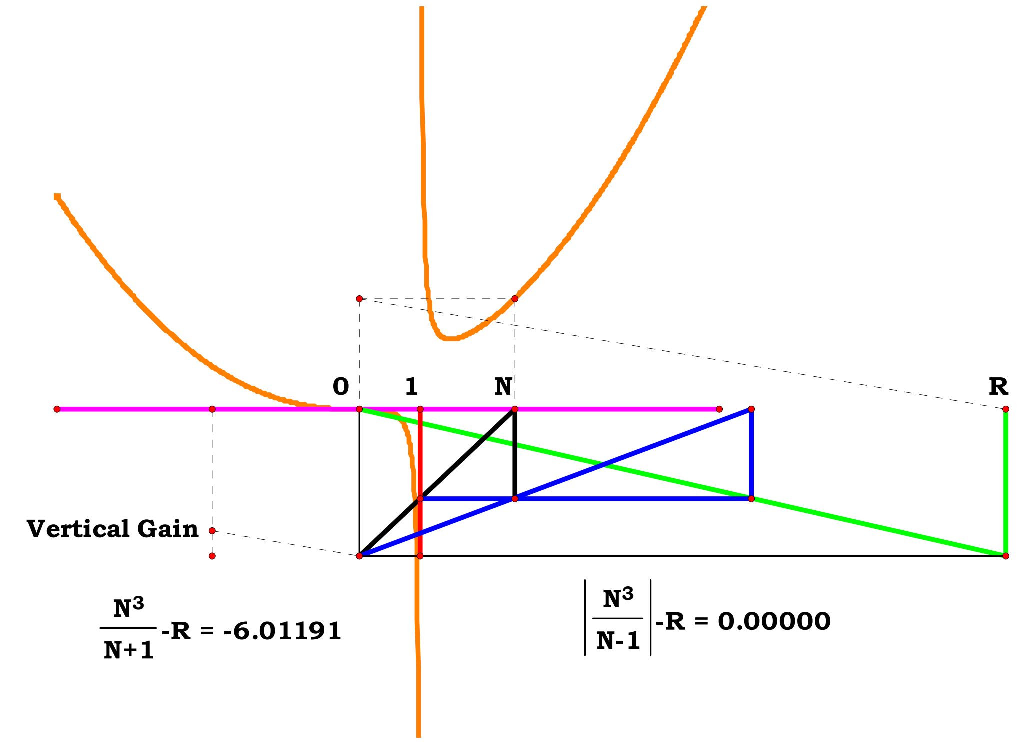
$$\mathbf{AB} := 1 \quad \mathbf{AN} := 3 \quad \mathbf{BE} := \frac{\mathbf{AN}-1}{\mathbf{AN}} \quad \mathbf{BF} := \mathbf{BE} \quad \mathbf{GJ} := \frac{\mathbf{AN}\cdot\mathbf{AB}}{\mathbf{BF}} \quad \mathbf{AR} := \mathbf{GJ} \quad \mathbf{AR} - \frac{\mathbf{AN}^2}{\mathbf{AN}-1} = 0$$

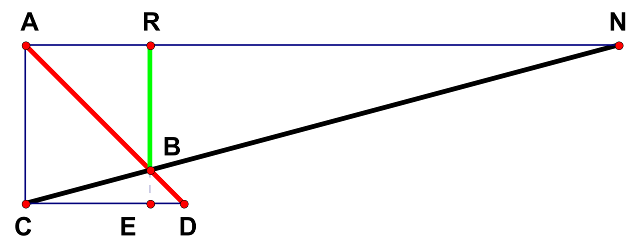


$$AB := 1 \quad AN := 3 \quad AD := AN^2 \quad BF := \frac{AN-1}{AN}$$

$$DH := BF \quad AR := \frac{AD \cdot AB}{DH} \quad AR - \frac{AN^3}{AN-1} = 0$$

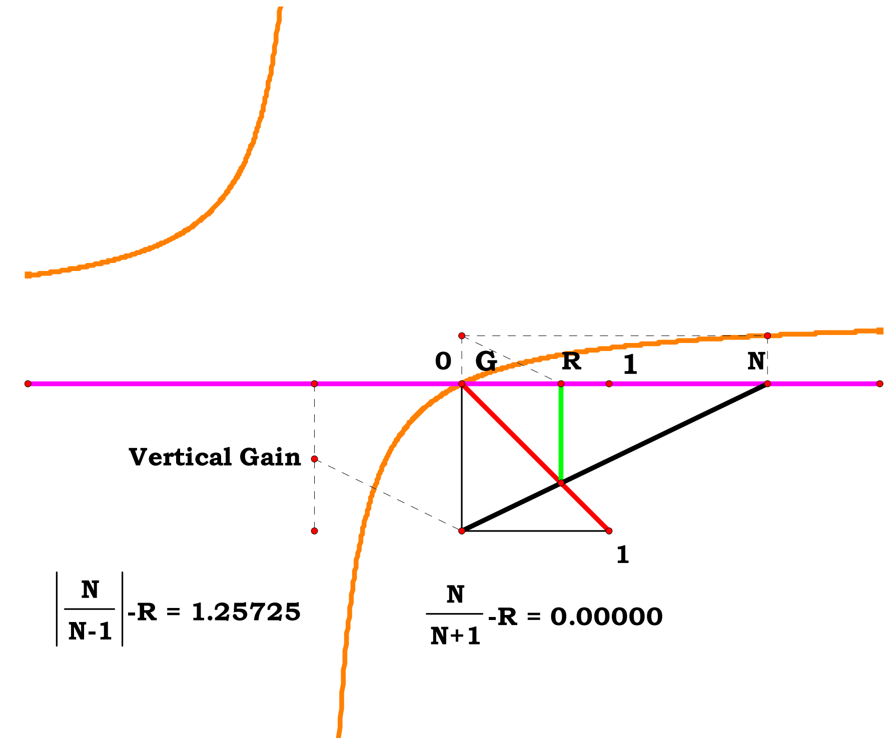


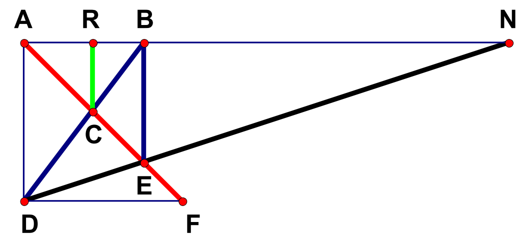




$$AC := 1 \quad CD := AC \quad AN := 4 \quad AR := \frac{CD \cdot AN}{CD + AN} \quad AR - \frac{AN}{AN + 1} = 0$$

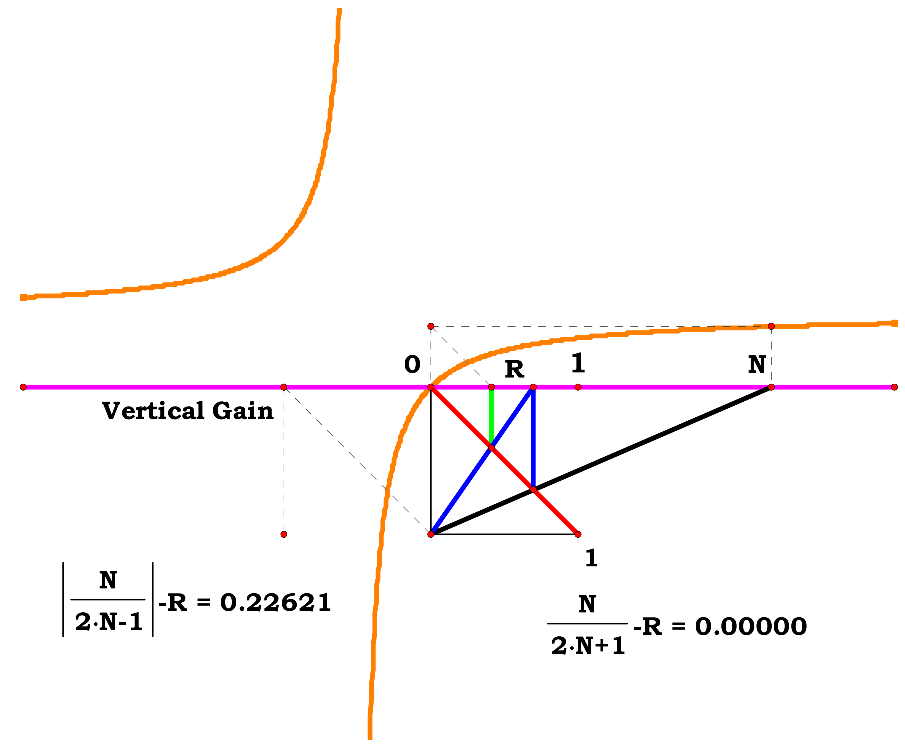
$$DE := AC - AR \qquad DE - \frac{1}{AN + 1} = 0$$

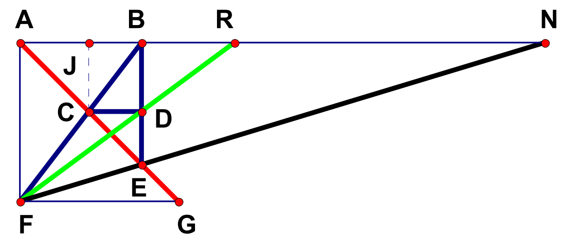




$$\mathbf{AD} := 1 \quad \mathbf{DF} := \mathbf{AD} \quad \mathbf{AN} := 3 \quad \mathbf{AB} := \frac{\mathbf{AN}}{\mathbf{AN}+1} \quad \mathbf{AR} := \frac{\mathbf{DF}\cdot\mathbf{AB}}{\mathbf{DF}+\mathbf{AB}}$$

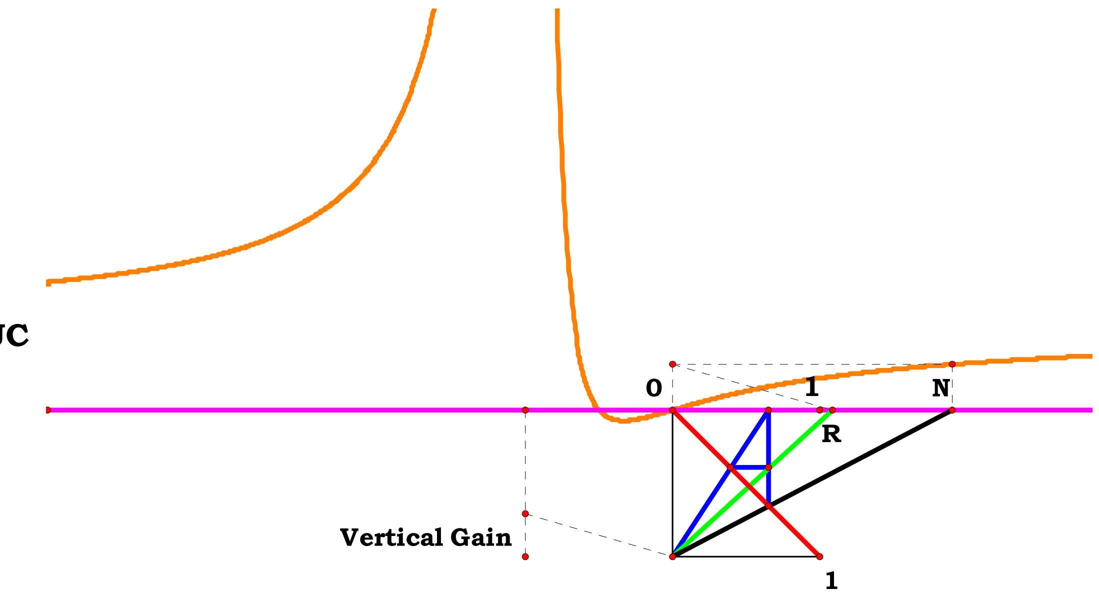
$$\mathbf{AR} - \frac{\mathbf{AN}}{2\mathbf{AN}+1} = 0$$





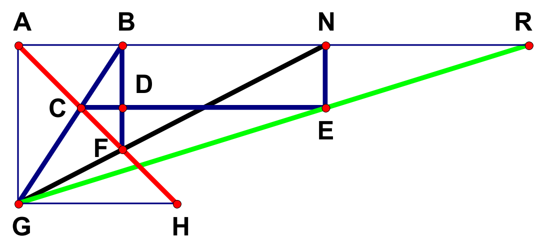
$AF := 1 \quad FG := AF \quad AN := 4 \quad AJ := \frac{AN}{2AN + 1} \quad JC := AJ \quad BD := JC$

$AB := \frac{AN}{AN + 1} \quad AR := \frac{AB \cdot AF}{AF - BD} \quad AR - \frac{2 \cdot AN^2 + AN}{AN^2 + 2 \cdot AN + 1} = 0$



$\left| \frac{2 \cdot N^2 - N}{(N^2 - 2 \cdot N) + 1} \right| - R = 5.43597$

$\frac{2 \cdot N^2 + N}{N^2 + 2 \cdot N + 1} - R = 0.00000$

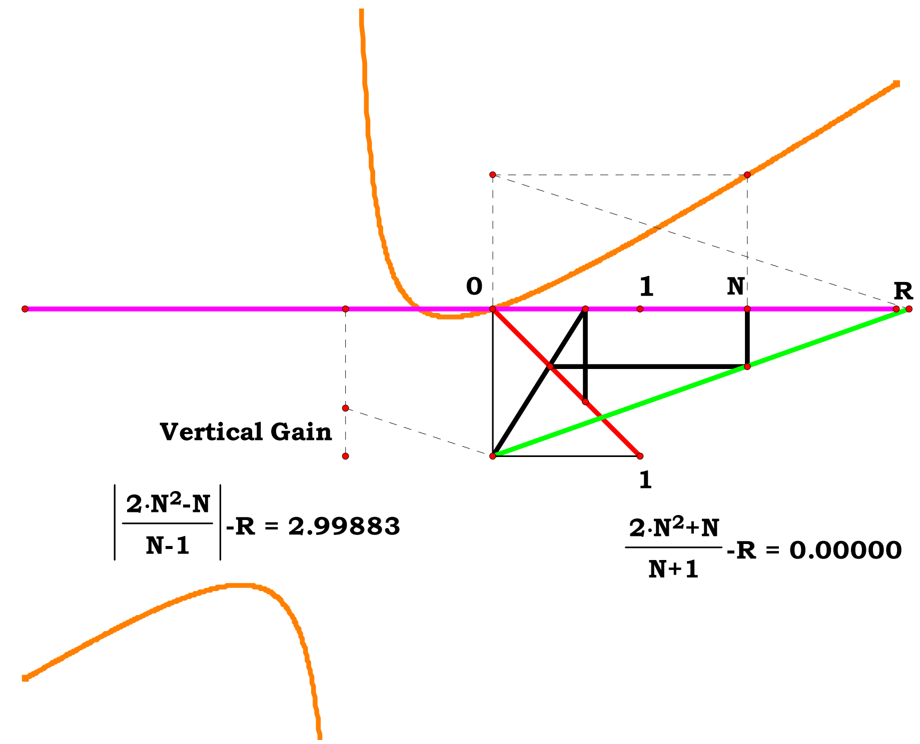


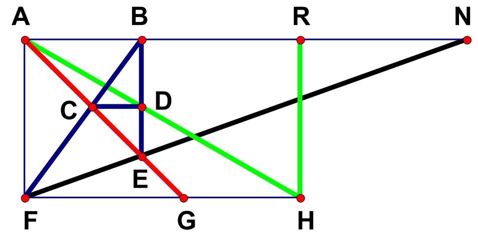
$AG := 1$

$GH := AG$

$AN := 2$

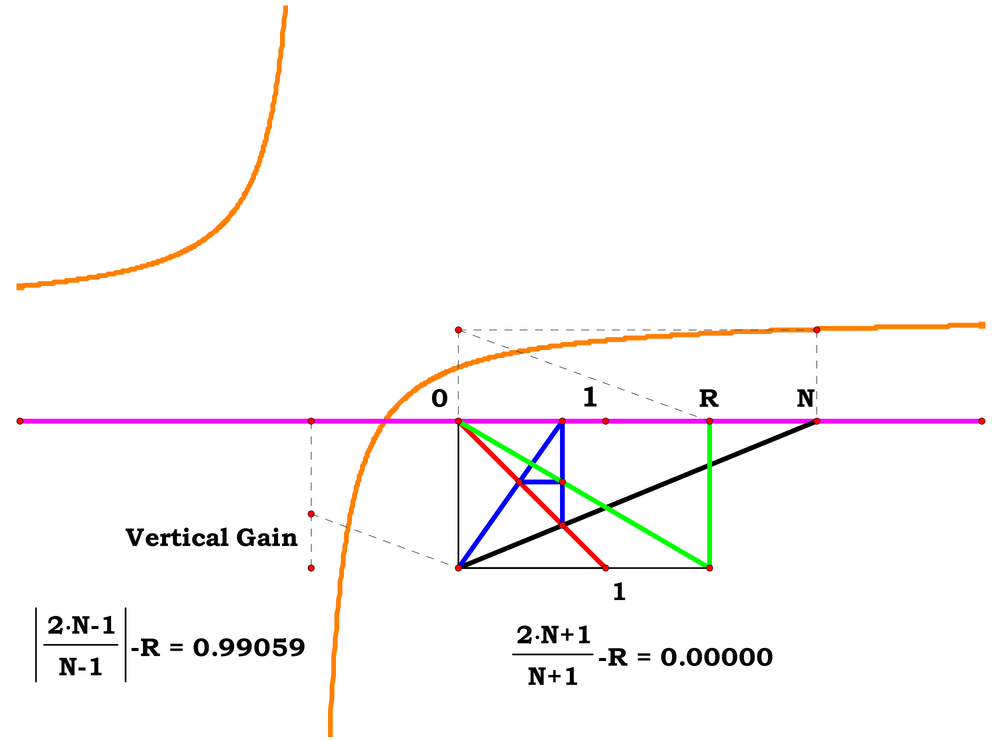
$$NE := \frac{AN}{2AN + 1} \qquad AR := \frac{AN \cdot AG}{AG - NE} \qquad AR - \frac{2 \cdot AN^2 + AN}{AN + 1} = 0$$

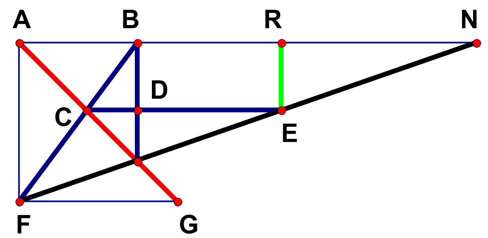




$$AF := 1 \quad FG := AF \quad AN := 3 \quad AB := \frac{AN}{AN+1} \quad BD := \frac{AN}{2AN+1}$$

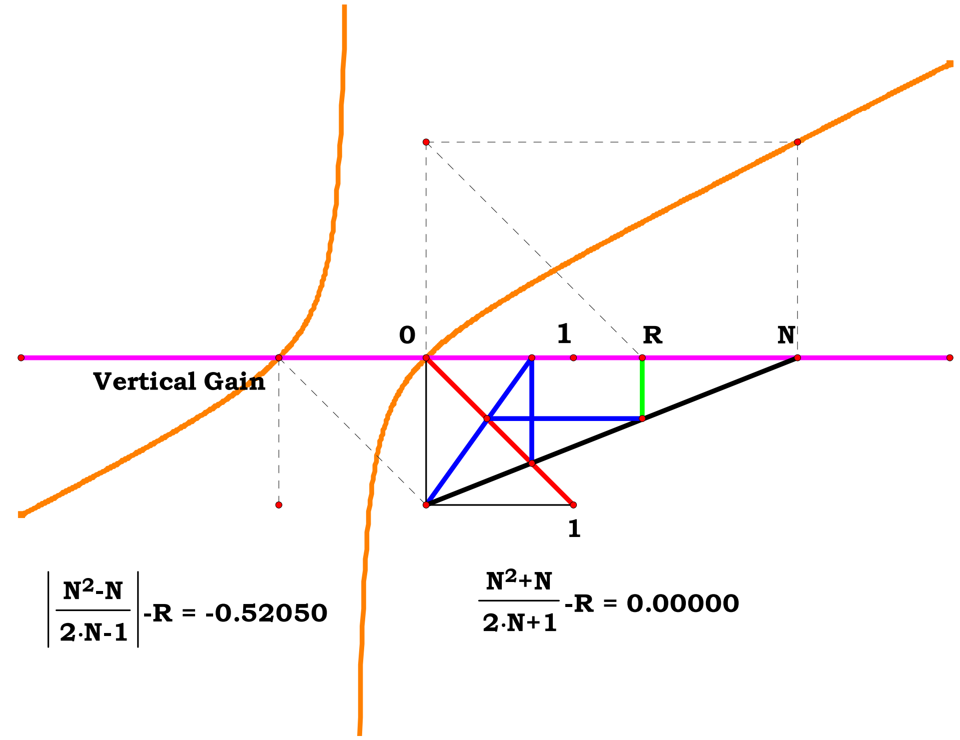
$$FH := \frac{AB \cdot AF}{BD} \quad AR := FH \quad AR - \frac{2 \cdot AN + 1}{AN+1} = 0$$





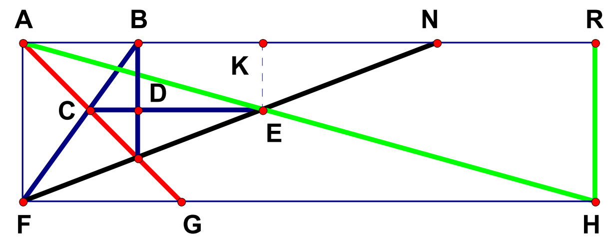
$$AF := 1 \qquad FG := AF \qquad AN := 3 \qquad BD := \frac{AN}{2AN+1} \qquad RE := BD$$

$$RN := \frac{AN \cdot RE}{AF} \qquad AR := AN - RN \qquad AR - \frac{AN^2 + AN}{2 \cdot AN + 1} = 0$$



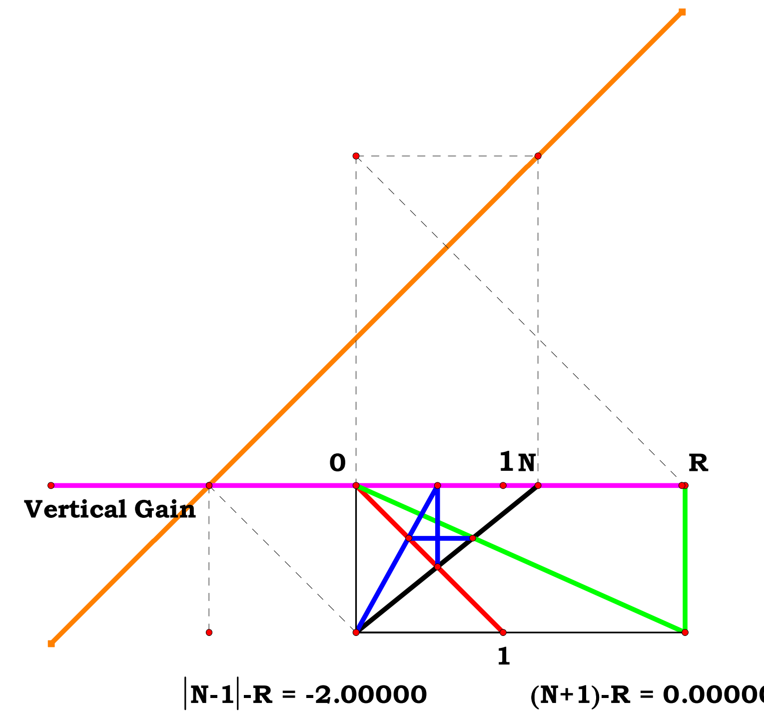
$$\left|\frac{N^2-N}{2 \cdot N-1}\right| - R = -0.52050$$

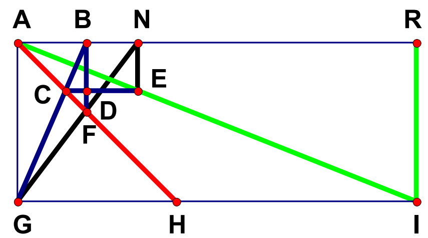
$$\frac{N^2+N}{2 \cdot N+1} - R = 0.00000$$



$$AF := 1 \quad FG := AF \quad AN := 3 \quad AK := \frac{AN^2 + AN}{2 \cdot AN + 1} \quad KE := \frac{AN}{2AN + 1}$$

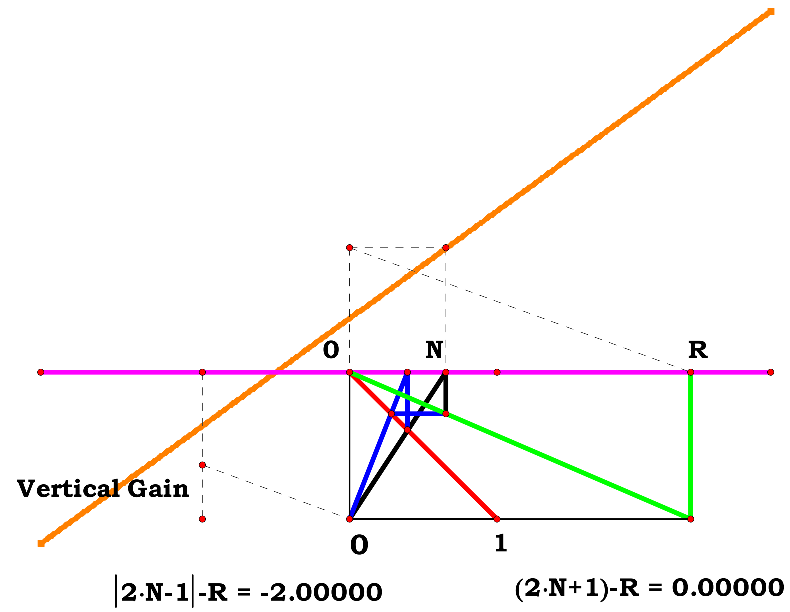
$$FH := \frac{AK \cdot AF}{KE} \quad AR := FH \quad AR - (AN + 1) = 0$$

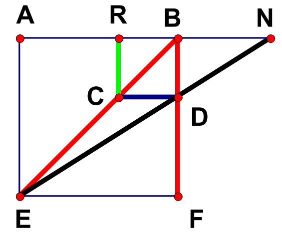




$$\mathbf{AG} := 1 \quad \mathbf{GH} := \mathbf{AG} \quad \mathbf{AN} := 5 \quad \mathbf{NE} := \frac{\mathbf{AN}}{2\mathbf{AN}+1}$$

$$\mathbf{GI} := \frac{\mathbf{AN}\cdot\mathbf{AG}}{\mathbf{NE}} \quad \mathbf{AR} := \mathbf{GI} \quad \mathbf{AR} - (2\mathbf{AN}+1) = 0$$



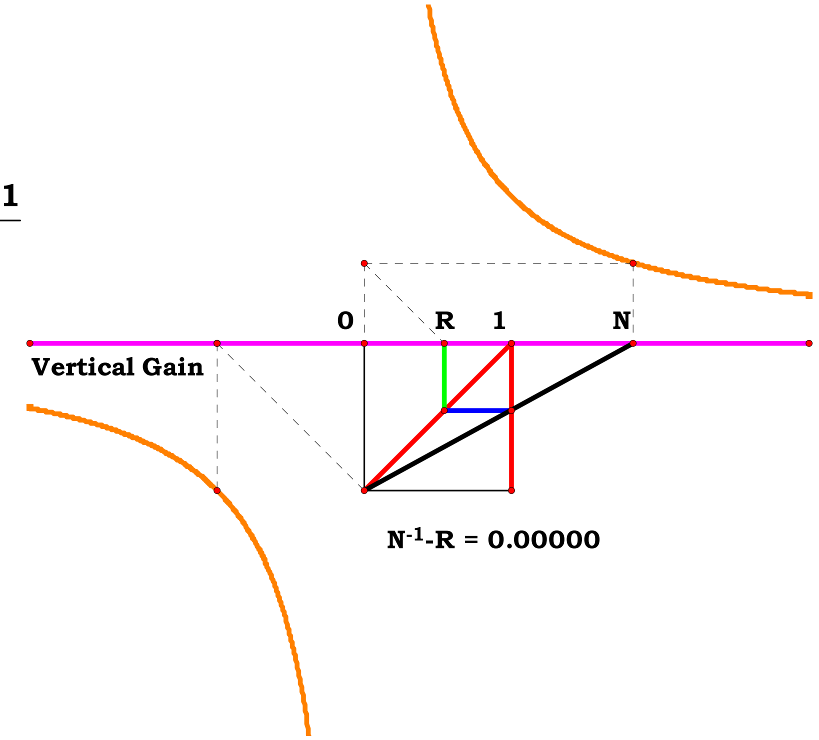


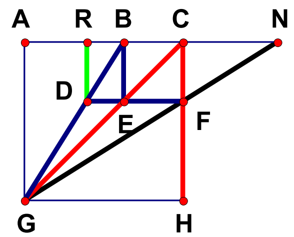
$$\mathbf{AB} := \mathbf{1}$$

$$\mathbf{AN} := \mathbf{3}$$

$$\mathbf{BD} := \frac{\mathbf{AN} - \mathbf{1}}{\mathbf{AN}}$$

$$\mathbf{AR} := \mathbf{AB} - \mathbf{BD} \qquad \mathbf{AR} - \frac{\mathbf{1}}{\mathbf{AN}} = \mathbf{0} \qquad \mathbf{AR} - \mathbf{AN}^{-\mathbf{1}} = \mathbf{0}$$





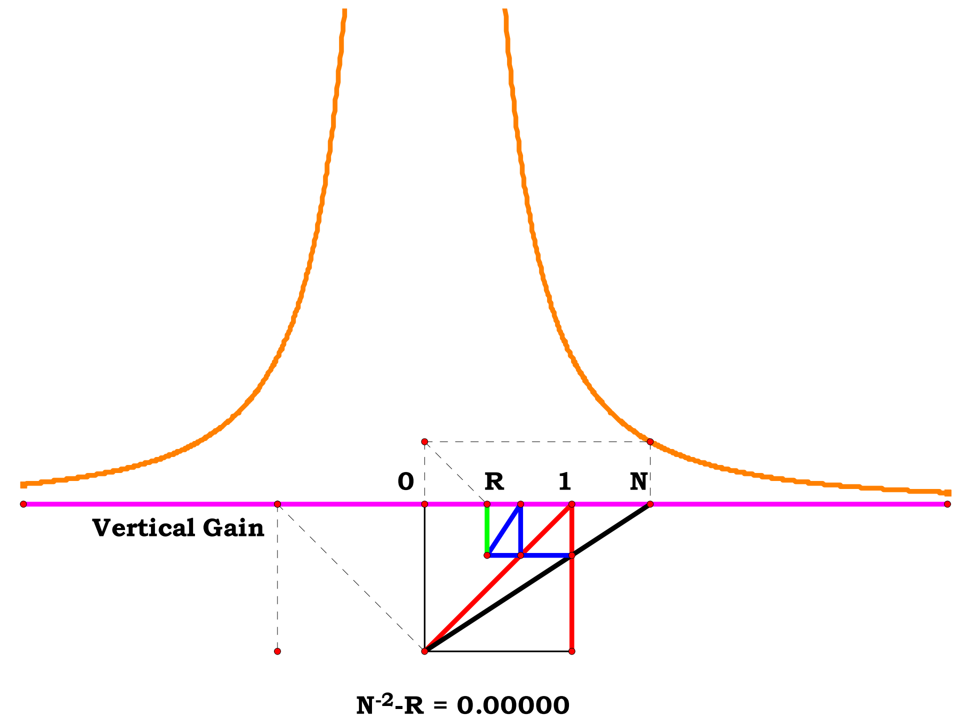
$AC := 1$

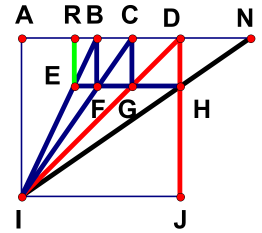
$AN := 3$

$AB := \dfrac{1}{AN}$

$BC := \dfrac{AN - 1}{AN}$ $\quad BR := \dfrac{AB \cdot BC}{AC}$ $\quad AR := AB - BR$ $\quad AR - \dfrac{1}{AN^2} = 0$

$AR - AN^{-2} = 0$



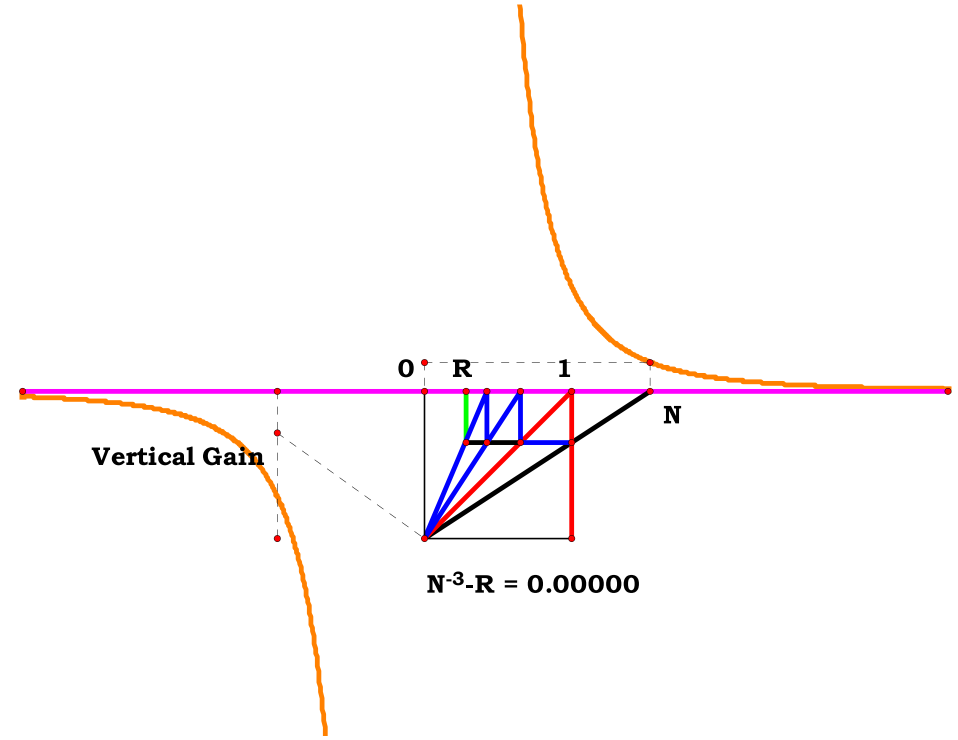


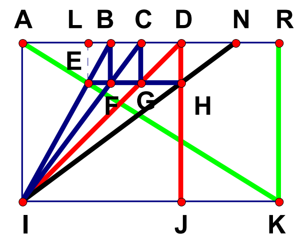
$AD := 1$

$AN := 2$

$DH := \dfrac{AN - 1}{AN}$

$AB := \dfrac{1}{AN^2}$ $\quad BR := \dfrac{AB \cdot DH}{AD}$ $\quad AR := AB - BR$ $\quad AR - \dfrac{1}{AN^3} = 0$ $\quad AR - AN^{-3} = 0$



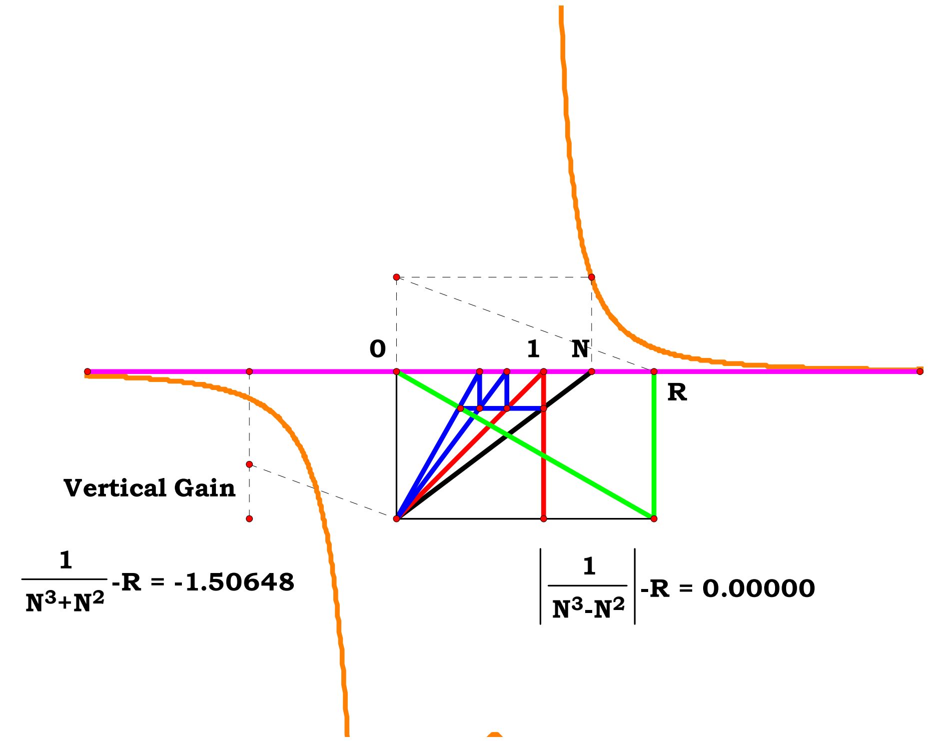


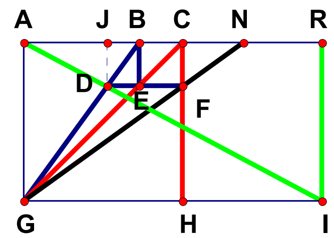
$AD := 1$

$AN := 3$

$$AL := \frac{1}{AN^3}$$

$$DH := \frac{AN - 1}{AN} \quad EL := DH \quad IK := \frac{AL \cdot AD}{EL} \quad AR := IK \quad AR - \frac{1}{AN^3 - AN^2} = 0$$



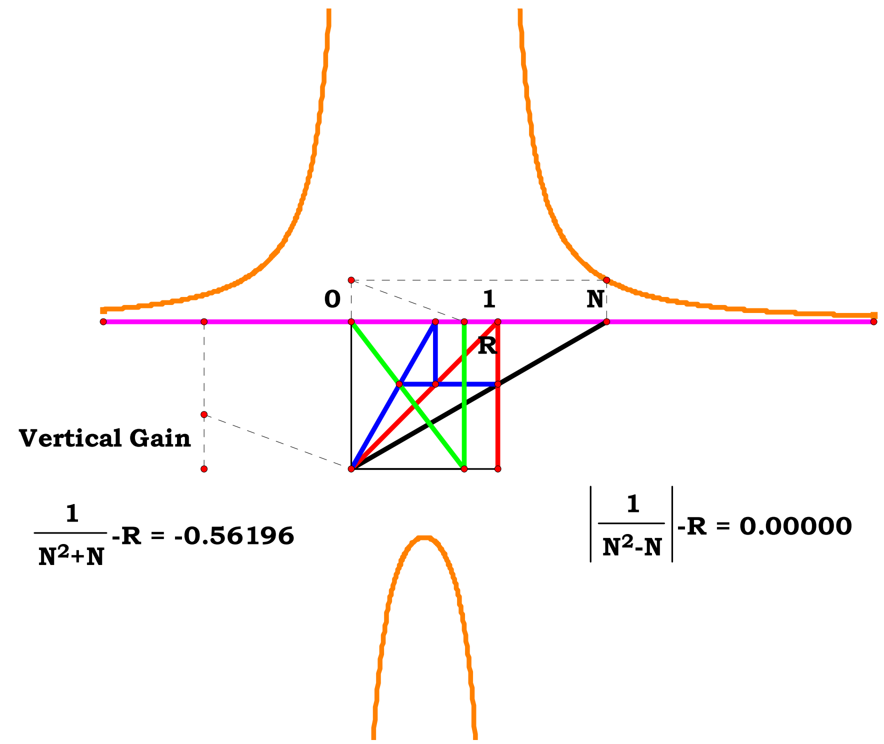


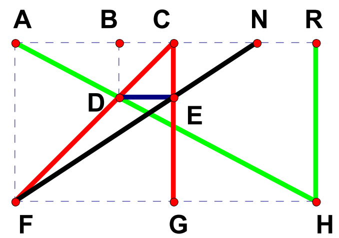
$AC := 1$

$AN := 3$

$AJ := \frac{1}{AN^2}$

$CF := \frac{AN - 1}{AN}$ $\quad DJ := CF$ $\quad GI := \frac{AJ \cdot AC}{DJ}$ $\quad AR := GI$ $\quad AR - \frac{1}{AN^2 - AN} = 0$



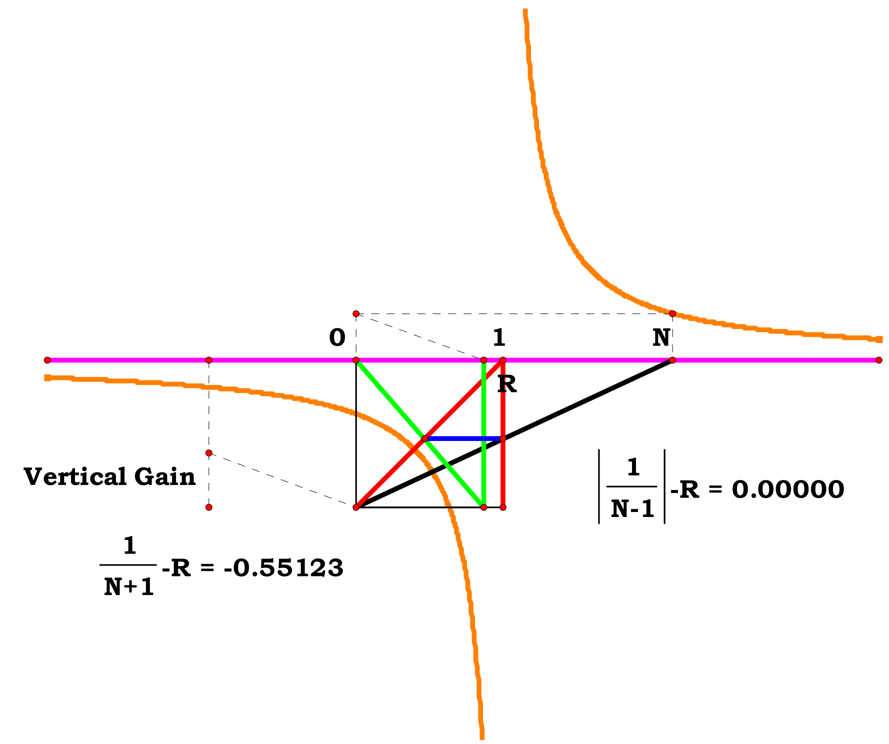


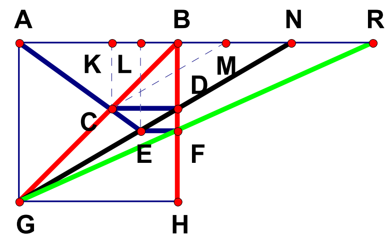
$$AC := 1$$

$$AN := 3$$

$$AB := \frac{1}{AN}$$

$$BD := \frac{AN-1}{AN} \qquad FH := \frac{AB \cdot AC}{BD} \qquad AR := FH \qquad AR - \frac{1}{AN-1} = 0$$





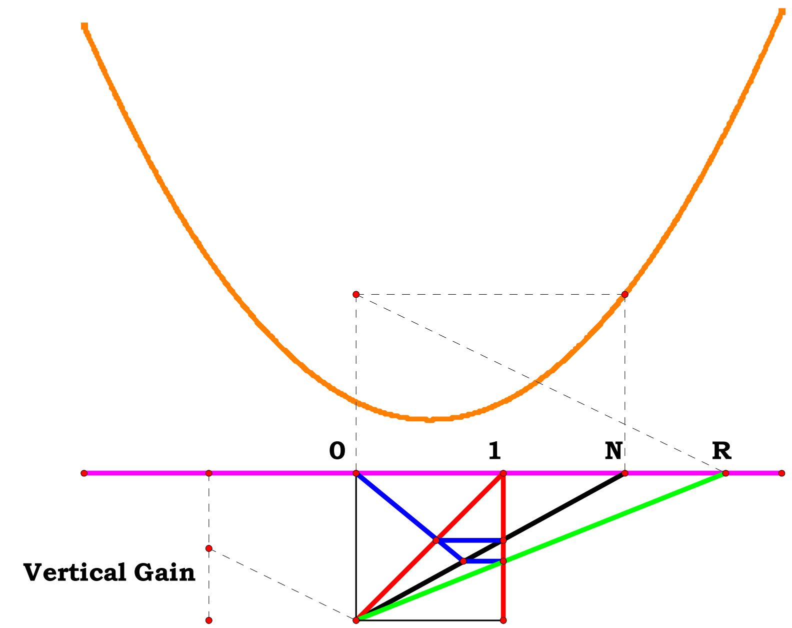
$AB := 1$

$AN := 2$

$AK := \frac{1}{AN}$

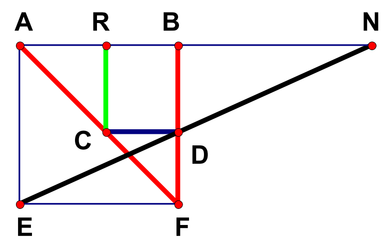
$KM := AN - AB \quad AM := AK + KM \quad AL := \frac{AK \cdot AN}{AM} \quad LN := AN - AL$

$EL := \frac{AB \cdot LN}{AN} \quad BF := EL \quad AR := \frac{AB^2}{AB - BF} \quad AR - \left(AN^2 - AN + 1\right) = 0$



$(N^2+N+1)-R = 3.65013$

$((N^2-N)+1)-R = 0.00000$

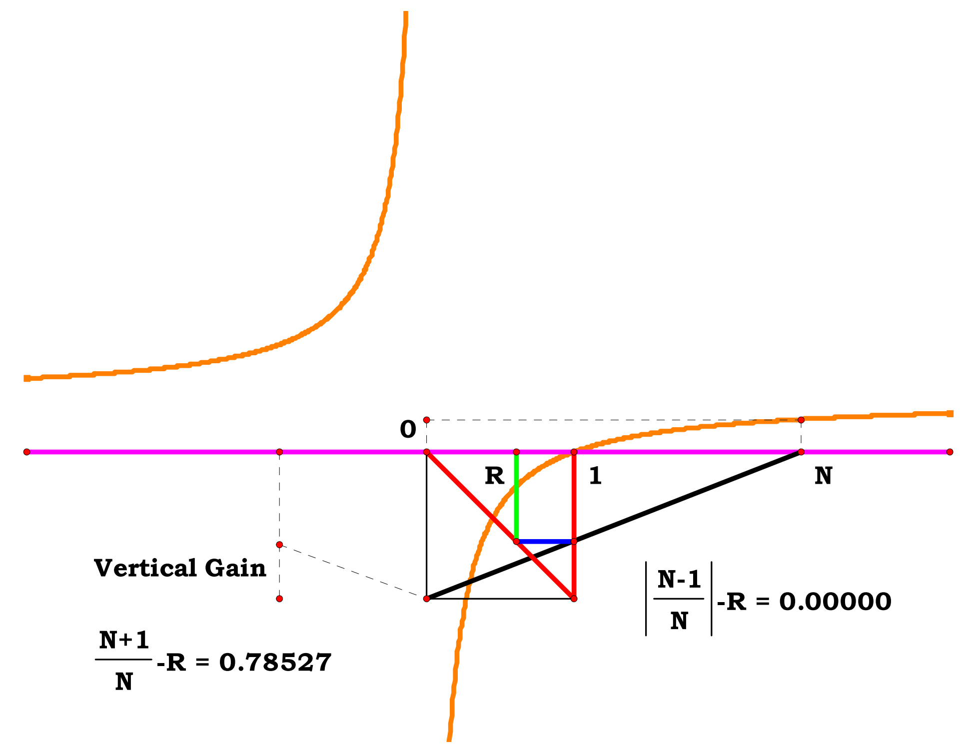


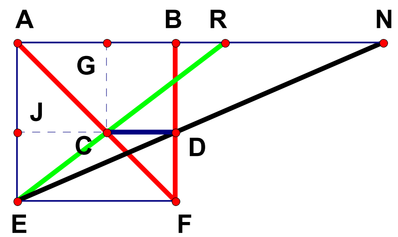
$AB := 1$

$AN := 3$

$BD := \dfrac{AN - 1}{AN}$

$AR := BD \qquad AR - \dfrac{AN - 1}{AN} = 0$



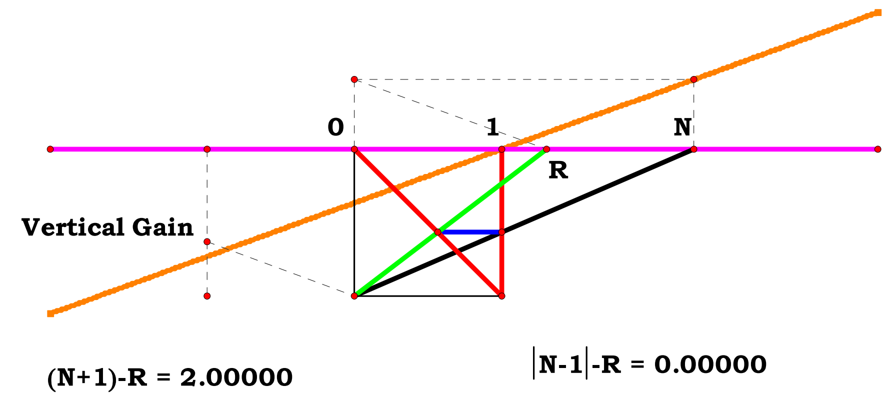


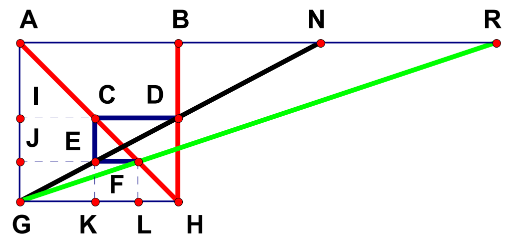
$$\mathbf{AB} := 1$$

$$\mathbf{AN} := 3$$

$$\mathbf{BD} := \frac{\mathbf{AN} - 1}{\mathbf{AN}}$$

$$\mathbf{CJ} := \mathbf{BD} \qquad \mathbf{EJ} := \mathbf{AB} - \mathbf{BD} \qquad \mathbf{AR} := \frac{\mathbf{CJ} \cdot \mathbf{AB}}{\mathbf{EJ}} \qquad \mathbf{AR} - (\mathbf{AN} - 1) = 0$$



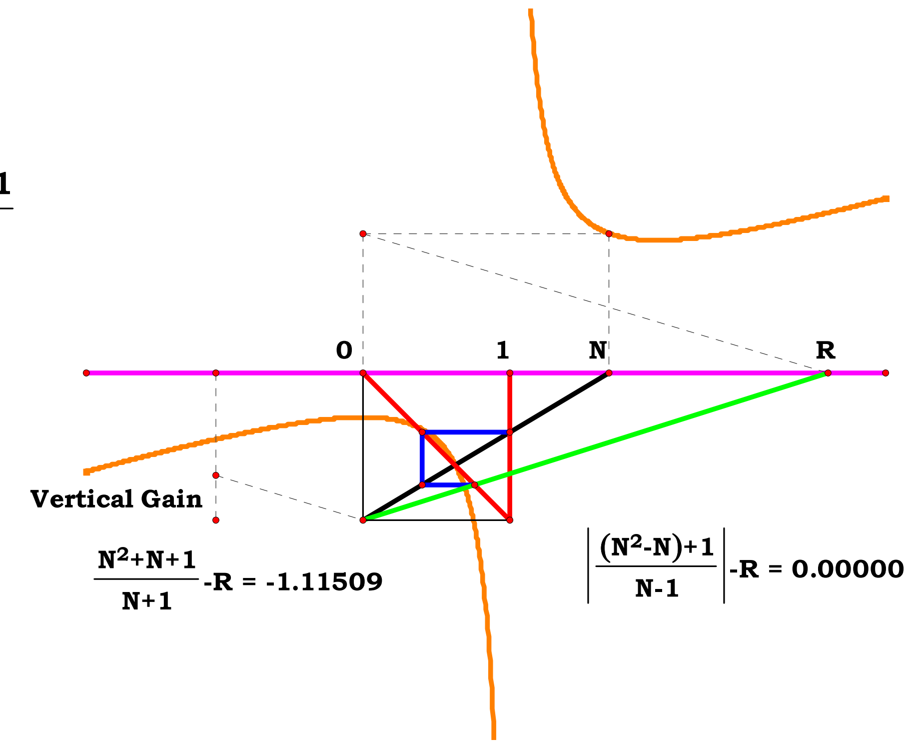


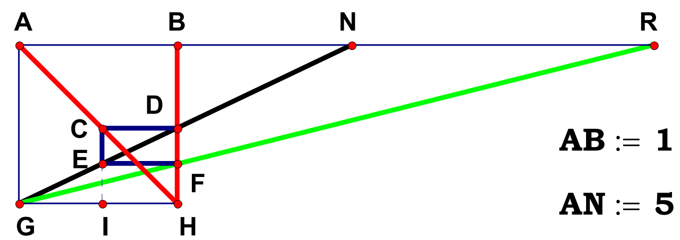
$$AB := 1$$

$$AN := 3$$

$$BD := \frac{AN - 1}{AN}$$

$$EK := \frac{AB \cdot BD}{AN} \qquad GL := AB - EK \qquad AR := \frac{GL \cdot AB}{EK} \qquad AR - \frac{AN^2 - AN + 1}{AN - 1} = 0$$

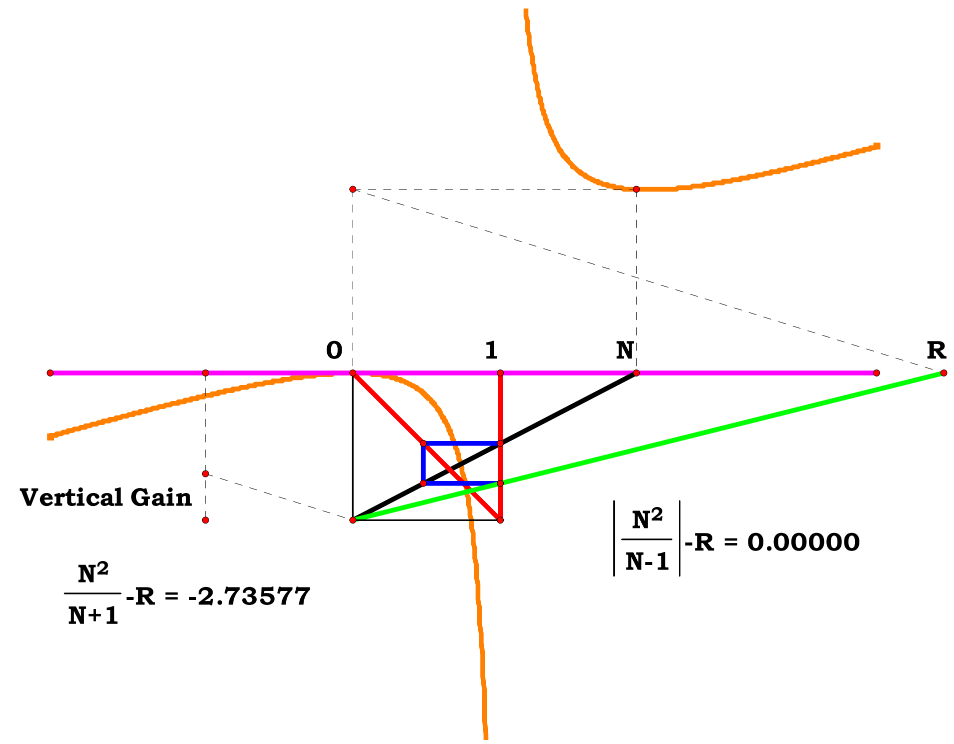


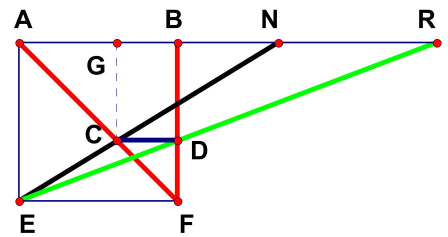


$AB := 1$

$AN := 5$

$BD := \frac{AN - 1}{AN}$ $\quad EI := \frac{AB \cdot BD}{AN}$ $\quad AR := \frac{AB^2}{EI}$ $\quad AR - \frac{AN^2}{AN - 1} = 0$

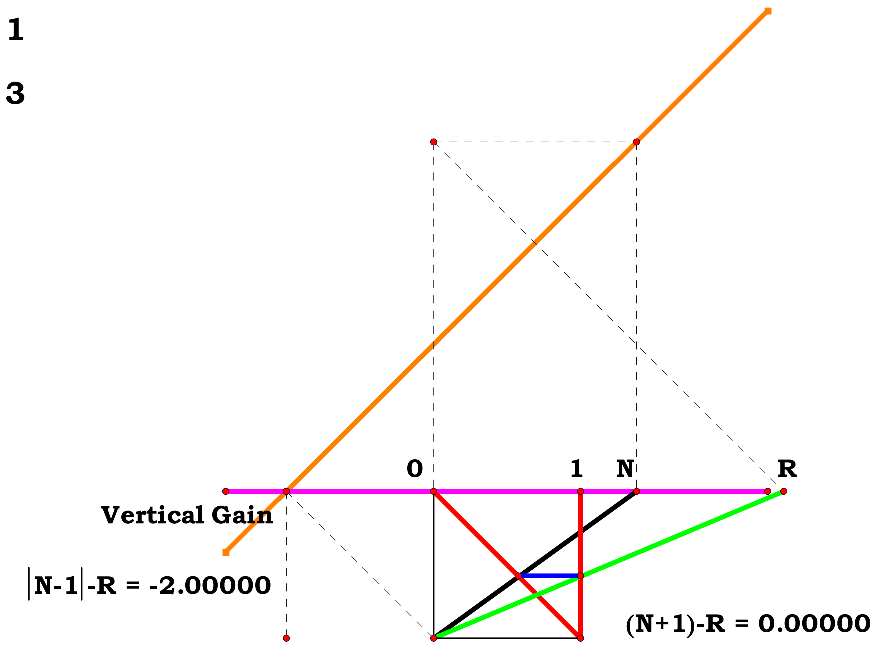


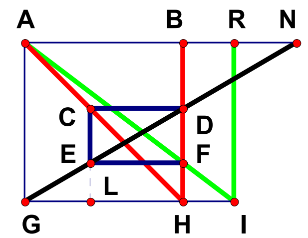


$\mathbf{AB} := \mathbf{1}$

$\mathbf{AN} := \mathbf{3}$

$$\mathbf{AG} := \frac{\mathbf{AB} \cdot \mathbf{AN}}{\mathbf{AB} + \mathbf{AN}} \qquad \mathbf{BD} := \mathbf{AG} \qquad \mathbf{AR} := \frac{\mathbf{AB}^2}{\mathbf{AB} - \mathbf{BD}} \qquad \mathbf{AR} - (\mathbf{AN} + \mathbf{1}) = \mathbf{0}$$

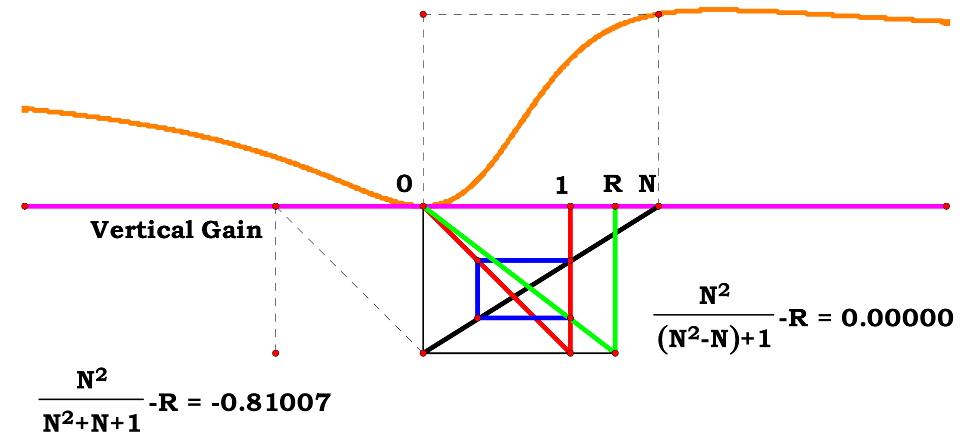




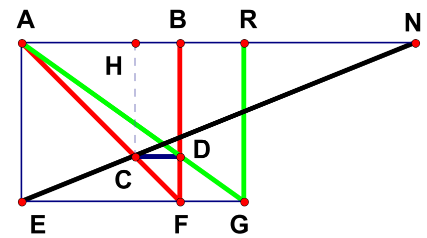
$$AB := 1$$

$$AN := 3$$

$$BD := \frac{AN - 1}{AN}$$



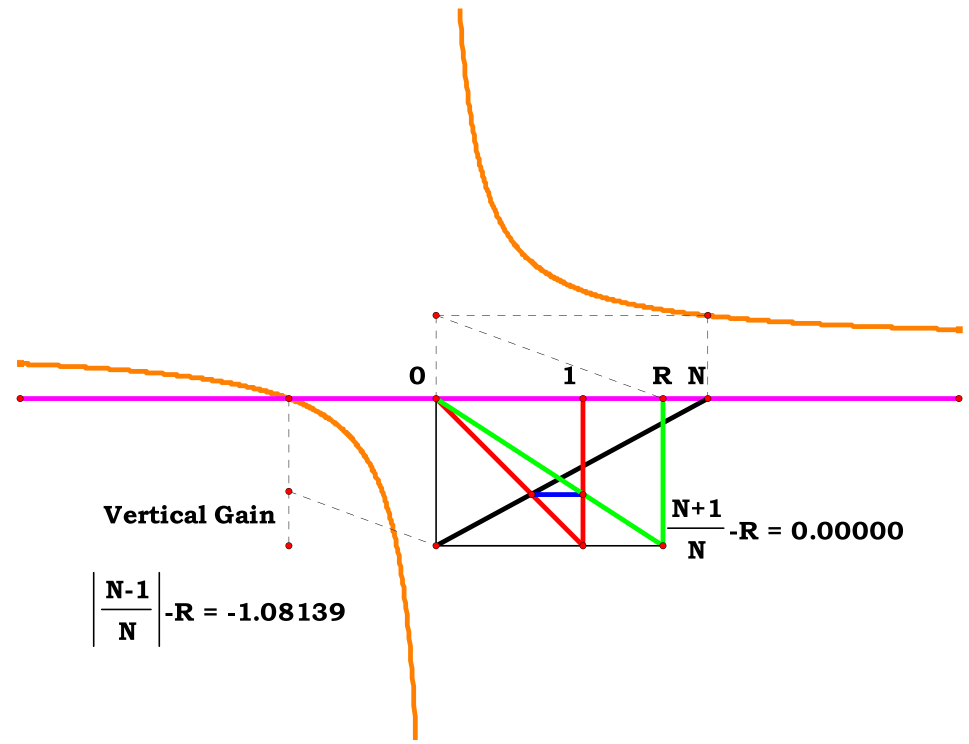
$$EL := \frac{AB \cdot BD}{AN} \qquad GI := \frac{AB^2}{AB - EL} \qquad AR := GI \qquad AR - \frac{AN^2}{AN^2 - AN + 1} = 0$$

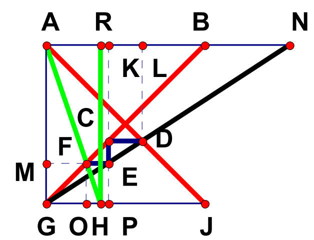


$AB := 1$

$AN := 3$

$$AH := \frac{AB \cdot AN}{AB + AN} \qquad EG := \frac{AB^2}{AH} \qquad AR := EG \qquad AR - \frac{AN + 1}{AN} = 0$$





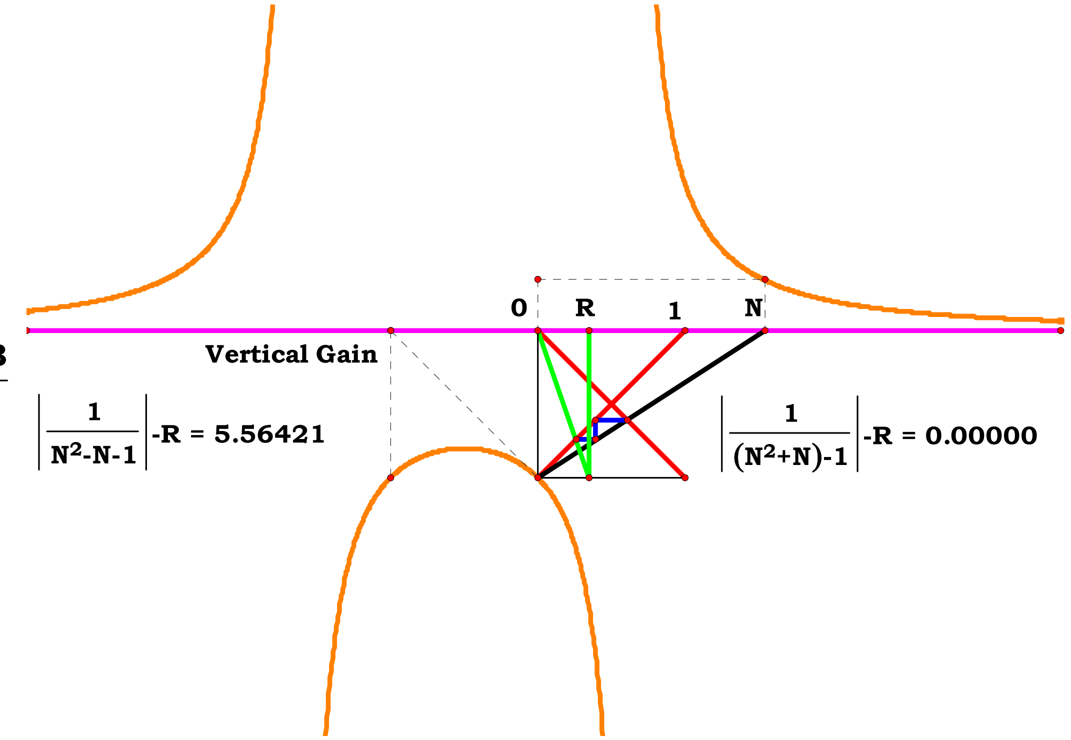
$$AB := 1$$

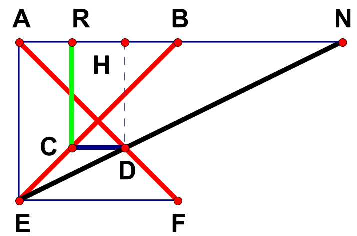
$$AN := 2$$

$$AL := \frac{AB \cdot AN}{AB + AN}$$

$GP := AB - AL \quad EP := \frac{AB \cdot GP}{AN} \quad AM := AB - EP \quad FM := EP \quad GH := \frac{FM \cdot AB}{AM}$

$AR := GH \qquad AR - \frac{1}{AN^2 + AN - 1} = 0$

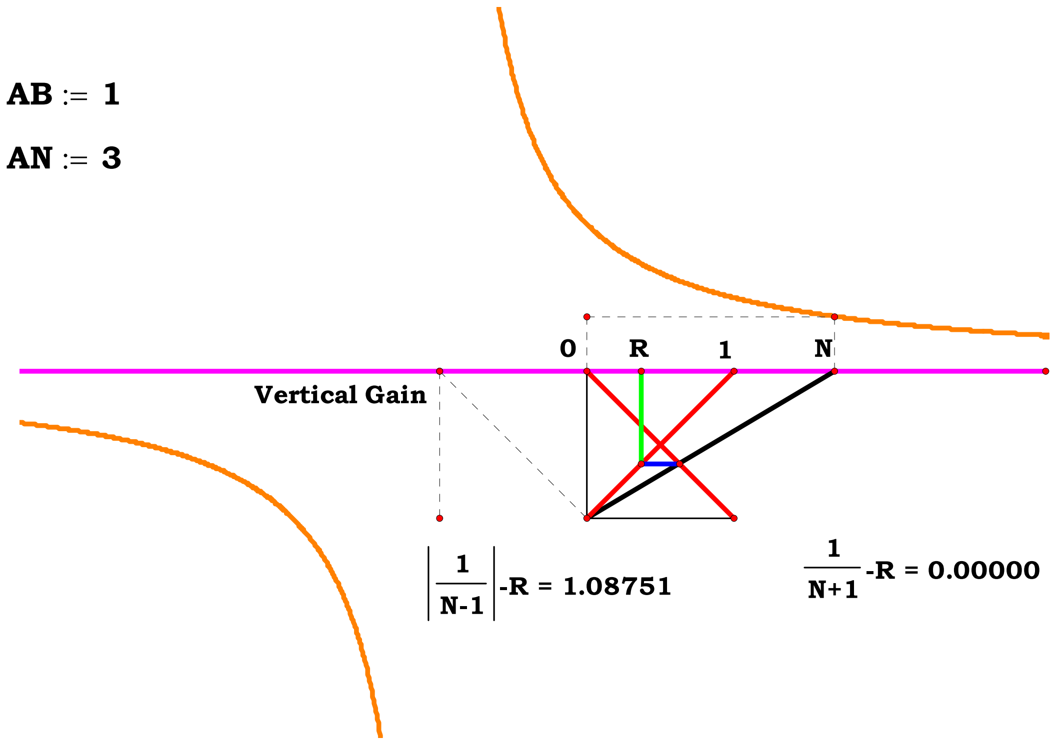


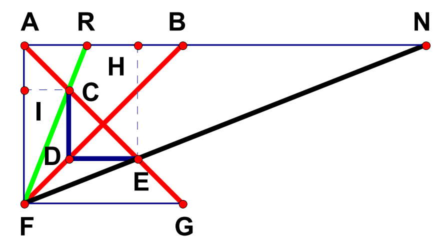


$$AB := 1$$

$$AN := 3$$

$$AH := \frac{AB \cdot AN}{AB + AN} \qquad AR := AB - AH \qquad AR - \frac{1}{AN + 1} = 0$$



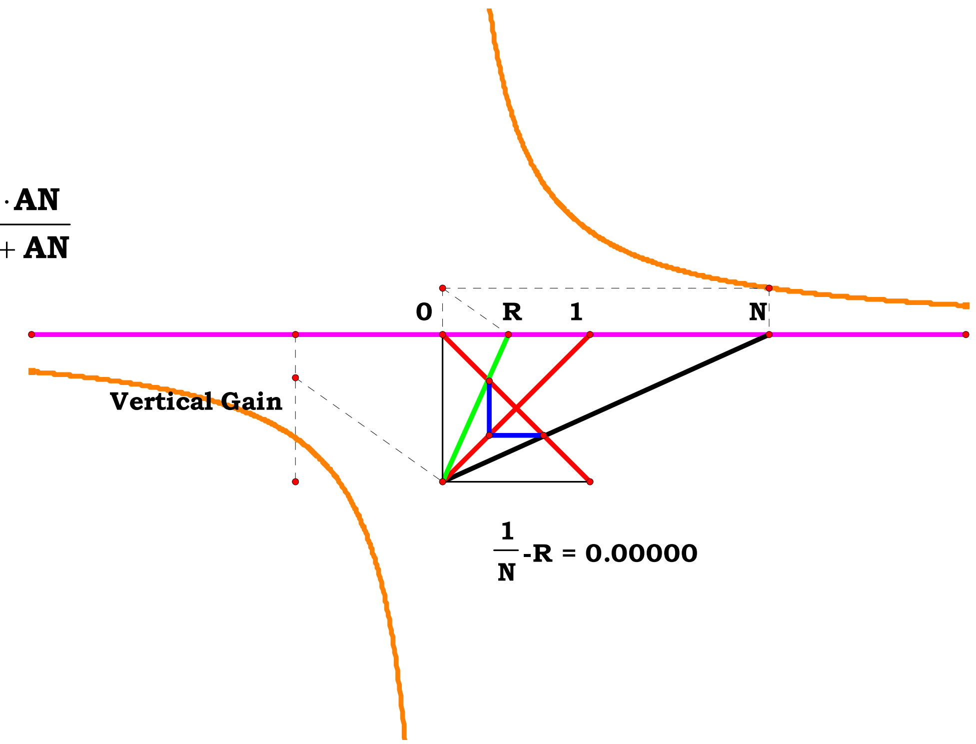


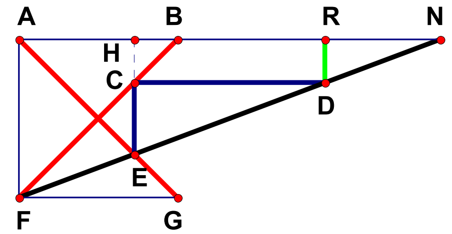
$$AB := 1$$

$$AN := 3$$

$$AH := \frac{AB \cdot AN}{AB + AN}$$

$BH := AB - AH$ $\quad CI := BH$ $\quad FI := AH$ $\quad AR := \frac{CI \cdot AB}{FI}$ $\quad AR - \frac{1}{AN} = 0$

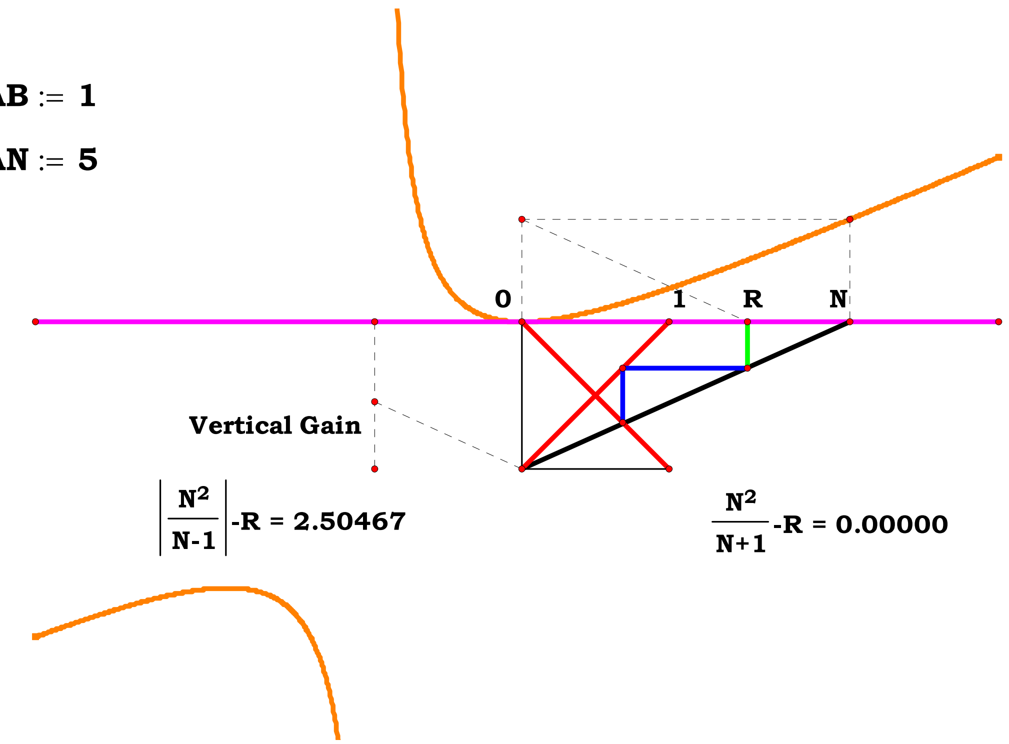


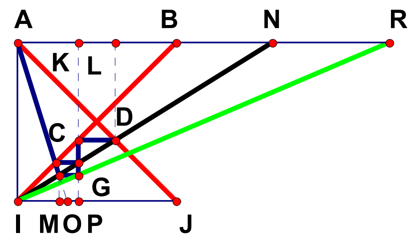


$AB := 1$

$AN := 5$

$$AH := \frac{AB \cdot AN}{AB + AN} \quad AR := AN - AH \quad AR - \frac{AN^2}{AN + 1} = 0$$





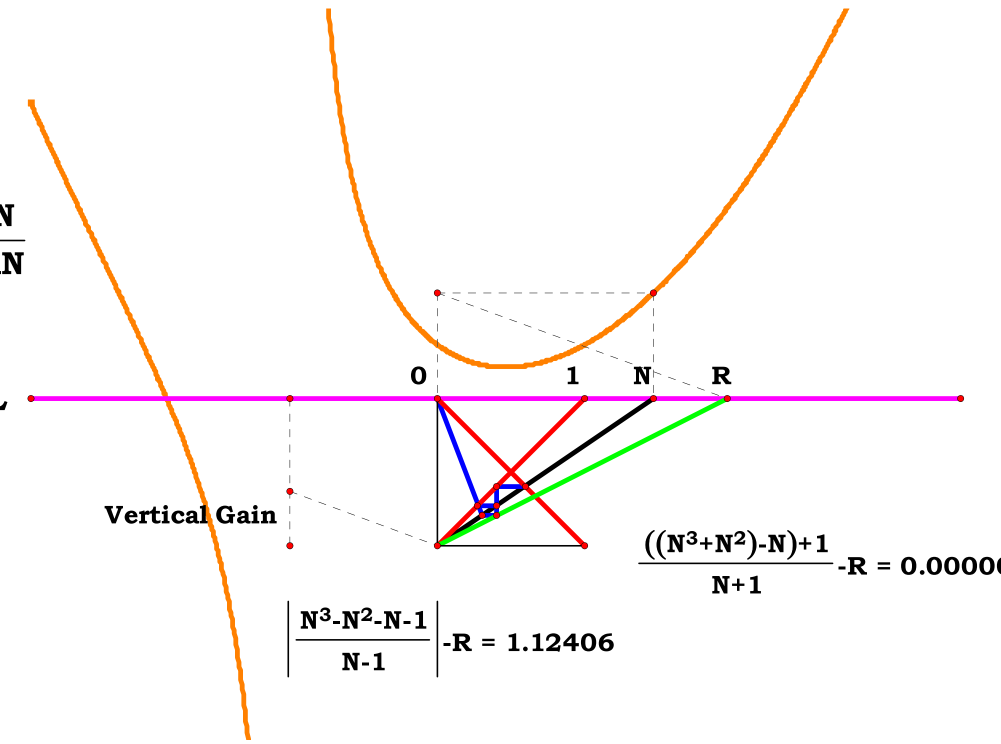
$AB := 1$

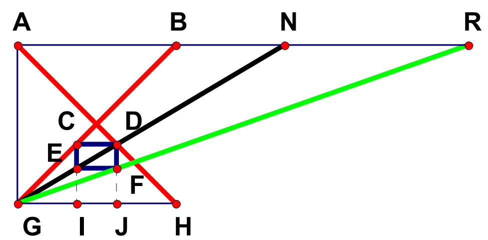
$AN := 3$

$$AL := \frac{AB \cdot AN}{AB + AN}$$

$$IO := \frac{1}{AN^2 + AN - 1} \quad IM := \frac{IO \cdot AN}{IO + AN} \quad HM := \frac{AB \cdot IM}{AN} \quad BL := AB - AL \quad IP := BL$$

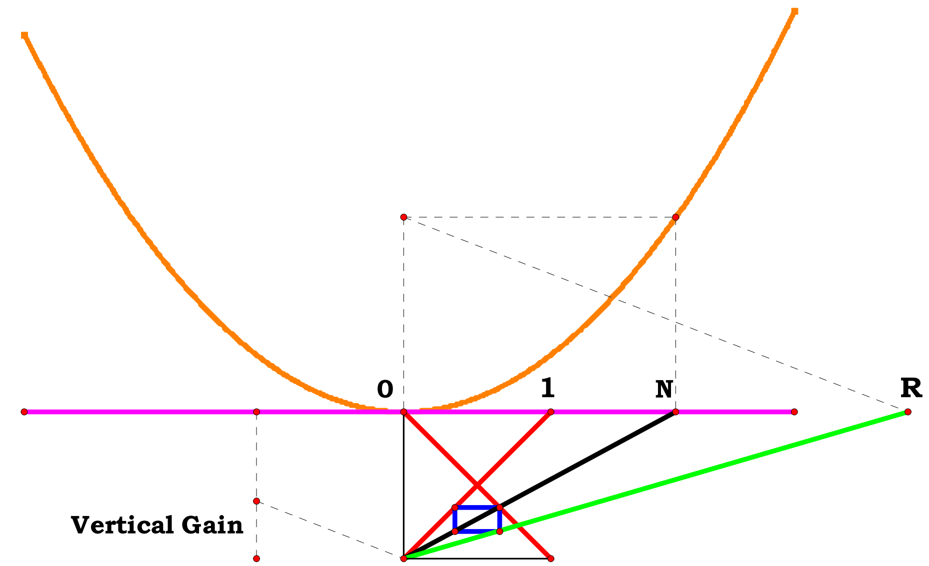
$$GP := HM \quad AR := \frac{IP \cdot AB}{GP} \quad AR - \frac{AN^3 + AN^2 - AN + 1}{AN + 1} = 0$$





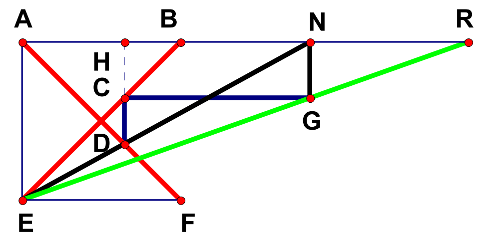
$$AB := 1$$

$$AN := 3$$



$$GJ := \frac{AB \cdot AN}{AB + AN} \qquad GI := AB - GJ \qquad EI := \frac{AB \cdot GI}{AN} \qquad AR := \frac{GJ \cdot AB}{EI} \qquad AR - AN^2 = 0$$

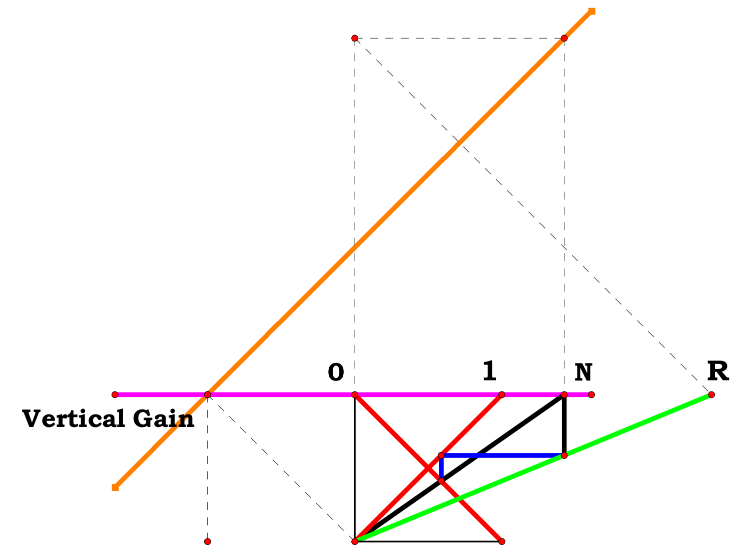
N^2-R = 0.00000



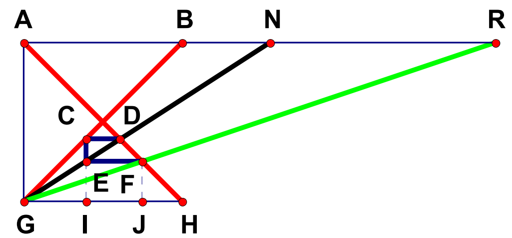
$AB := 1$

$AN := 3$

$AH := \dfrac{AB \cdot AN}{AB + AN}$ $\quad NG := AB - AH$ $\quad AR := \dfrac{AN \cdot AB}{AH}$ $\quad AR - (AN + 1) = 0$



$|N\text{-}1|\text{-}R$ = -2.00000 $\qquad$ (N+1)-R = 0.00000



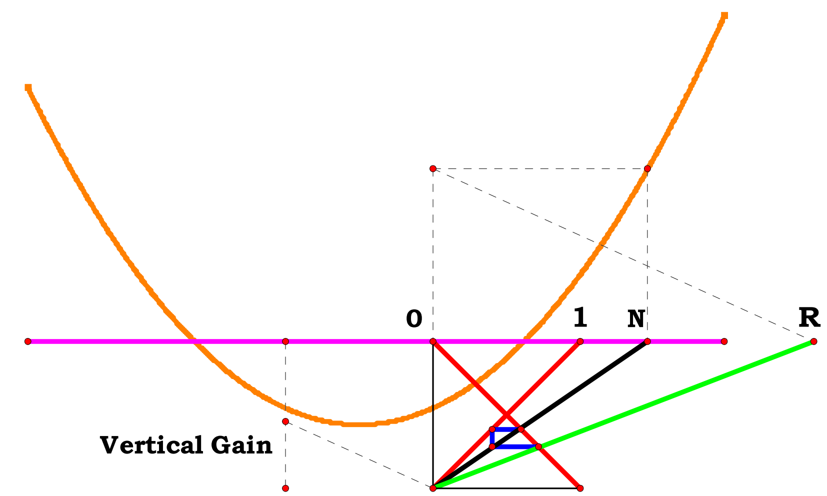
$$AB := 1$$

$$AN := 3$$

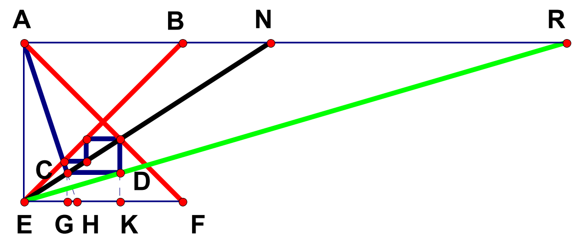
$$HI := \frac{AB \cdot AN}{AB + AN}$$

$$GI := AB - HI \qquad EI := \frac{AB \cdot GI}{AN} \qquad GJ := AB - EI \qquad AR := \frac{GJ \cdot AB}{EI} \qquad AR - \left(AN^2 + AN - 1\right) = 0$$

$$GI - \frac{1}{AN + 1} = 0 \qquad EI - \frac{1}{AN^2 + AN} = 0$$



$|-N^2+N+1|-R = -2.24930$ $\qquad ((N^2+N)-1)-R = 0.00000$



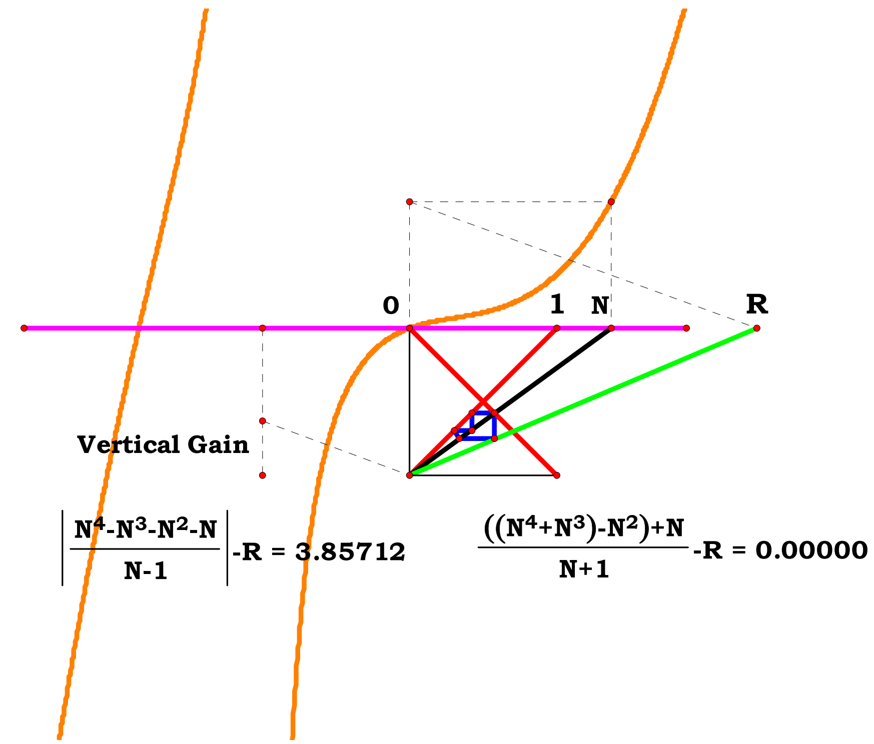
$$AB := 1$$

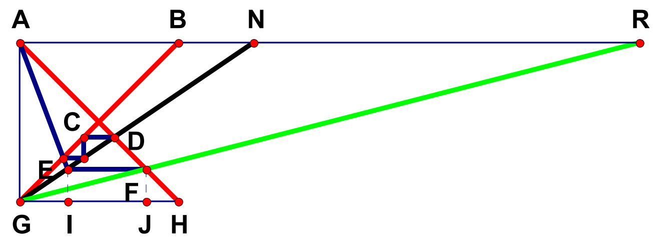
$$AN := 3$$

$$EK := \frac{AB \cdot AN}{AB + AN} \qquad EH := \frac{1}{AN^2 + AN - 1} \qquad EG := \frac{EH \cdot AN}{EH + AN} \qquad CG := \frac{AB \cdot EG}{AN} \qquad AR := \frac{EK \cdot AB}{CG}$$

$$AR - \frac{AN^4 + AN^3 - AN^2 + AN}{AN + 1} = 0$$

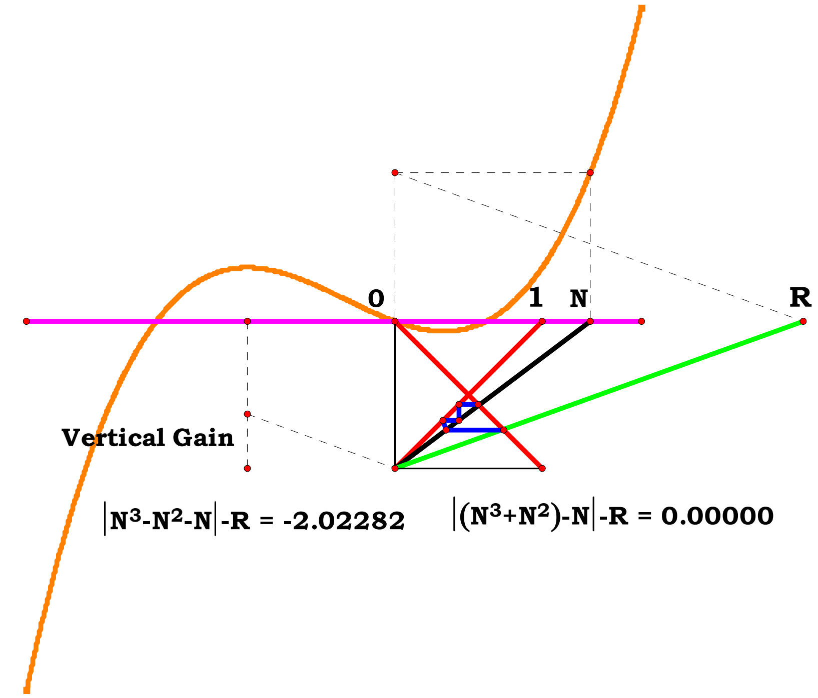
$$EG - \frac{AN}{AN^3 + AN^2 - AN + 1} = 0 \qquad CG - \frac{1}{AN^3 + AN^2 - AN + 1} = 0$$

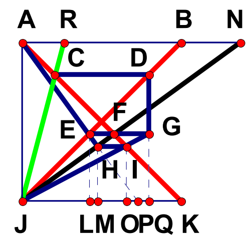




$AB := 1 \quad AN := 4 \quad EI := \dfrac{1}{AN^3 + AN^2 - AN + 1} \quad GJ := AB - EI \quad FJ := EI$

$AR := \dfrac{GJ \cdot AB}{FJ} \quad AR - \left(AN^3 + AN^2 - AN\right) = 0$





$AB := 1$

$AN := 8$

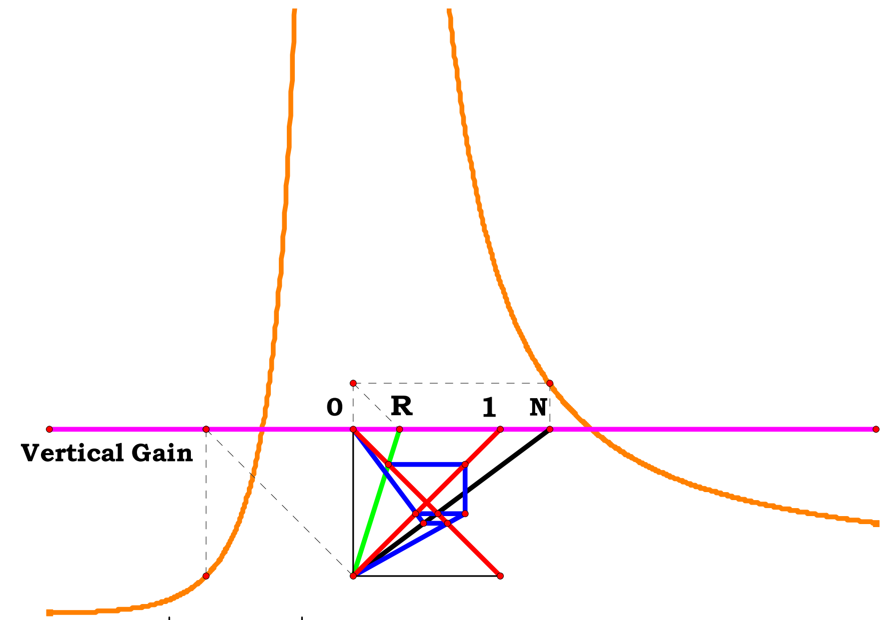
$KL := \dfrac{AB \cdot AN}{AB + AN}$

$JL := AB - KL$

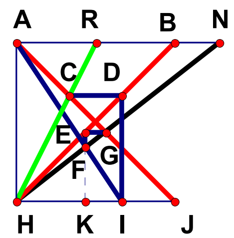
$JP := \dfrac{JL \cdot AB}{KL}$ $\quad JM := \dfrac{JP \cdot AN}{JP + AN}$ $\quad HM := \dfrac{AB \cdot JM}{AN}$ $\quad JO := AB - HM$

$JQ := \dfrac{JO \cdot JL}{HM}$ $\quad KQ := AB - JQ$ $\quad AR := \dfrac{KQ \cdot AB}{JQ}$ $\quad AR - \dfrac{AN - AN^2 + 1}{AN^2} = 0$

$JQ - \dfrac{AN^2}{AN + 1} = 0$



$\left|\dfrac{(N-1)+N^2}{N^2}\right| \text{-R} = 0.87543$ $\qquad \left|\dfrac{(N+1)-N^2}{N^2}\right| \text{-R} = 0.00000$



$$AB := 1$$

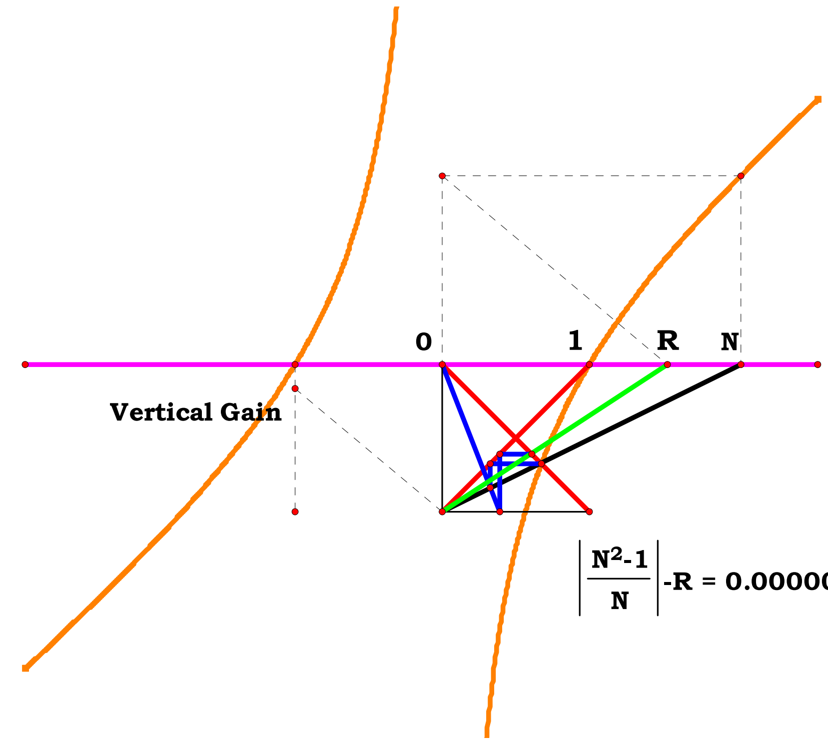
$$AN := 3$$

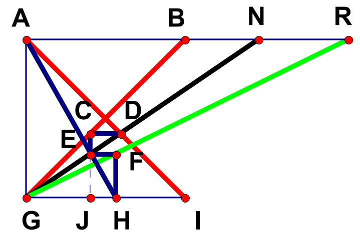
$$JK := \frac{AB \cdot AN}{AB + AN}$$

$$HK := AB - JK \qquad FK := \frac{AB \cdot HK}{AN} \qquad HI := \frac{HK \cdot AB}{AB - FK} \qquad JI := AB - HI$$

$$AR := \frac{JI \cdot AB}{HI} \qquad AR - \frac{AN^2 - 1}{AN} = 0$$

$$HK - \frac{1}{1 + AN} = 0 \qquad FK - \frac{1}{AN^2 + AN} = 0 \qquad HI - \frac{AN}{AN^2 + AN - 1} = 0$$

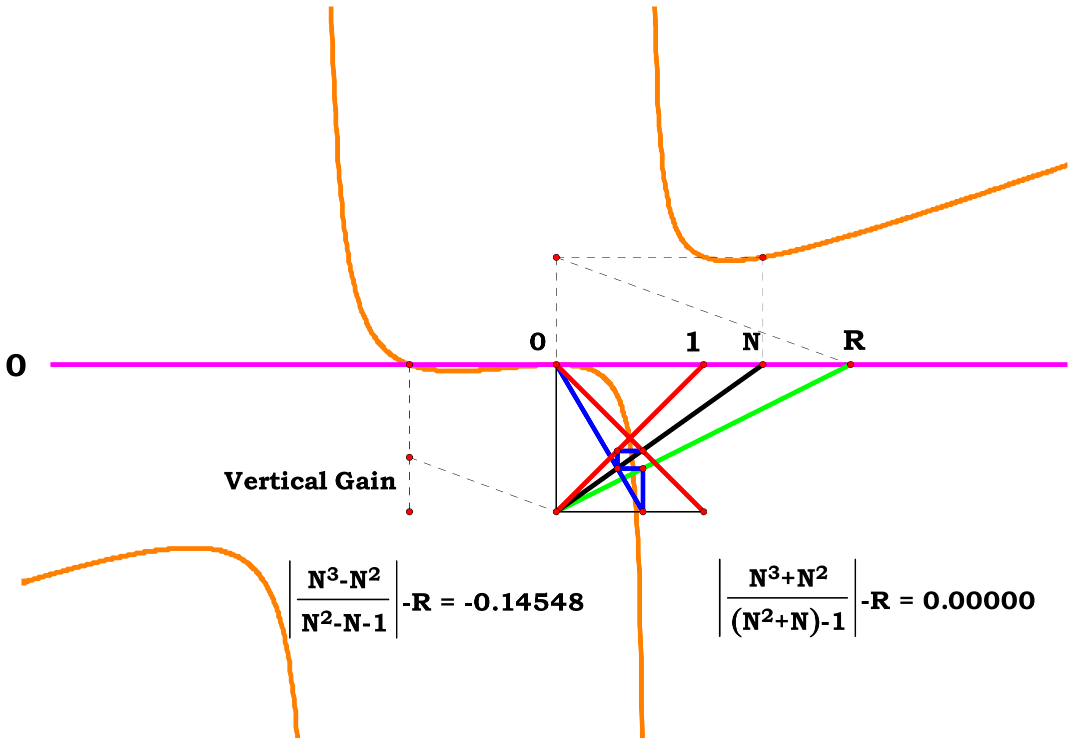


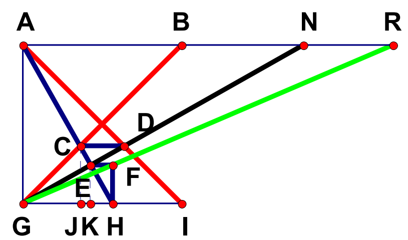


$AB := 1$

$AN := 3$

$$EJ := \frac{1}{AN^2 + AN} \qquad GH := \frac{AN}{AN^2 + AN - 1} \qquad AR := \frac{GH \cdot AB}{EJ} \qquad AR - \frac{AN^3 + AN^2}{AN^2 + AN - 1} = 0$$





$AB := 1$

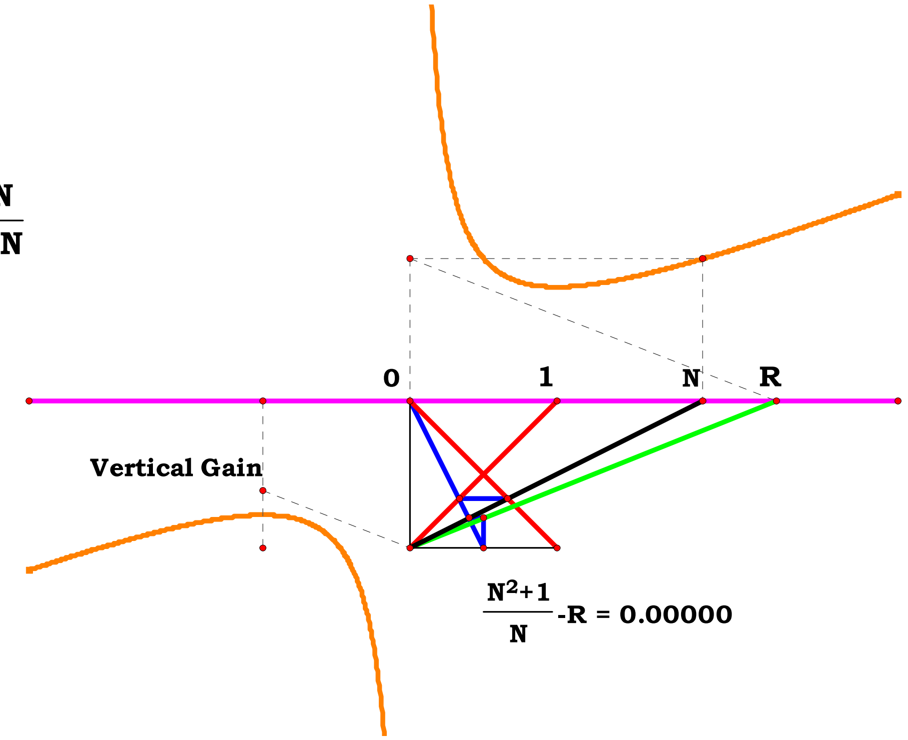
$AN := 3$

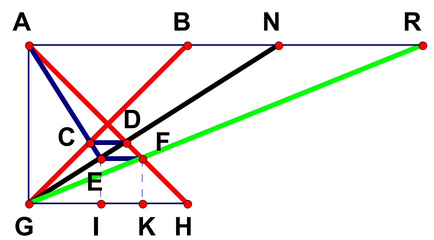
$IJ := \dfrac{AB \cdot AN}{AB + AN}$

$GJ := AB - IJ \qquad GH := \dfrac{GJ \cdot AB}{IJ} \qquad GK := \dfrac{GH \cdot AN}{GH + AN} \qquad EK := \dfrac{AB \cdot GK}{AN}$

$AR := \dfrac{GH \cdot AB}{EK} \qquad AR - \dfrac{AN^2 + 1}{AN} = 0$

$EK - \dfrac{1}{AN^2 + 1} = 0$



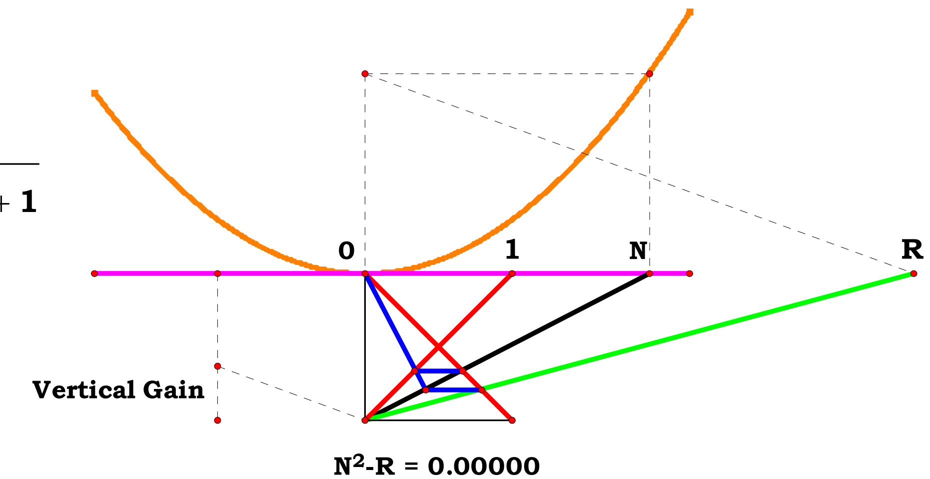


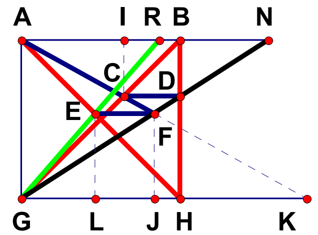
$$AB := 1$$

$$AN := 3$$

$$EI := \frac{1}{AN^2 + 1}$$

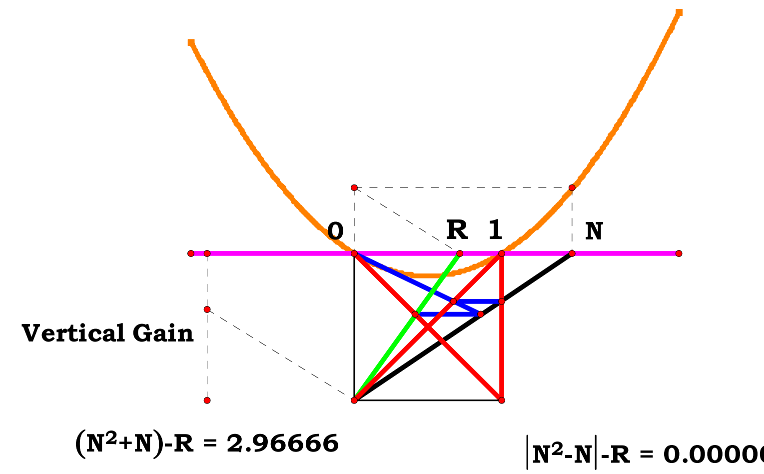
$$HK := EI \qquad GK := AB - EI \qquad AR := \frac{GK \cdot AB}{HK} \qquad AR - AN^2 = 0$$





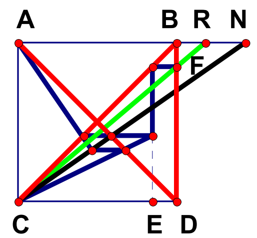
$$AB := 1$$

$$AN := 3$$



$$AI := \frac{1}{AN} \qquad CI := \frac{AN-1}{AN} \qquad GK := \frac{AI \cdot AB}{CI} \qquad GJ := \frac{GK \cdot AN}{GK + AN} \qquad FJ := \frac{AB \cdot GJ}{AN}$$

$$GL := AB - FJ \qquad AR := \frac{GL \cdot AB}{FJ} \qquad AR - \left(AN^2 - AN\right) = 0$$

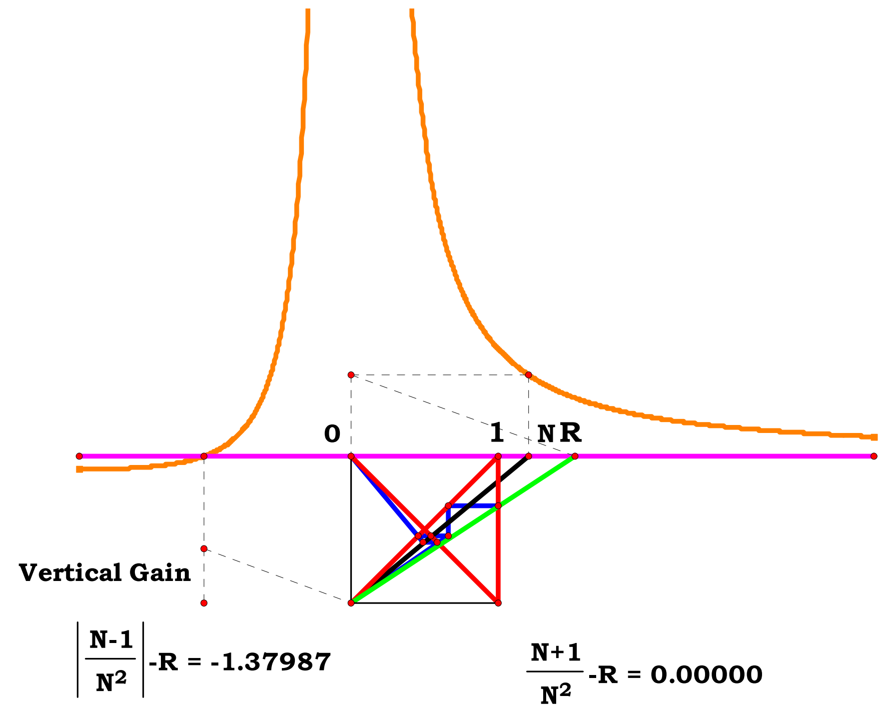


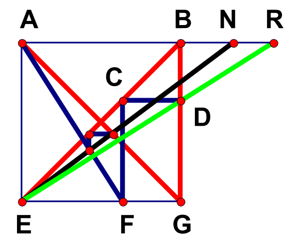
$$AB := 1$$

$$AN := 8$$

$$CE := \frac{AN^2}{AN + 1}$$

$$DE := AB - CE \qquad BF := DE \qquad AR := \frac{AB}{CE} \qquad AR - \frac{AN + 1}{AN^2} = 0$$



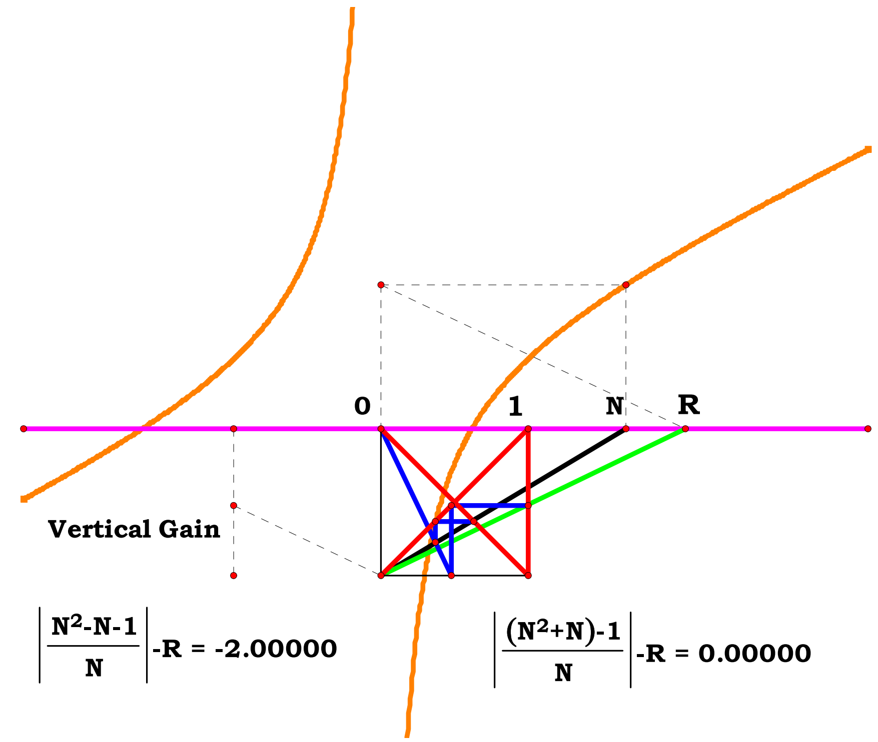


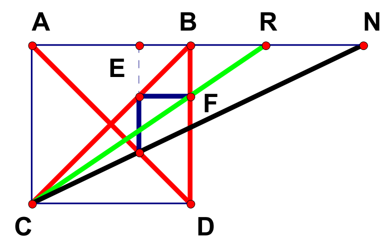
$$AB := 1$$

$$AN := 3$$

$$EF := \frac{AN}{AN^2 + AN - 1}$$

$$BD := AB - EF \quad AR := \frac{AB^2}{EF} \quad AR - \frac{AN^2 + AN - 1}{AN} = 0$$

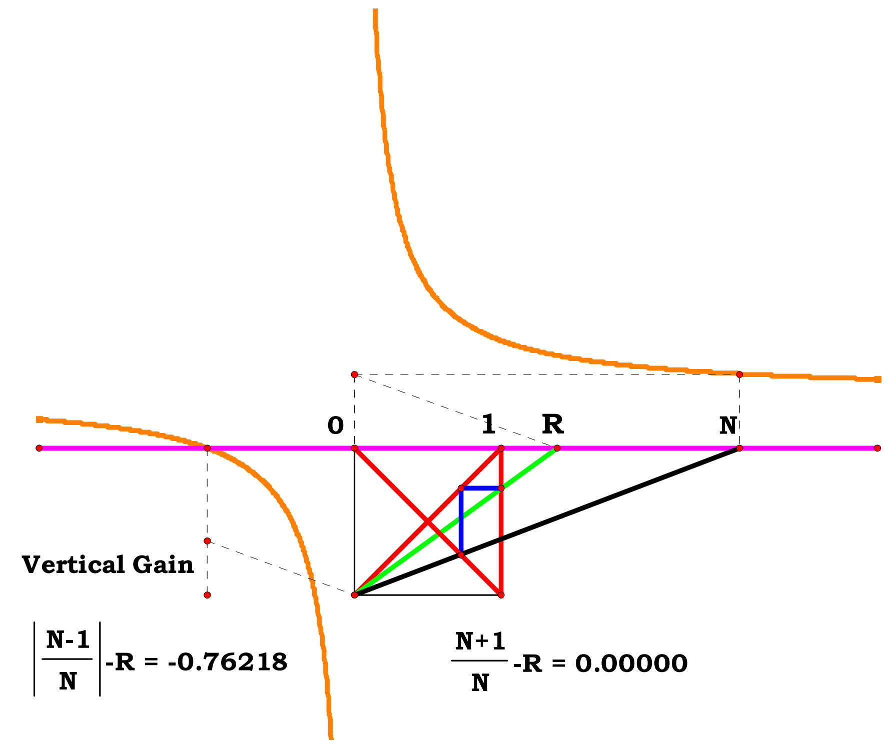


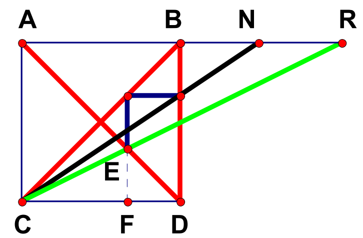


$$AB := 1$$

$$AN := 3$$

$$AE := \frac{AB \cdot AN}{AB + AN} \qquad BF := AB - AE \qquad AR := \frac{AB^2}{AE} \qquad AR - \frac{AN + 1}{AN} = 0$$

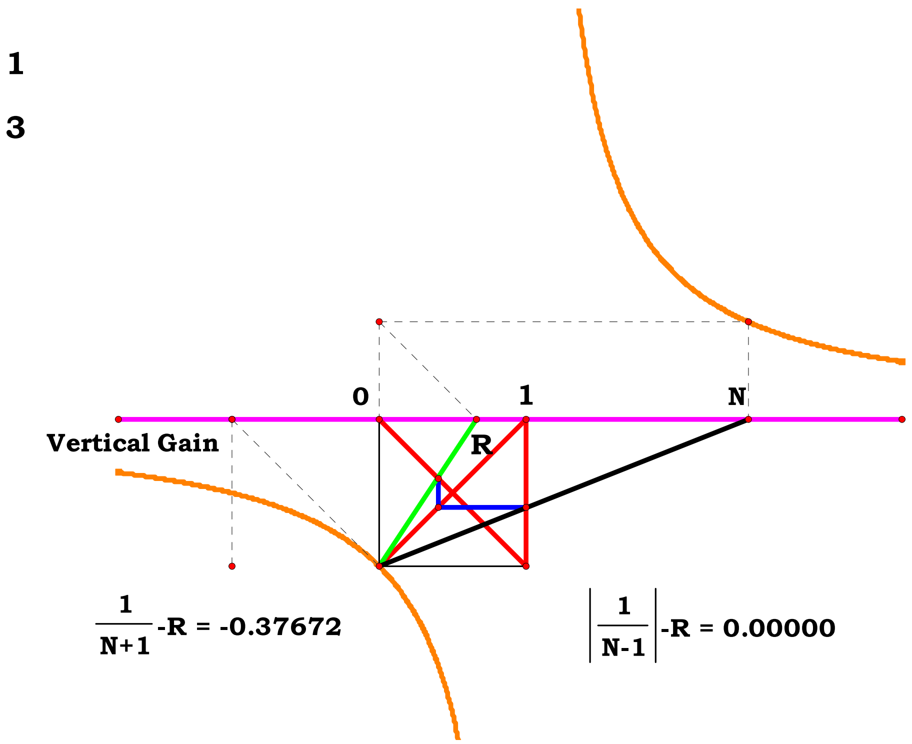


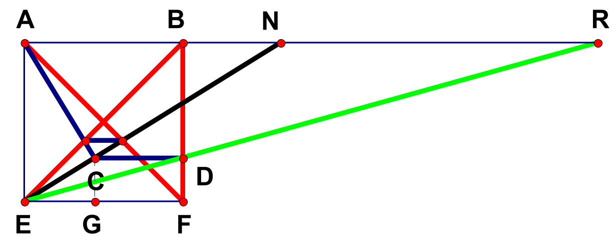


$AB := 1$

$AN := 3$

$$CF := \frac{1}{AN} \qquad EF := AB - CF \qquad AR := \frac{CF \cdot AB}{EF} \qquad AR - \frac{1}{AN - 1} = 0$$

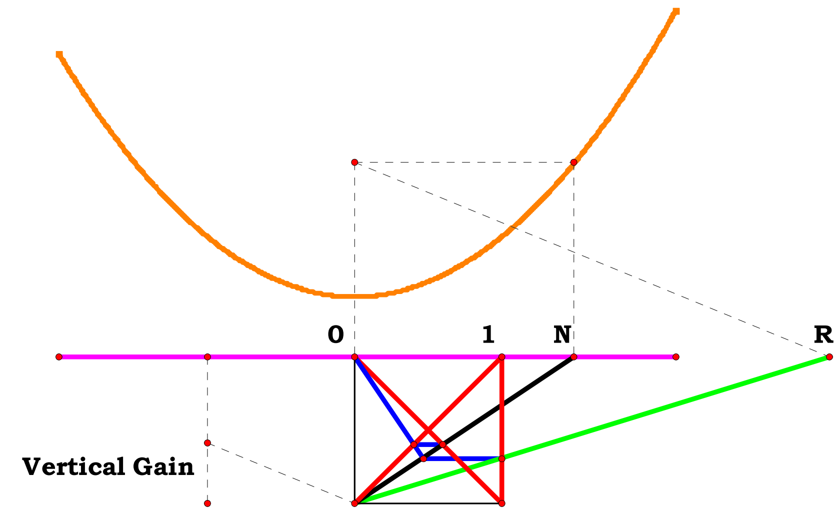




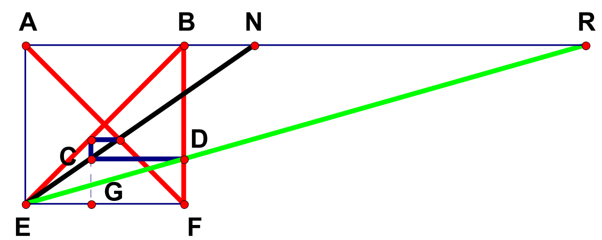
$AB := 1$

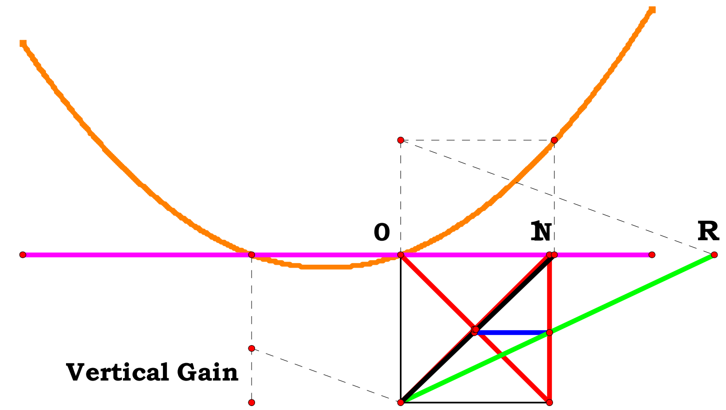
$AN := 3$

$$CG := \frac{1}{AN^2 + 1} \qquad DF := CG \qquad AR := \frac{AB^2}{DF} \qquad AR - \left(AN^2 + 1\right) = 0$$



$(N^2+1)-R = 0.00000$

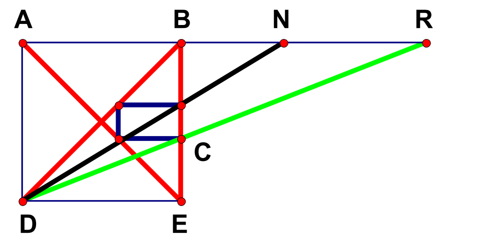




$\mathbf{AB} := \mathbf{1} \quad \mathbf{AN} := \mathbf{3} \quad \mathbf{CG} := \dfrac{\mathbf{1}}{\mathbf{AN}^2 + \mathbf{AN}} \quad \mathbf{DF} := \mathbf{CG} \quad \mathbf{AR} := \dfrac{\mathbf{AB}^2}{\mathbf{DF}}$

$\mathbf{AR} - \left(\mathbf{AN}^2 + \mathbf{AN}\right) = \mathbf{0}$

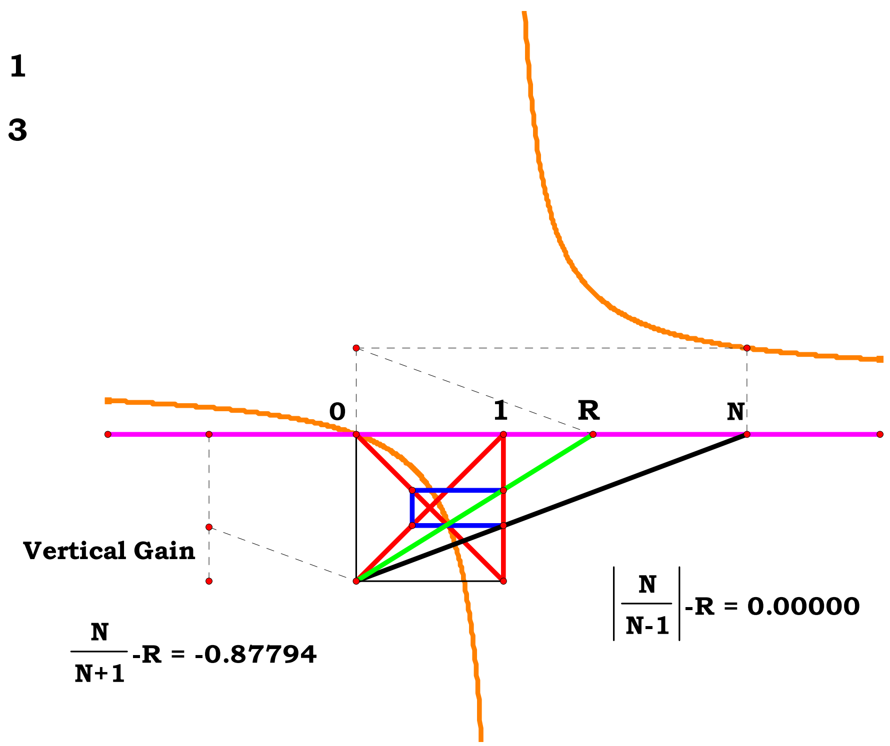
$|N^2\text{-}N|\text{-}R = -2.07389$ $\quad (N^2+N)\text{-}R = 0.00000$

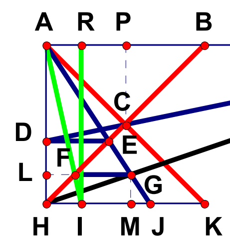


$\mathbf{AB} := 1$

$\mathbf{AN} := 3$

$$\mathbf{BC} := \frac{1}{\mathbf{AN}} \qquad \mathbf{CE} := \mathbf{AB} - \mathbf{BC} \qquad \mathbf{AR} := \frac{\mathbf{AB}^2}{\mathbf{CE}} \qquad \mathbf{AR} - \frac{\mathbf{AN}}{\mathbf{AN} - 1} = 0$$





$AB := 1$

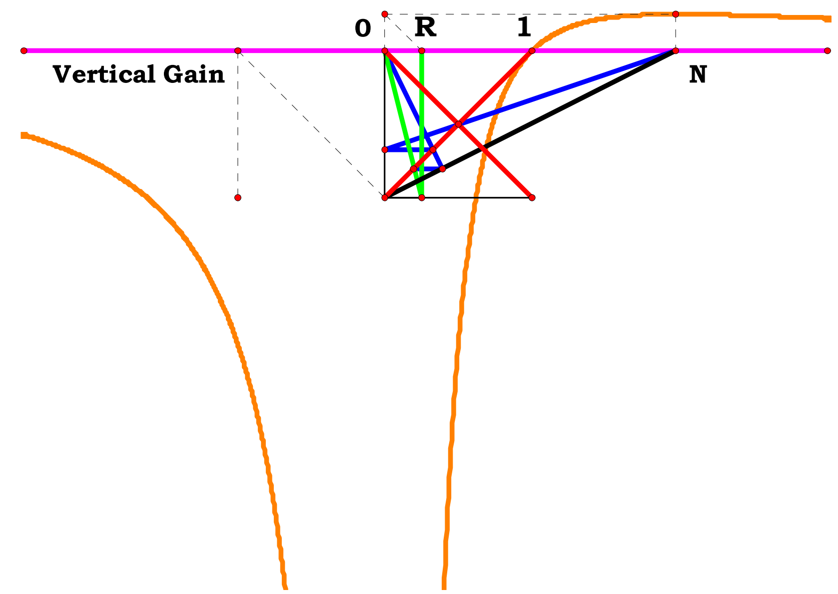
$AN := 3$

$AP := \dfrac{AB}{2}$

$NP := AN - AP$

$\dfrac{N+1}{N^2}$-R = 0.51129

$\left|\dfrac{N-1}{N^2}\right|$-R = 0.00000

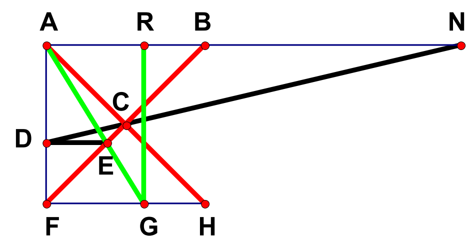


$CP := AP \quad AD := \dfrac{CP \cdot AN}{NP} \quad DH := AB - AD \quad DE := DH \quad HJ := \dfrac{DE \cdot AB}{AD}$

$HM := \dfrac{HJ \cdot AN}{HJ + AN} \quad GM := \dfrac{AB \cdot HM}{AN} \quad AL := AB - GM \quad FL := GM \quad HI := \dfrac{FL \cdot AB}{AL}$

$AR := HI \quad AR - \dfrac{AN - 1}{AN^2} = 0$

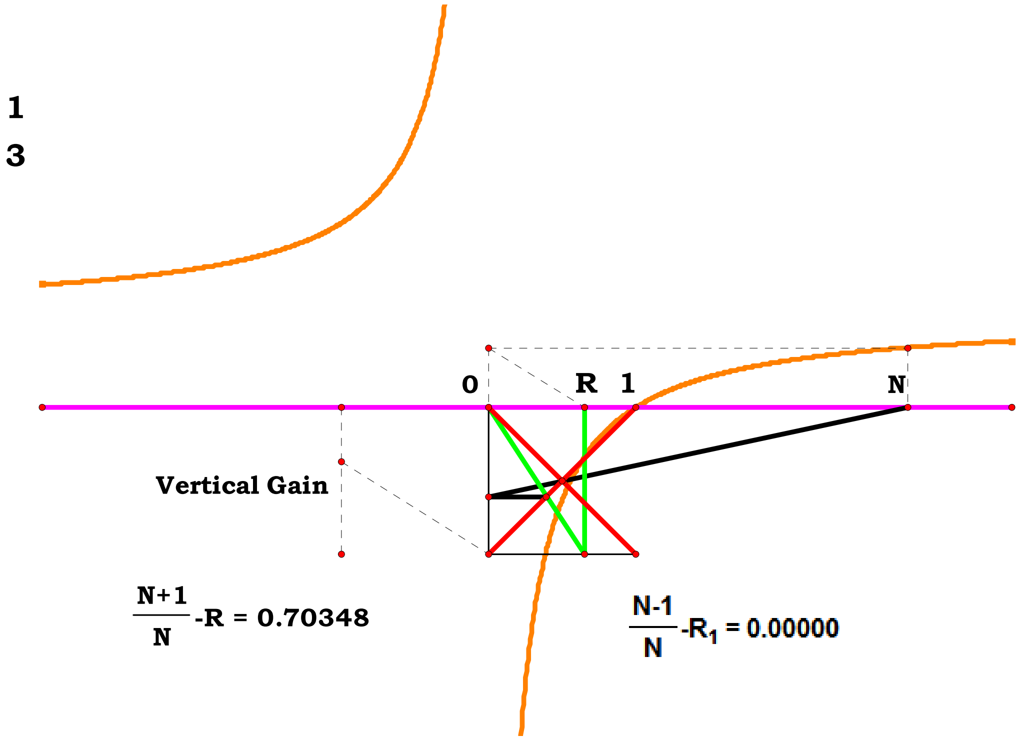
$AD - \dfrac{AN}{2 \cdot AN - 1} = 0 \quad HJ - \dfrac{AN - 1}{AN} = 0 \quad DH - \dfrac{AN - 1}{2 \cdot AN - 1} = 0 \quad HM - \dfrac{AN^2 - AN}{AN^2 + AN - 1} = 0$

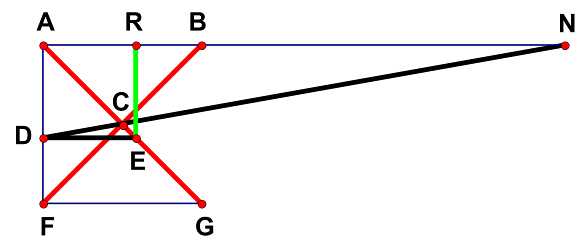


$\mathbf{AB} := 1$

$\mathbf{AN} := 3$

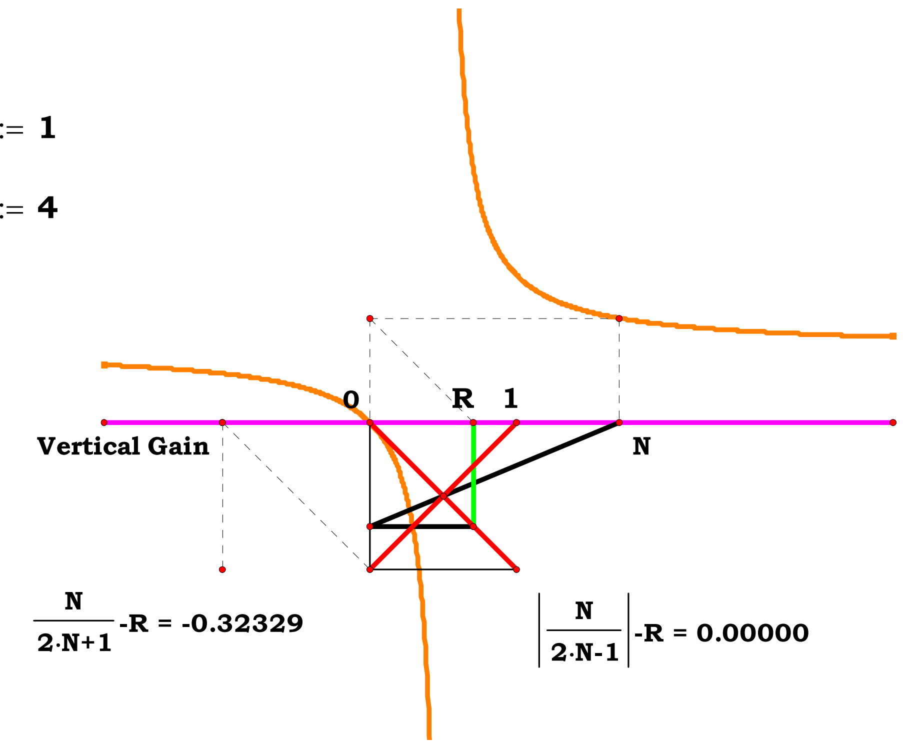
$$\mathbf{FG} := \frac{\mathbf{AN} - 1}{\mathbf{AN}} \qquad \mathbf{AR} := \mathbf{FG} \qquad \mathbf{AR} - \frac{\mathbf{AN} - 1}{\mathbf{AN}} = 0$$



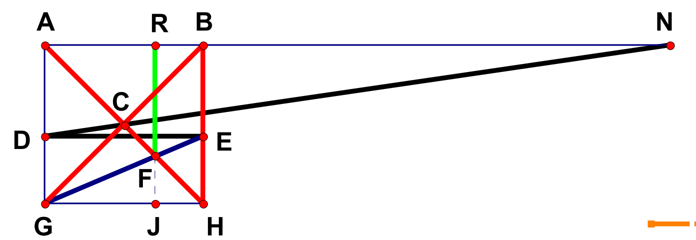


$$AB := 1$$

$$AN := 4$$

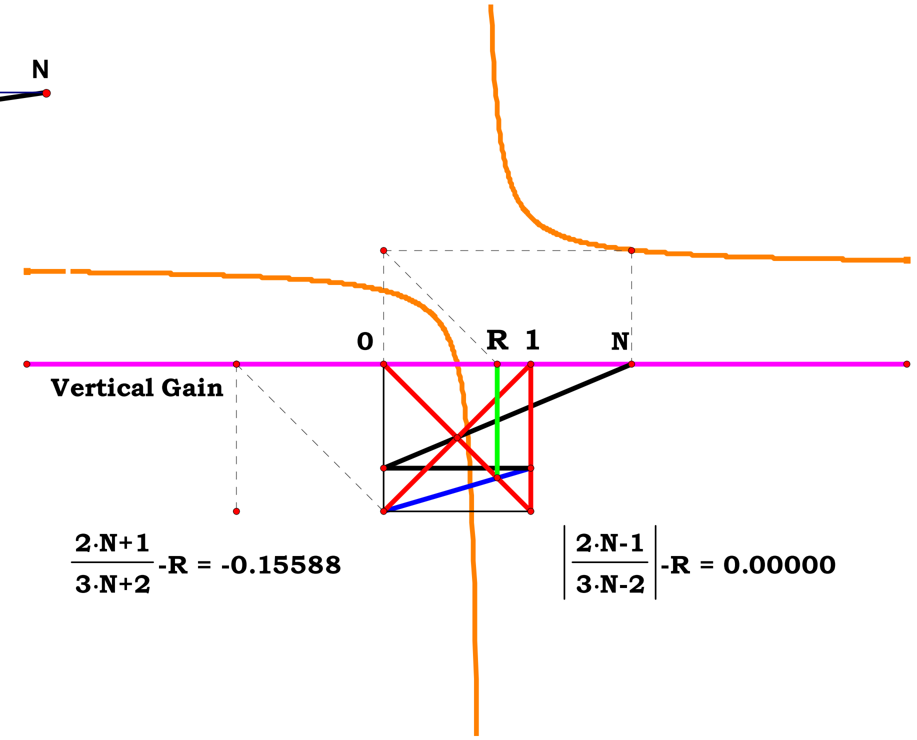


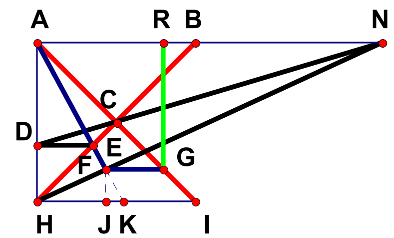
$$AD := \frac{AN}{2 \cdot AN - 1} \qquad AR := AD \qquad AD - \frac{AN}{2 \cdot AN - 1} = 0$$



$$\mathbf{AB} := 1 \quad \mathbf{AN} := 4 \quad \mathbf{DG} := \frac{\mathbf{AN}-1}{2\cdot \mathbf{AN}-1} \quad \mathbf{EH} := \mathbf{DG} \quad \mathbf{FJ} := \frac{\mathbf{AB}\cdot \mathbf{EH}}{\mathbf{AB}+\mathbf{EH}}$$

$$\mathbf{GJ} := \mathbf{AB} - \mathbf{FJ} \qquad \mathbf{AR} := \mathbf{GJ} \qquad \mathbf{AR} - \frac{2\mathbf{AN}-1}{3\mathbf{AN}-2} = 0$$

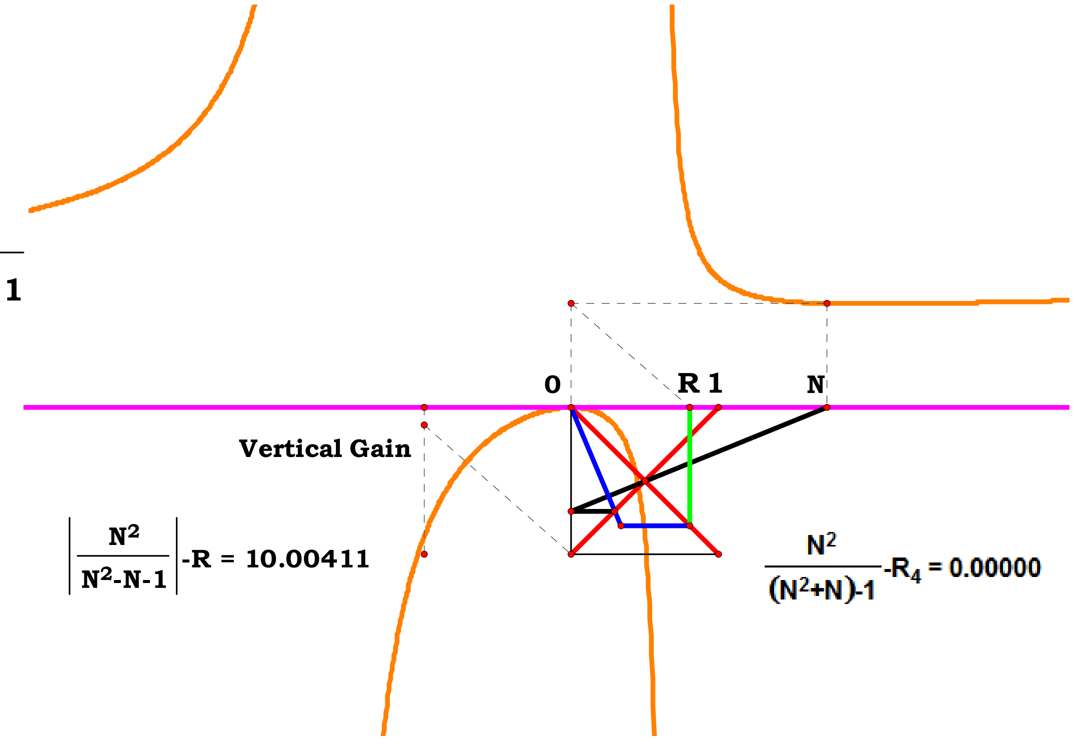




$$AB := 1$$

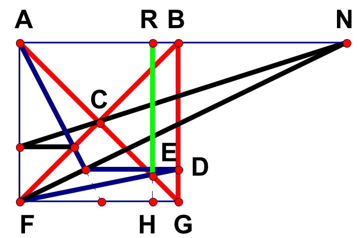
$$AN := 3$$

$$HJ := \frac{AN^2 - AN}{AN^2 + AN - 1}$$



$$FJ := \frac{AB \cdot HJ}{AN} \qquad BR := FJ \qquad AR := AB - BR \qquad AR - \frac{AN^2}{AN^2 + AN - 1} = 0$$

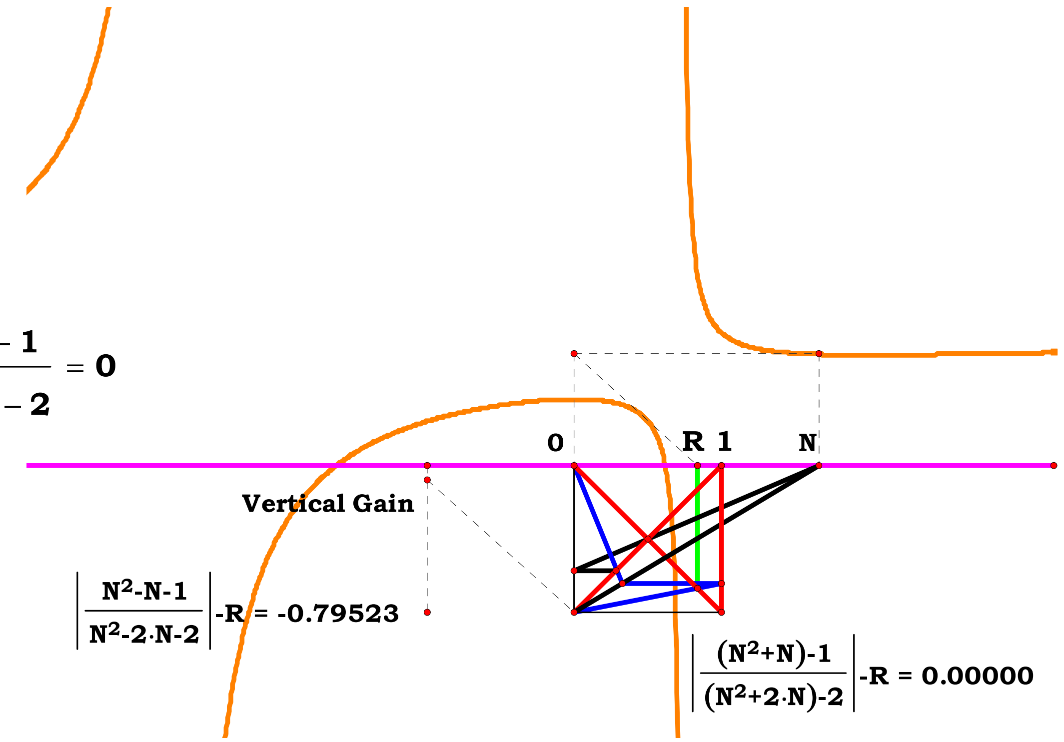
$$FJ - \frac{AN - 1}{AN^2 + AN - 1} = 0$$

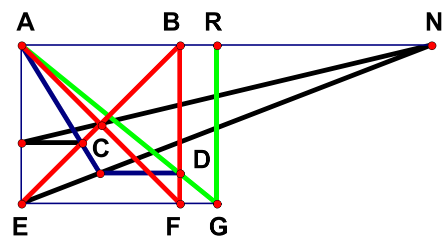


$$AB := 1$$

$$AN := 3$$

$$DG := \frac{AN - 1}{AN^2 + AN - 1} \qquad EH := \frac{AB \cdot DG}{AB + DG} \qquad AR := AB - EH \qquad AR - \frac{AN^2 + AN - 1}{AN^2 + 2AN - 2} = 0$$



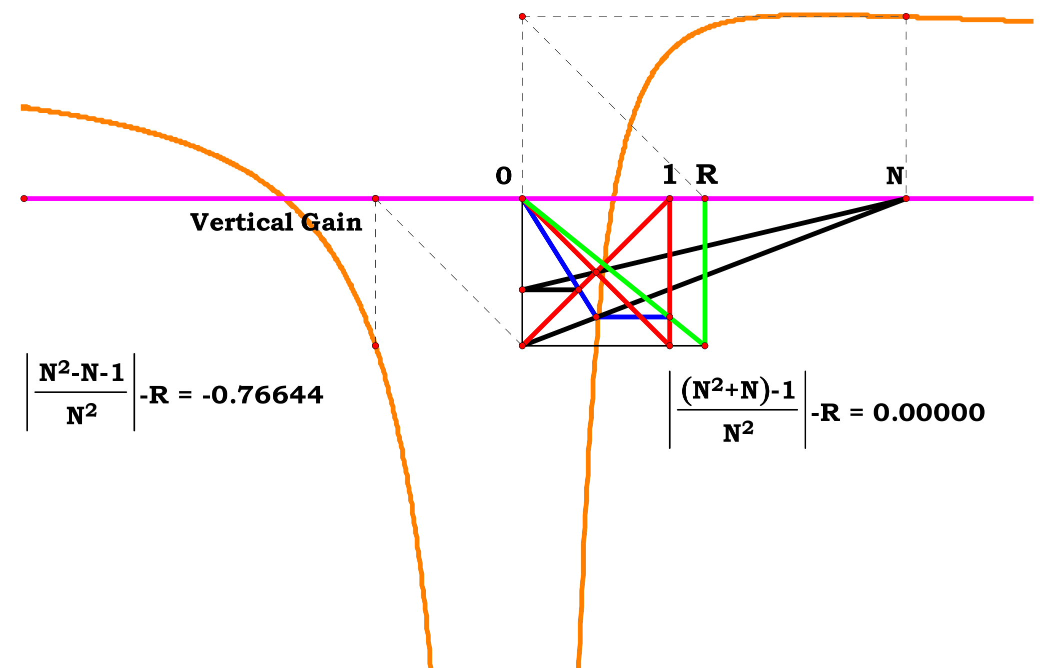


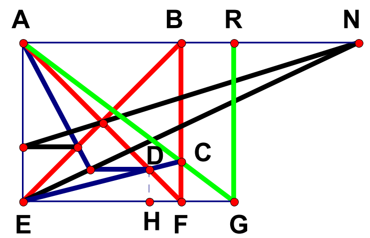
$$AB := 1$$

$$AN := 3$$

$$DF := \frac{AN - 1}{AN^2 + AN - 1}$$

$$BD := AB - DF \qquad EG := \frac{AB^2}{BD} \qquad AR := EG \qquad AR - \frac{AN^2 + AN - 1}{AN^2} = 0$$





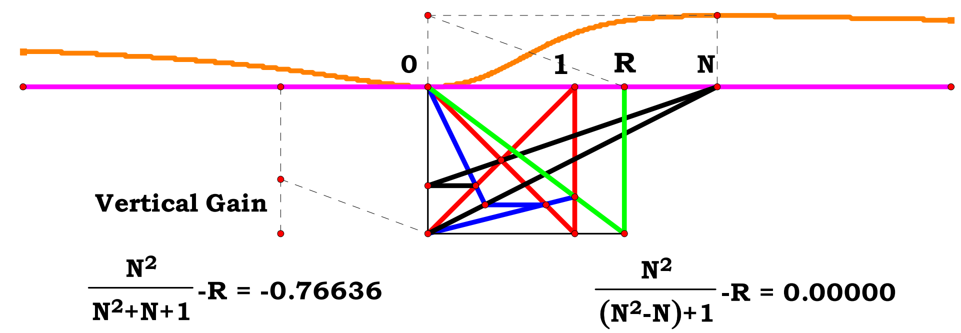
$$AB := 1$$

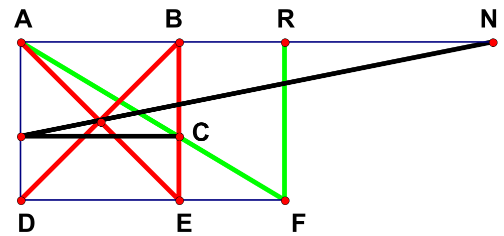
$$AN := 3$$

$$DH := \frac{AN - 1}{AN^2 + AN - 1}$$

$$EH := AB - DH \qquad CF := \frac{DH \cdot AB}{EH} \qquad BC := AB - CF \qquad EG := \frac{AB^2}{BC} \qquad AR := EG$$

$$AR - \frac{AN^2}{AN^2 - AN + 1} = 0$$

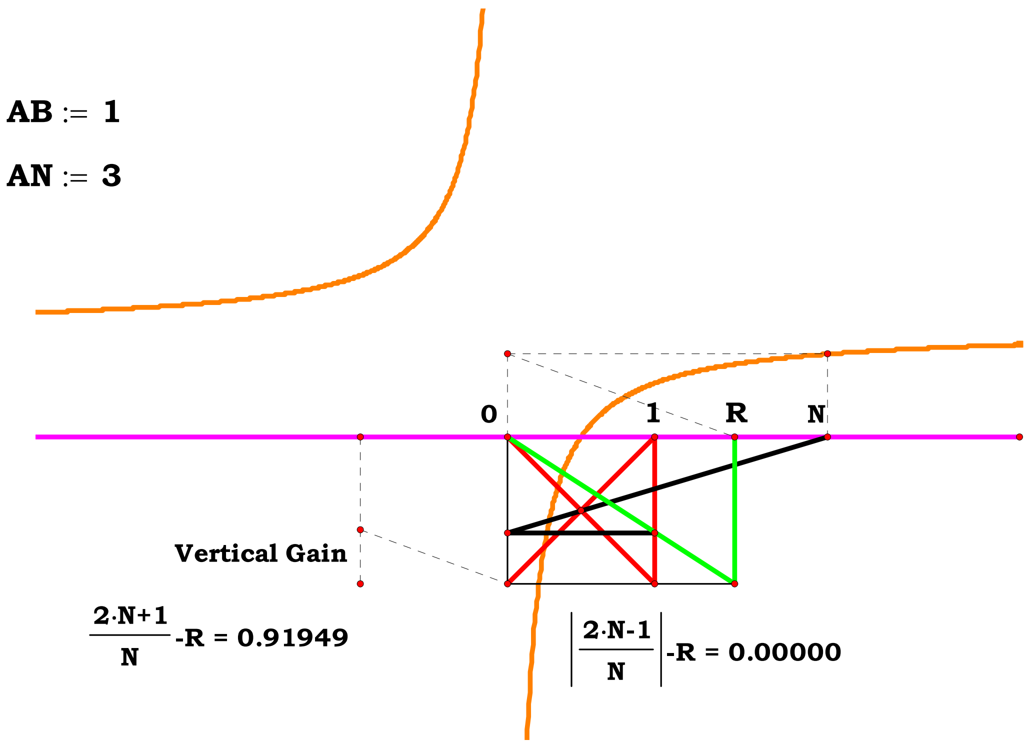


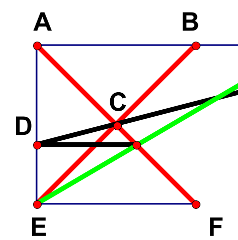


$AB := 1$

$AN := 3$

$$BC := \frac{AN}{2 \cdot AN - 1} \qquad DF := \frac{AB^2}{BC} \qquad AR := DF \qquad AR - \frac{2AN - 1}{AN} = 0$$

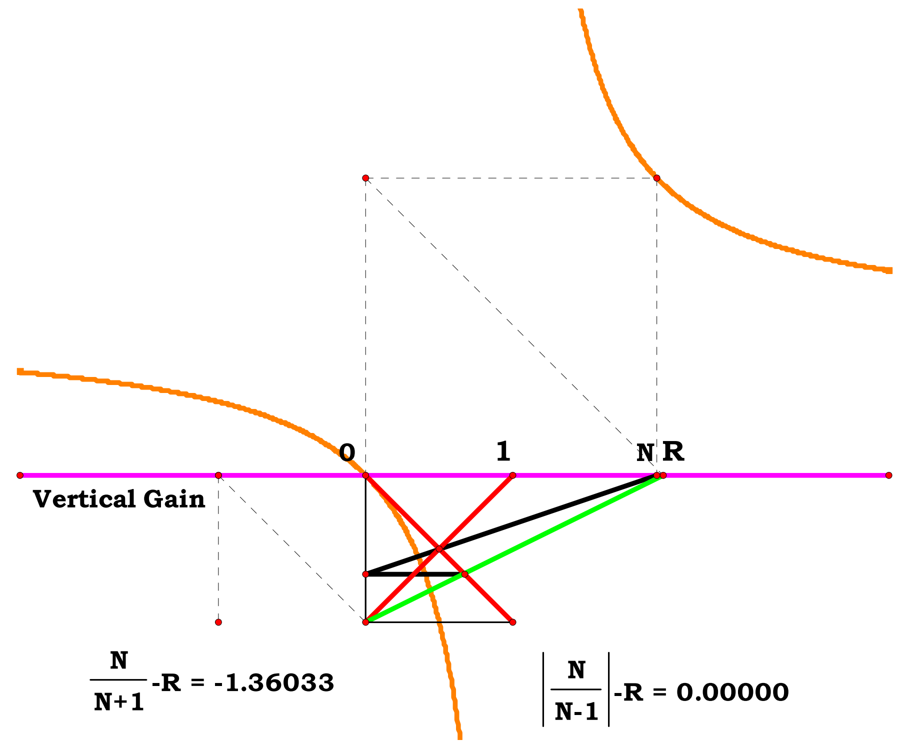


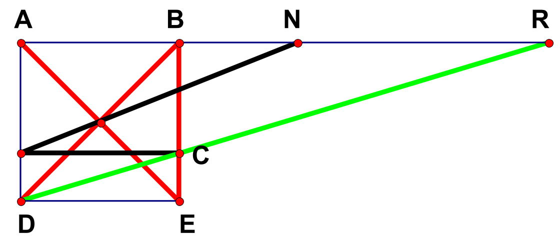


$$AB := 1$$

$$AN := 3$$

$$AD := \frac{AN}{2 \cdot AN - 1} \qquad DE := AB - AD \qquad AR := \frac{AD \cdot AB}{DE} \qquad AR - \frac{AN}{AN - 1} = 0$$

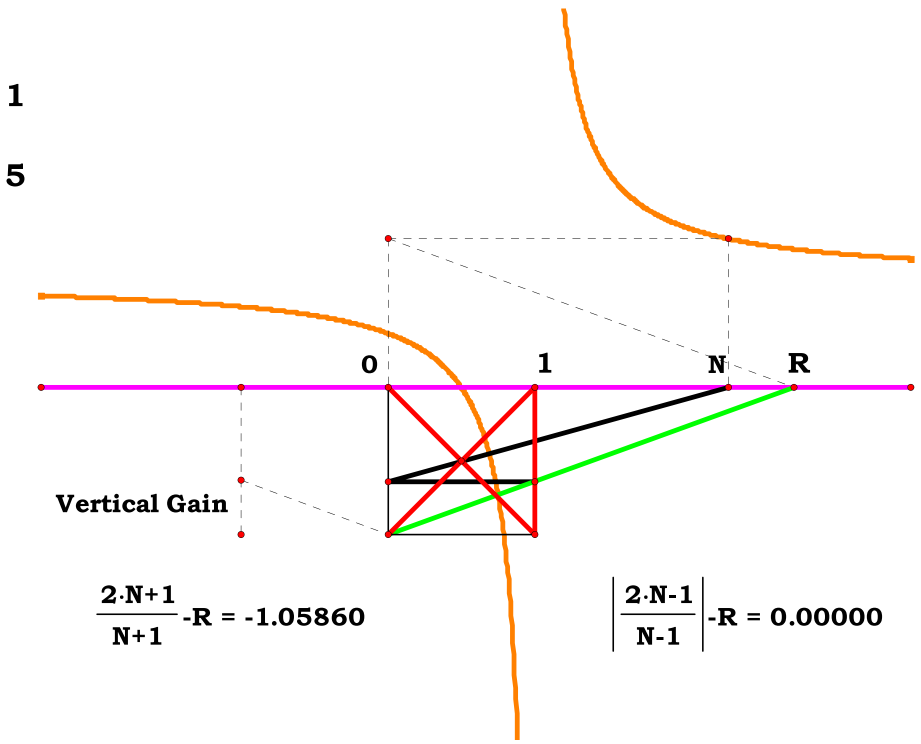


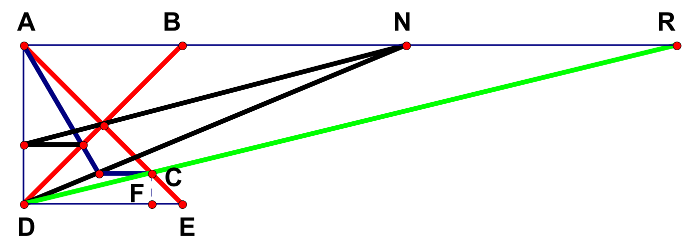


$AB := 1$

$AN := 5$

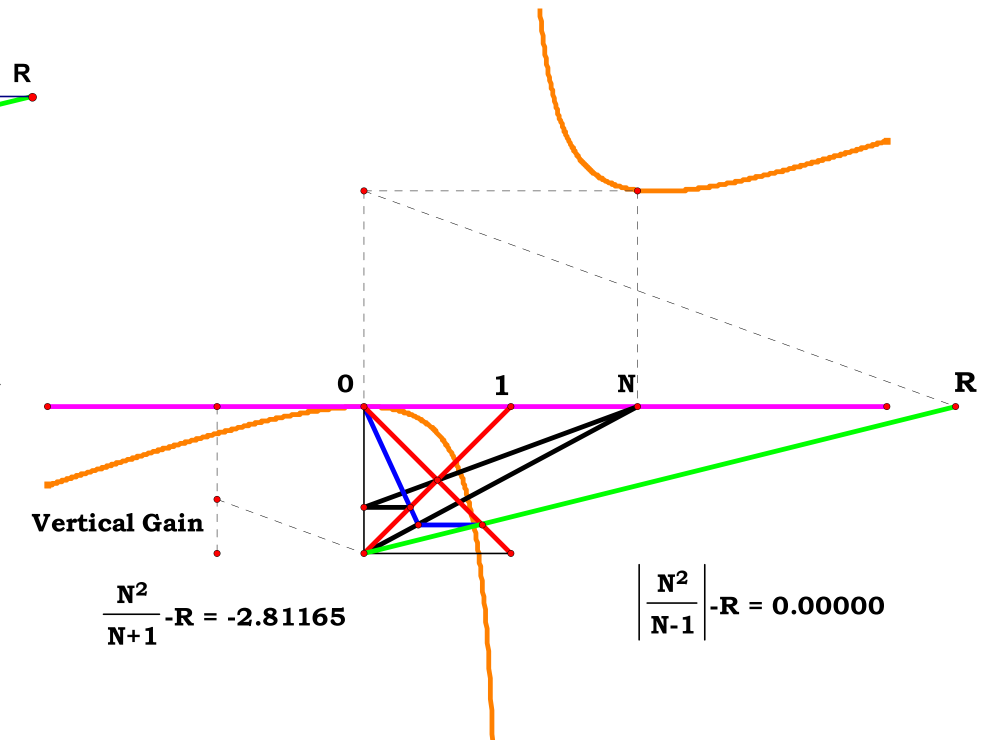
$$BC := \frac{AN}{2 \cdot AN - 1} \qquad CE := AB - BC \qquad AR := \frac{AB^2}{CE} \qquad AR - \frac{2AN - 1}{AN - 1} = 0$$

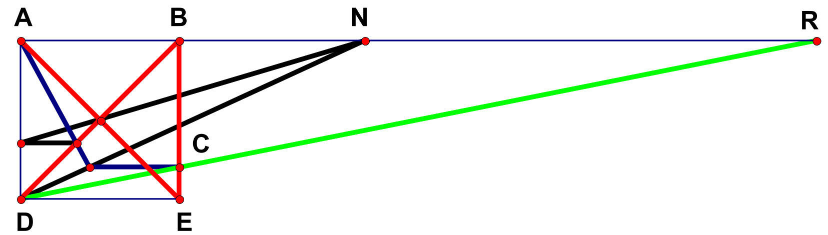




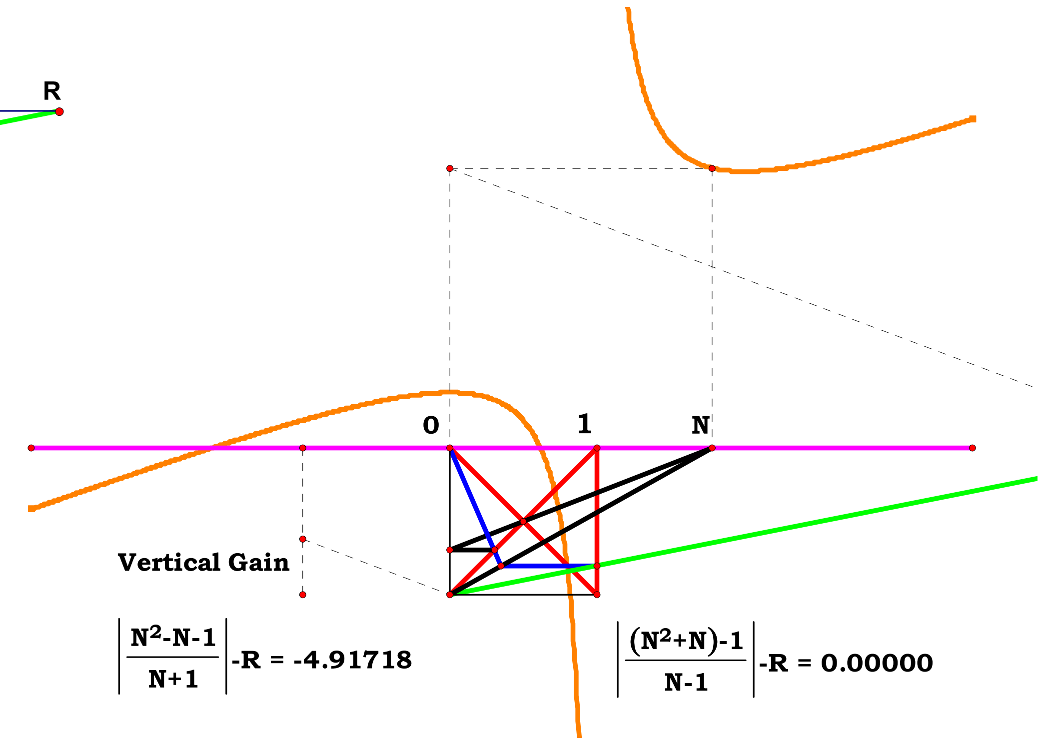
$\mathbf{AB} := 1 \quad \mathbf{AN} := 3 \quad \mathbf{CF} := \frac{\mathbf{AN}-1}{\mathbf{AN}^2+\mathbf{AN}-1} \quad \mathbf{DF} := \mathbf{AB}-\mathbf{CF} \quad \mathbf{AR} := \frac{\mathbf{DF}\cdot\mathbf{AB}}{\mathbf{CF}}$

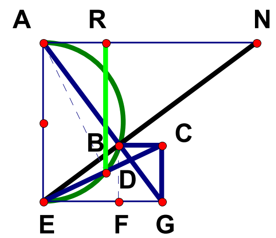
$\mathbf{AR} - \frac{\mathbf{AN}^2}{\mathbf{AN}-1} = 0$





$$\mathbf{AB} := 1 \qquad \mathbf{AN} := 3 \qquad \mathbf{CE} := \frac{\mathbf{AN} - 1}{\mathbf{AN}^2 + \mathbf{AN} - 1} \qquad \mathbf{AR} := \frac{\mathbf{AB}^2}{\mathbf{CE}} \qquad \mathbf{AR} - \frac{\mathbf{AN}^2 + \mathbf{AN} - 1}{\mathbf{AN} - 1} = 0$$

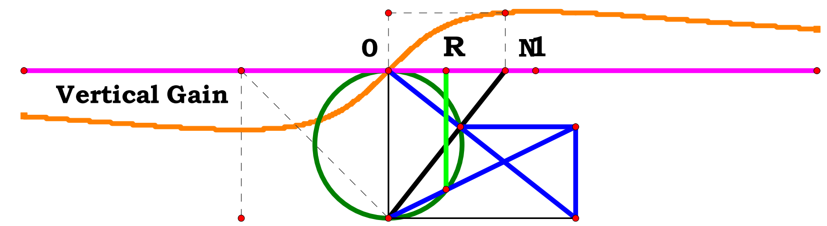




$$AE := 1$$

$$AN := 3$$

$$EN := \sqrt{AE^2 + AN^2}$$

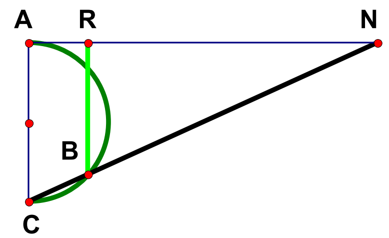


$$\frac{N^3+N}{N^4+3\cdot N^2+1} - R = 0.00000$$

$$BE := \frac{AE^2}{EN} \quad BF := \frac{AE \cdot BE}{EN} \quad EG := \frac{EN \cdot BE}{AN} \quad CG := BF \quad CE := \sqrt{EG^2 + CG^2}$$

$$DE := \frac{CG \cdot AE}{CE} \quad AR := \frac{EG \cdot DE}{CE} \quad AR - \frac{AN^3 + AN}{AN^4 + 3 \cdot AN^2 + 1} = 0$$

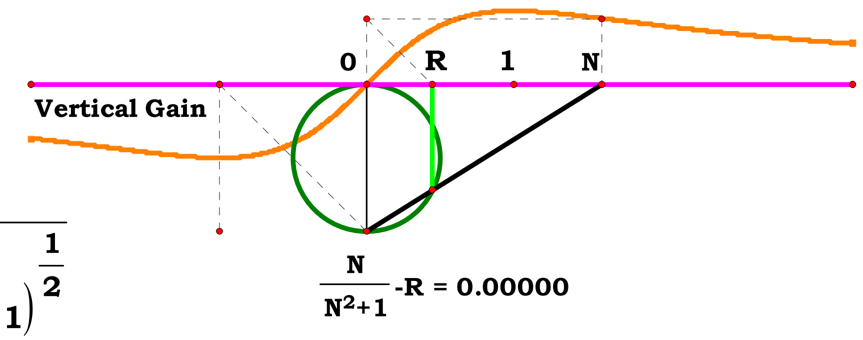
$$BE - \frac{1}{\left(AN^2 + 1\right)^{\frac{1}{2}}} = 0 \quad BF - \frac{1}{AN^2 + 1} = 0 \quad EG - \frac{1}{AN} = 0$$



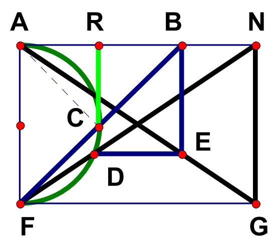
$$AC := 1$$

$$AN := 3$$

$$BC := \frac{1}{\left(AN^2 + 1\right)^{\frac{1}{2}}}$$



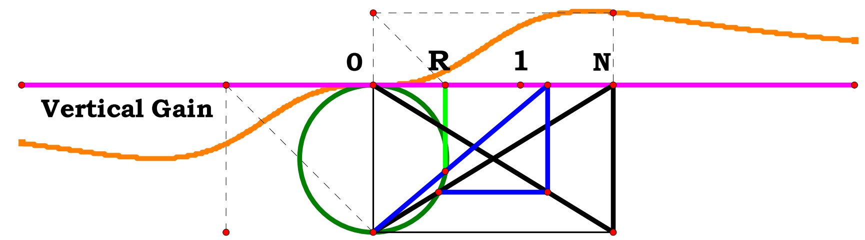
$$CN := \sqrt{AN^2 + AC^2} \qquad AR := \frac{AN \cdot BC}{CN} \qquad AR - \frac{AN}{AN^2 + 1} = 0$$



$AF := 1$

$AN := 3$

$FN := \sqrt{AN^2 + AF^2}$

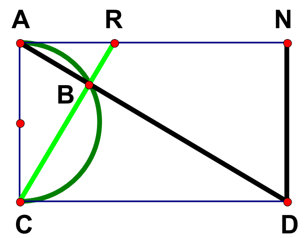


$$\frac{N^5+N^3}{N^6+N^4+2\cdot N^2+1}-R = 0.00000$$

$$DF := \frac{1}{\left(AN^2+1\right)^{\frac{1}{2}}} \qquad BN := \frac{AN \cdot DF}{FN} \qquad AB := AN - BN \qquad CF := \frac{1}{\left(AB^2+1\right)^{\frac{1}{2}}}$$

$$AC := \sqrt{AF^2 - CF^2} \qquad AR := \frac{CF \cdot AC}{AF} \qquad AR - \frac{AN^5 + AN^3}{AN^6 + AN^4 + 2AN^2 + 1} = 0$$

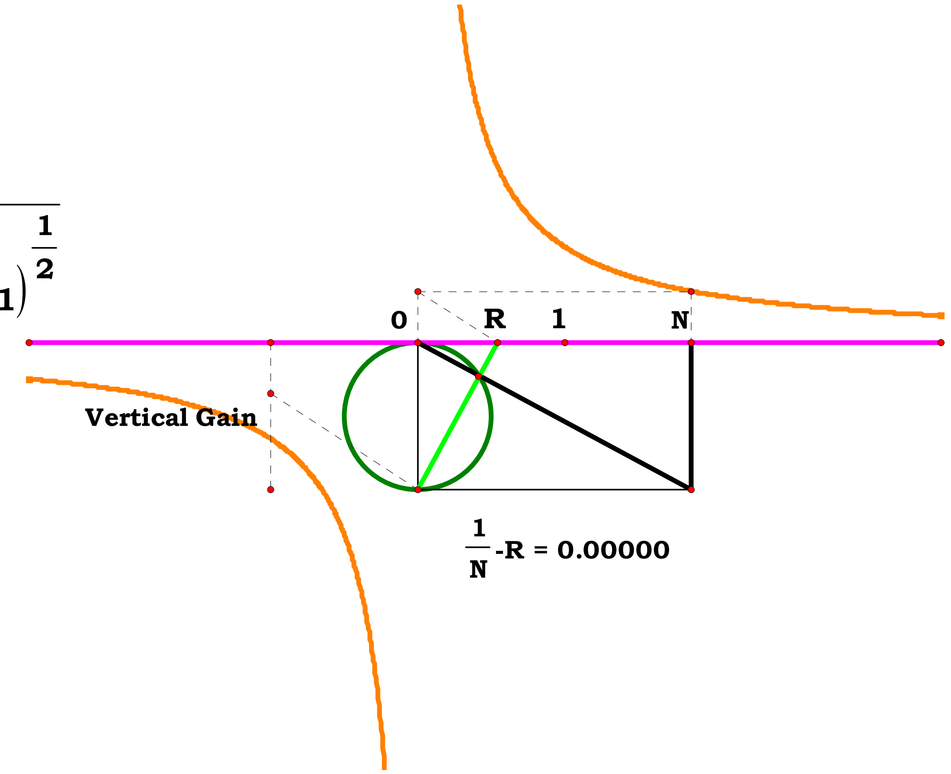
$$CF - \frac{AN^2 + 1}{\left(AN^6 + AN^4 + 2 \cdot AN^2 + 1\right)^{\frac{1}{2}}} = 0$$



$$AC := 1$$

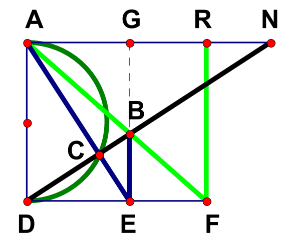
$$AN := 3$$

$$AB := \frac{1}{\left(AN^2 + 1\right)^{\frac{1}{2}}}$$



$$\frac{1}{N}\text{-}R = 0.00000$$

$$AD := \sqrt{AN^2 + AC^2} \qquad AR := \frac{AD \cdot AB}{AN} \qquad AR - \frac{1}{AN} = 0$$



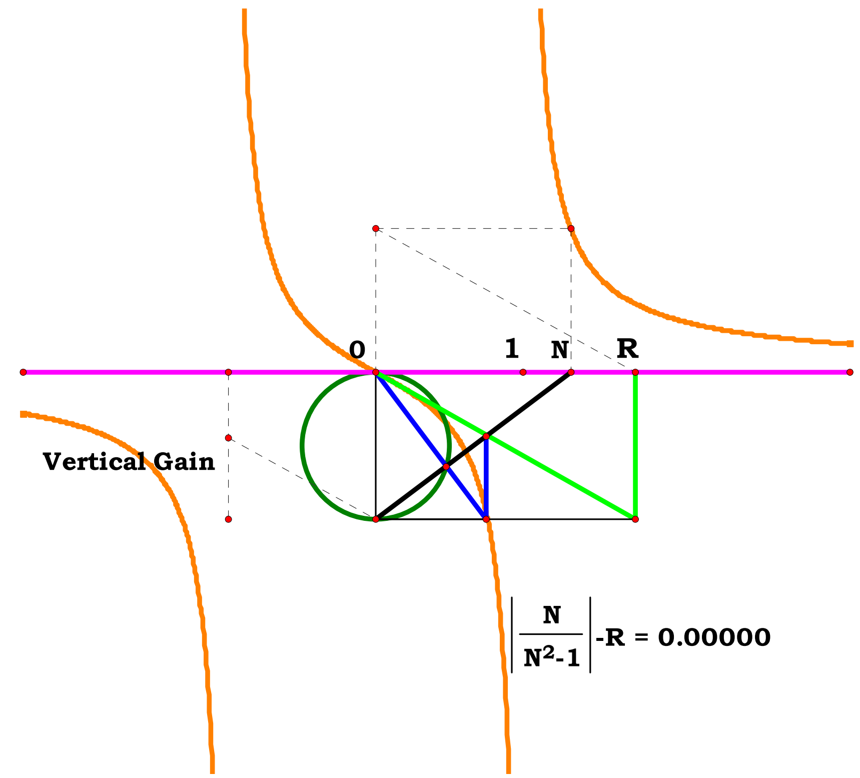
$$AD := 1$$

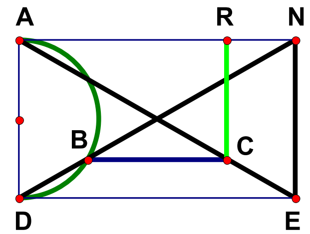
$$AN := 3$$

$$DE := \frac{1}{AN}$$

$$BE := \frac{AD \cdot DE}{AN} \qquad BG := AD - BE \qquad DF := \frac{DE \cdot AD}{BG} \qquad AR := DF$$

$$AR - \frac{AN}{AN^2 - 1} = 0$$



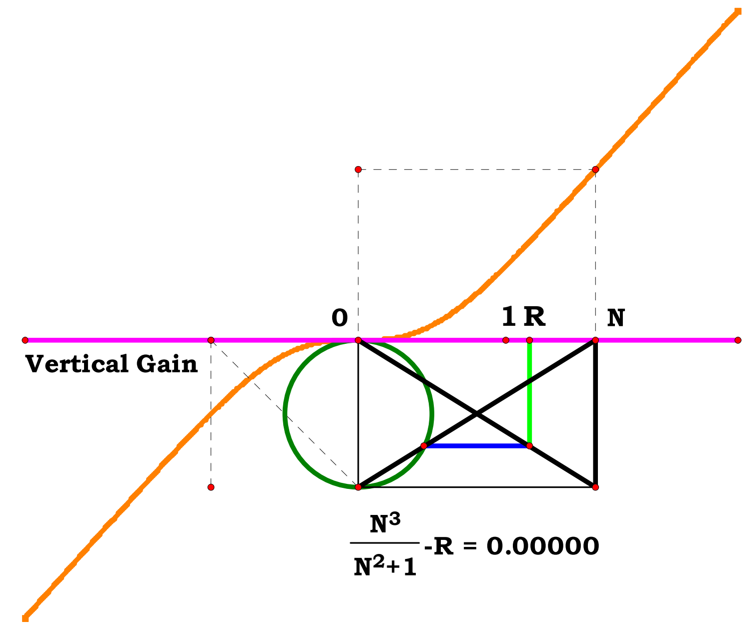


$$AD := 1$$

$$AN := 3$$

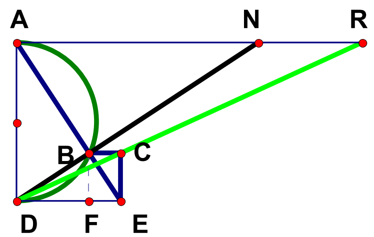
$$DN := \sqrt{AN^2 + AD^2}$$

$$BD := \frac{1}{\left(AN^2 + 1\right)^{\frac{1}{2}}} \qquad NR := \frac{AN \cdot BD}{DN} \qquad AR := AN - NR \qquad AR - \frac{AN^3}{AN^2 + 1} = 0$$



$$\frac{N^3}{N^2+1} - R = 0.00000$$

$$NR - \frac{AN}{AN^2 + 1} = 0$$

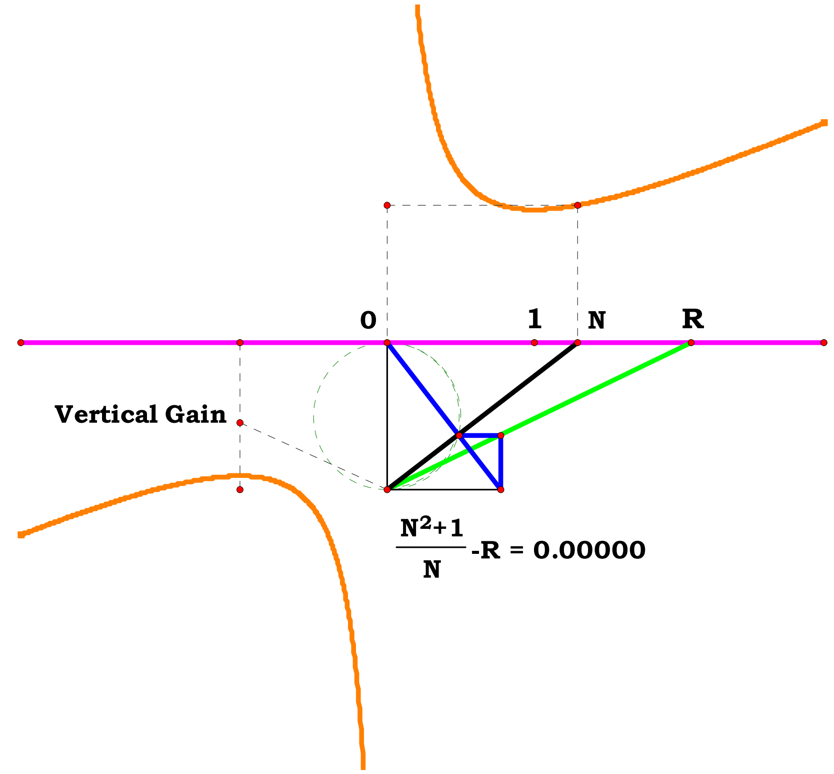


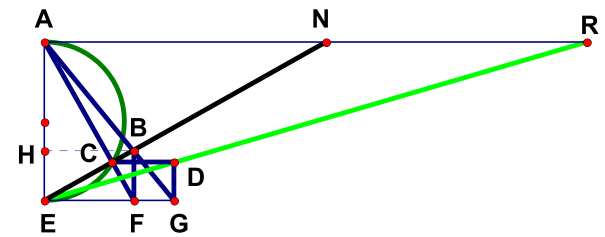
$$AD := 1$$

$$AN := 3$$

$$BF := \frac{1}{1 + AN^2}$$

$$CE := BF \quad DE := \frac{1}{AN} \quad AR := \frac{DE \cdot AD}{CE} \quad AR - \frac{AN^2 + 1}{AN} = 0$$





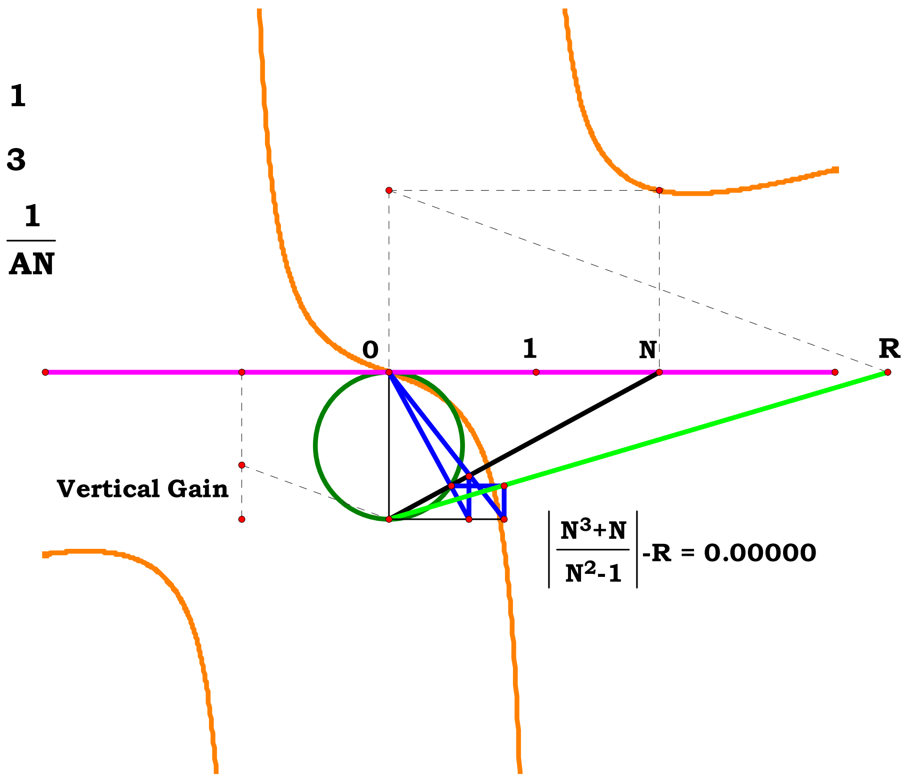
$$AE := 1$$

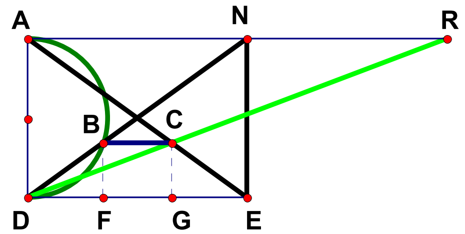
$$AN := 3$$

$$EF := \frac{1}{AN}$$

$$BF := \frac{AE \cdot EF}{AN} \qquad AH := AE - BF \qquad EG := \frac{EF \cdot AE}{AH} \qquad DG := \frac{1}{1 + AN^2} \qquad AR := \frac{EG \cdot AE}{DG}$$

$$AR - \frac{AN^3 + AN}{AN^2 - 1} = 0$$



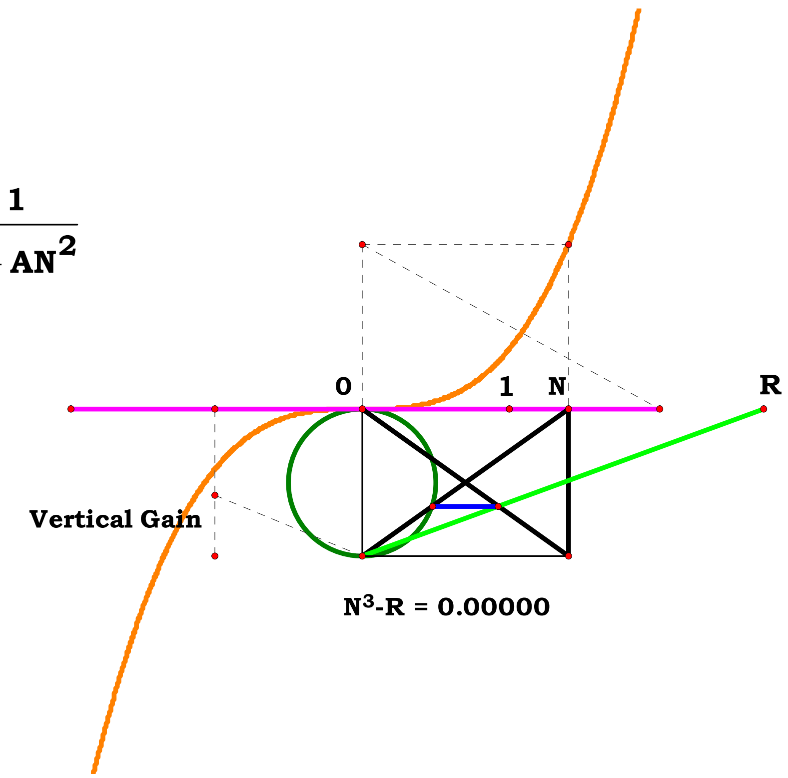


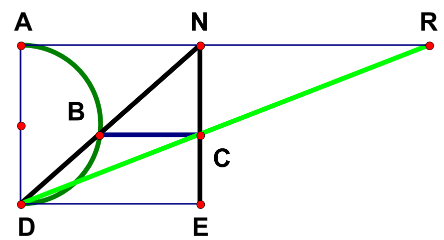
$$AD := 1$$

$$AN := 3$$

$$BF := \frac{1}{1 + AN^2}$$

$$DF := \frac{AN}{AN^2 + 1} \qquad DG := AN - DF \qquad AR := \frac{DG \cdot AD}{BF} \qquad AR - AN^3 = 0$$

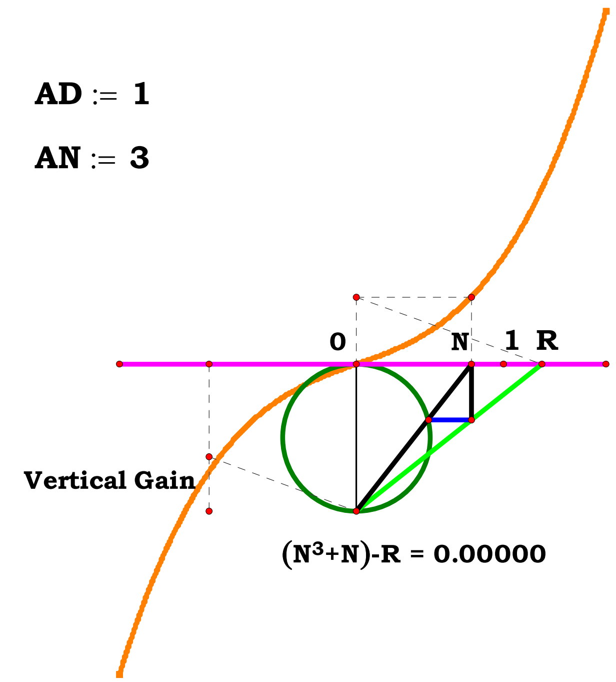


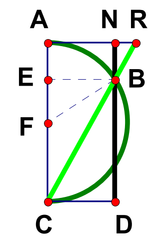


$\mathbf{AD} := \mathbf{1}$

$\mathbf{AN} := \mathbf{3}$

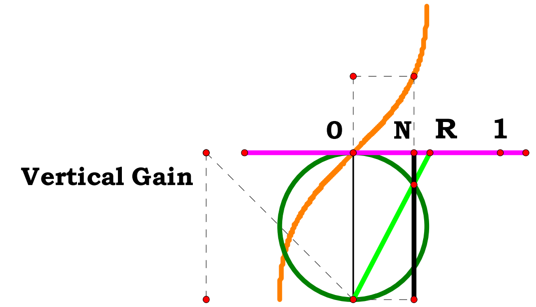
$$\mathbf{CE} := \frac{\mathbf{1}}{\mathbf{AN}^{\mathbf{2}} + \mathbf{1}} \qquad \mathbf{AR} := \frac{\mathbf{AN} \cdot \mathbf{AD}}{\mathbf{CE}} \qquad \mathbf{AR} - \left(\mathbf{AN}^{\mathbf{3}} + \mathbf{AN}\right) = \mathbf{0}$$





$$AC := 1$$

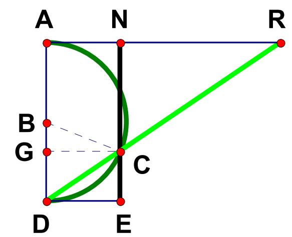
$$AN := .3$$



$$BF := \frac{AC}{2} \quad BE := AN \quad EF := \sqrt{BF^2 - BE^2} \quad BD := \frac{AC}{2} + EF \quad AR := \frac{AN \cdot AC}{BD}$$

$$\frac{2 \cdot N}{\sqrt{1 - 4 \cdot N^2} + 1} - R = 0.00000$$

$$AR - \frac{2AN}{\left(1 - 4 \cdot AN^2\right)^{\frac{1}{2}} + 1} = 0$$

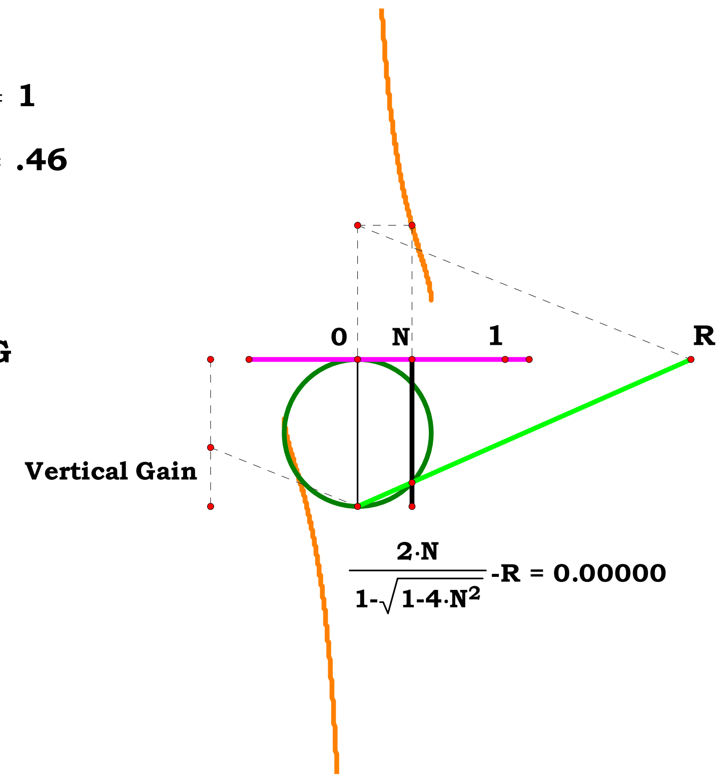


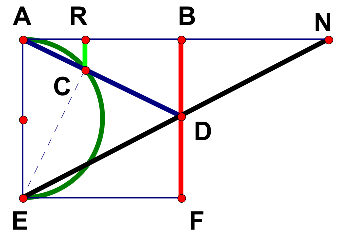
$AD := 1$

$AN := .46$

$BC := \dfrac{AD}{2} \qquad CG := AN \qquad BG := \sqrt{BC^2 - CG^2} \qquad CE := \dfrac{AD}{2} - BG$

$AR := \dfrac{AN \cdot AD}{CE} \qquad AR - \dfrac{2AN}{1 - \sqrt{1 - 4AN^2}} = 0$

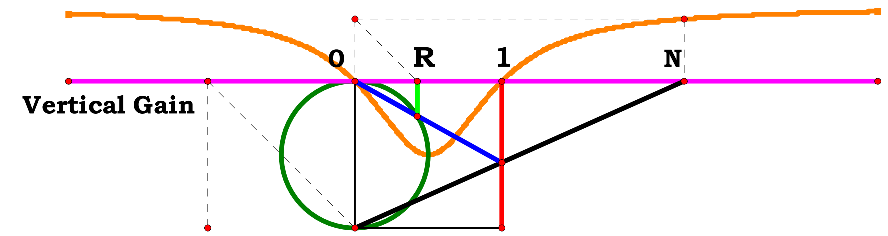




$$AB := 1$$

$$AN := 2$$

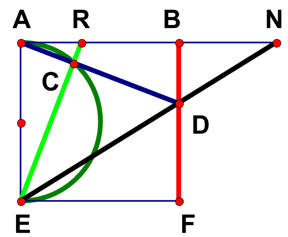
$$BD := \frac{AN - 1}{AN}$$



$$\frac{N^2+N}{N^2+2\cdot N+1}-R = 0.04441 \qquad \left|\frac{N^2-N}{(2\cdot N^2-2\cdot N)+1}\right|-R = 0.00000$$

$$AD := \sqrt{AB^2 + BD^2} \qquad AC := \frac{BD \cdot AB}{AD} \qquad AR := \frac{AB \cdot AC}{AD} \qquad AR - \frac{AN^2 - AN}{2 \cdot AN^2 - 2 \cdot AN + 1} = 0$$

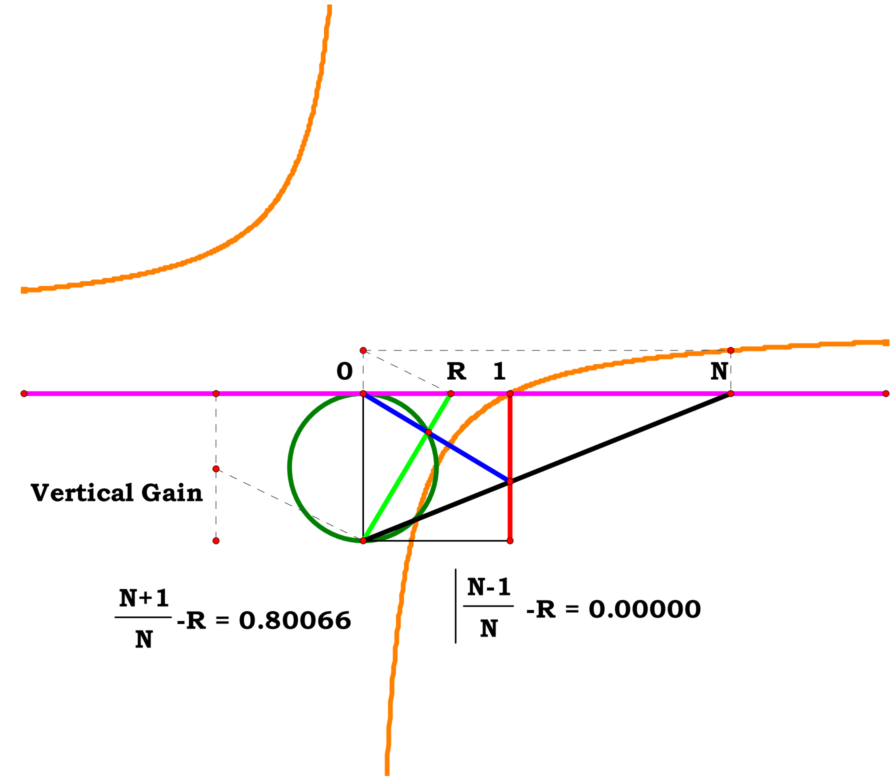
$$AC - \frac{AN - 1}{\left(2 \cdot AN^2 - 2 \cdot AN + 1\right)^{\frac{1}{2}}} = 0$$



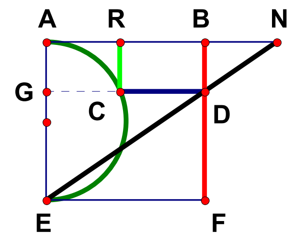
$AB := 1$

$AN := 3$

$BD := \dfrac{AN - 1}{AN}$ $\quad AD := \sqrt{AB^2 + BD^2}$ $\quad AC := \dfrac{BD \cdot AB}{AD}$ $\quad AR := \dfrac{AD \cdot AC}{AB}$



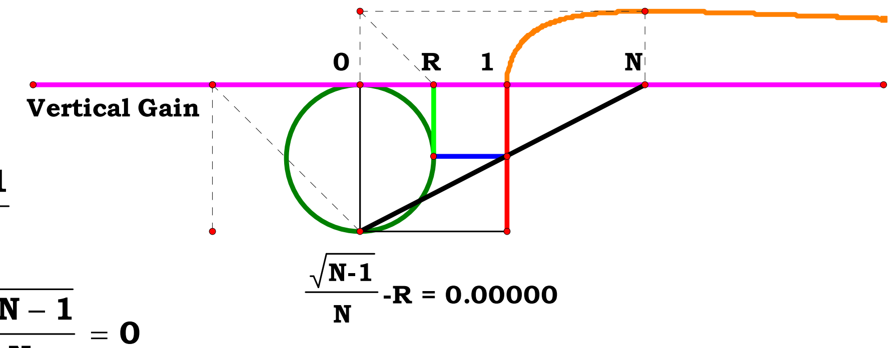
$$AR - \frac{AN - 1}{AN} = 0$$



$$AB := 1$$

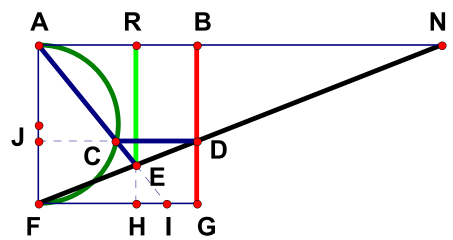
$$AN := 2$$

$$BD := \frac{AN - 1}{AN}$$



$AG := BD \quad GE := AB - BD \quad CG := \sqrt{AG \cdot GE} \quad AR := CG \quad AR - \frac{\sqrt{AN-1}}{AN} = 0$

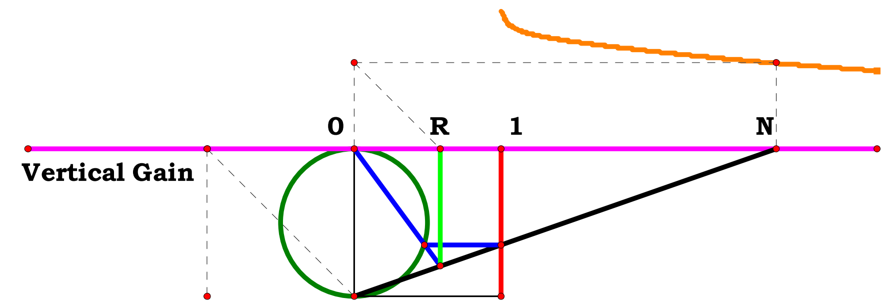
$CD := AB - CG \quad CD - \frac{AN - (AN-1)^{\frac{1}{2}}}{AN} = 0$



$$AB := 1$$

$$AN := 3$$

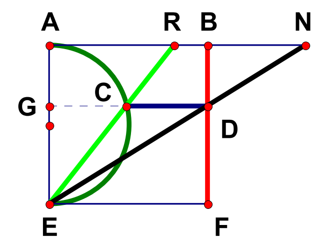
$$BD := \frac{AN - 1}{AN}$$



$$\frac{N}{N \cdot \sqrt{N-1} + 1} - R = 0.00000$$

$$AJ := BD \qquad JF := AB - BD \qquad CJ := \sqrt{AJ \cdot JF} \qquad FI := \frac{CJ \cdot AB}{AJ} \qquad FH := \frac{FI \cdot AN}{FI + AN}$$

$$AR := FH \qquad AR - \frac{AN}{AN \cdot \sqrt{AN - 1} + 1} = 0$$

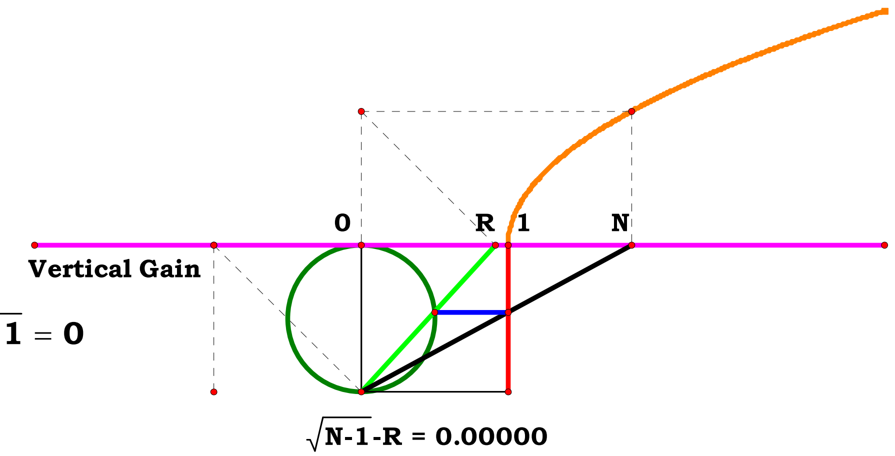


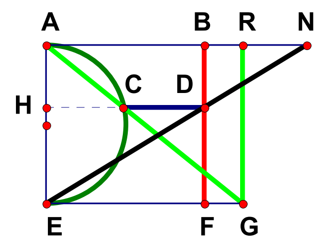
$$\mathbf{AB} := 1$$

$$\mathbf{AN} := 3$$

$$\mathbf{BD} := \frac{\mathbf{AN} - 1}{\mathbf{AN}}$$

$$\mathbf{AG} := \mathbf{BD} \qquad \mathbf{GE} := \mathbf{AB} - \mathbf{BD} \qquad \mathbf{CG} := \sqrt{\mathbf{AG} \cdot \mathbf{GE}} \qquad \mathbf{AR} := \frac{\mathbf{CG} \cdot \mathbf{AB}}{\mathbf{GE}} \qquad \mathbf{AR} - \sqrt{\mathbf{AN} - 1} = 0$$





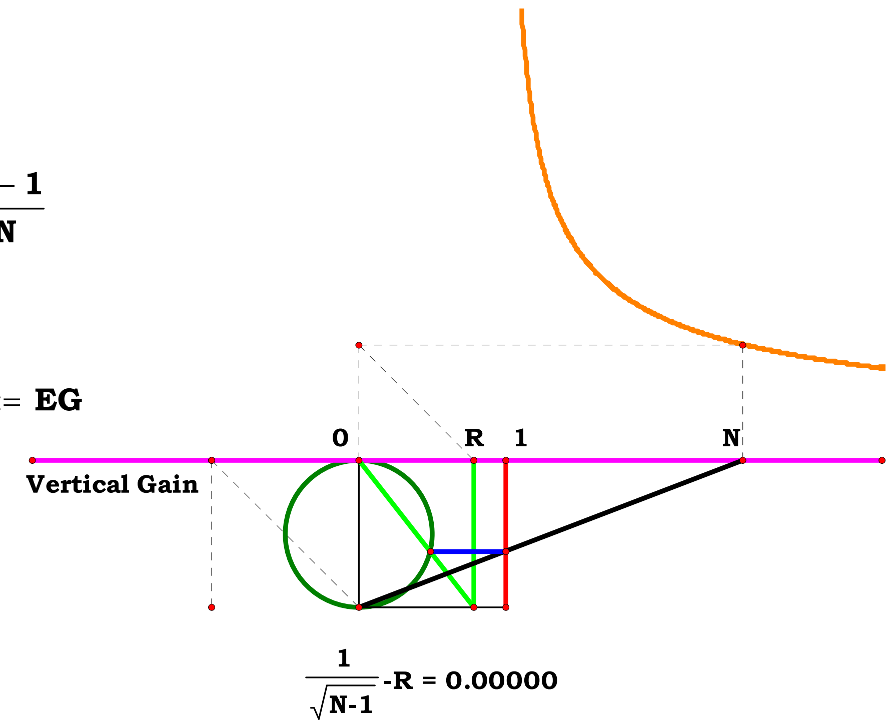
$$AB := 1$$

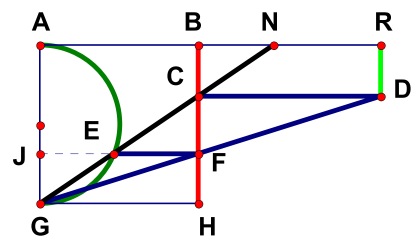
$$AN := 3$$

$$BD := \frac{AN - 1}{AN}$$

$$AH := BD \quad HE := AB - BD \quad CH := \sqrt{AH \cdot HE} \quad EG := \frac{CH \cdot AB}{BD} \quad AR := EG$$

$$AR - \frac{1}{\sqrt{AN - 1}} = 0$$

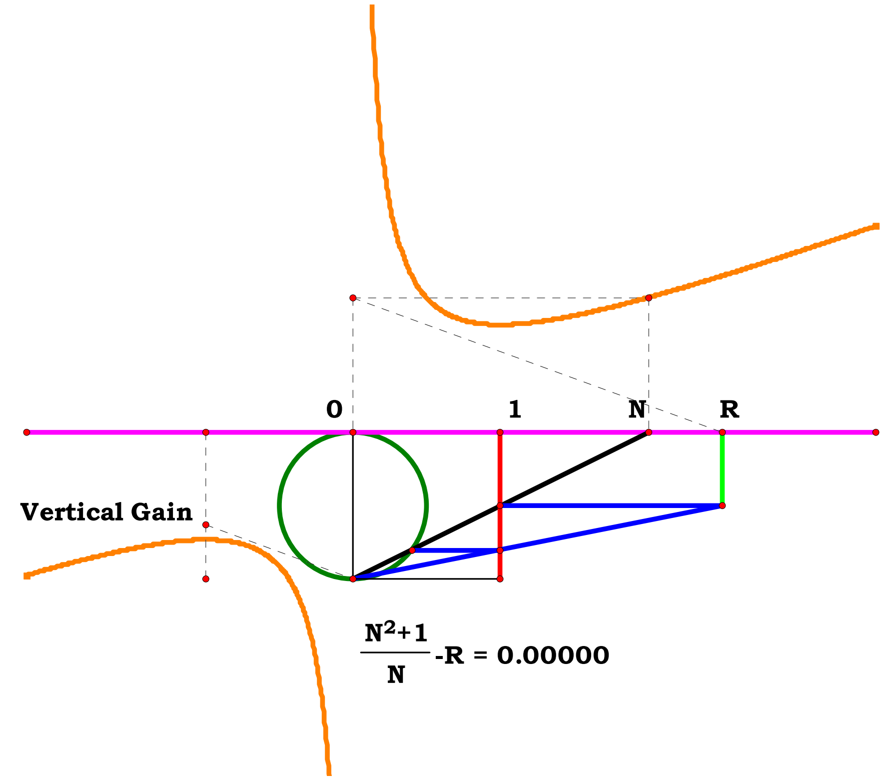




$$\mathbf{AB} := 1$$

$$\mathbf{AN} := 3$$

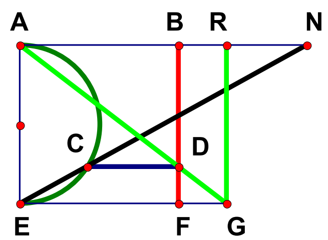
$$\mathbf{BC} := \frac{\mathbf{AN} - 1}{\mathbf{AN}}$$



$$\mathbf{FH} := \frac{1}{\mathbf{AN}^2 + 1} \qquad \mathbf{JG} := \mathbf{FH} \qquad \mathbf{AJ} := \mathbf{AB} - \mathbf{JG} \qquad \mathbf{EJ} := \sqrt{\mathbf{AJ} \cdot \mathbf{JG}} \qquad \mathbf{EF} := \mathbf{AB} - \mathbf{EJ}$$

$$\mathbf{CH} := \mathbf{AB} - \mathbf{BC} \qquad \mathbf{CD} := \frac{\mathbf{EF} \cdot \mathbf{CH}}{\mathbf{FH}} \qquad \mathbf{AR} := \mathbf{AB} + \mathbf{CD} \qquad \mathbf{AR} - \frac{\mathbf{AN}^2 + 1}{\mathbf{AN}} = 0$$

$$\mathbf{EF} - \frac{\mathbf{AN}^2 - \mathbf{AN} + 1}{\mathbf{AN}^2 + 1} = 0$$

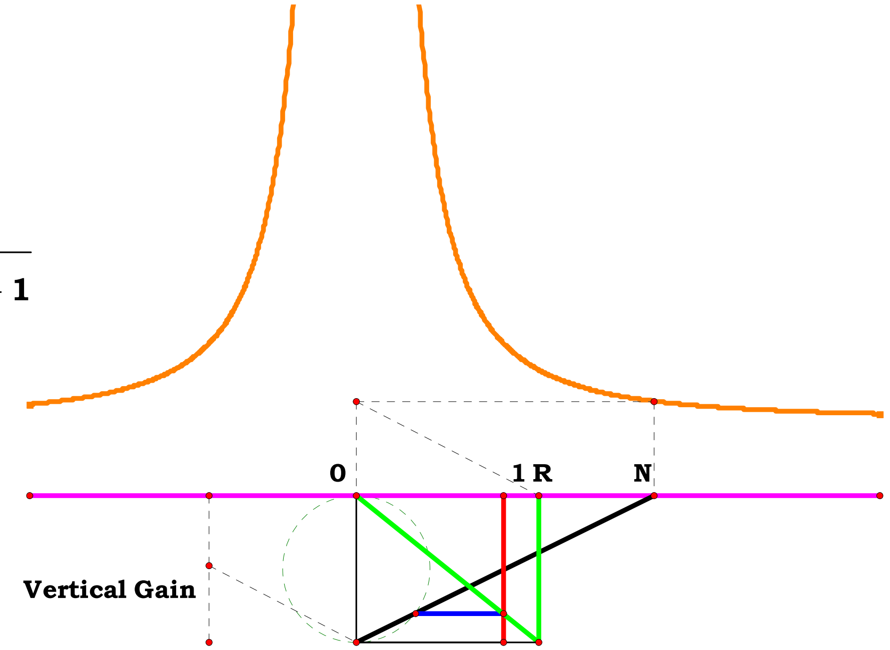


$AB := 1$

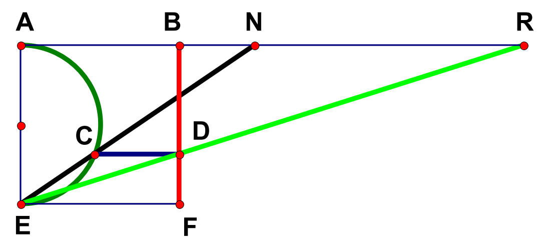
$AN := 3$

$DF := \dfrac{1}{AN^2 + 1}$

$BD := AB - DF \quad EG := \dfrac{AB^2}{BD} \quad AR := EG \quad AR - \dfrac{AN^2 + 1}{AN^2} = 0$

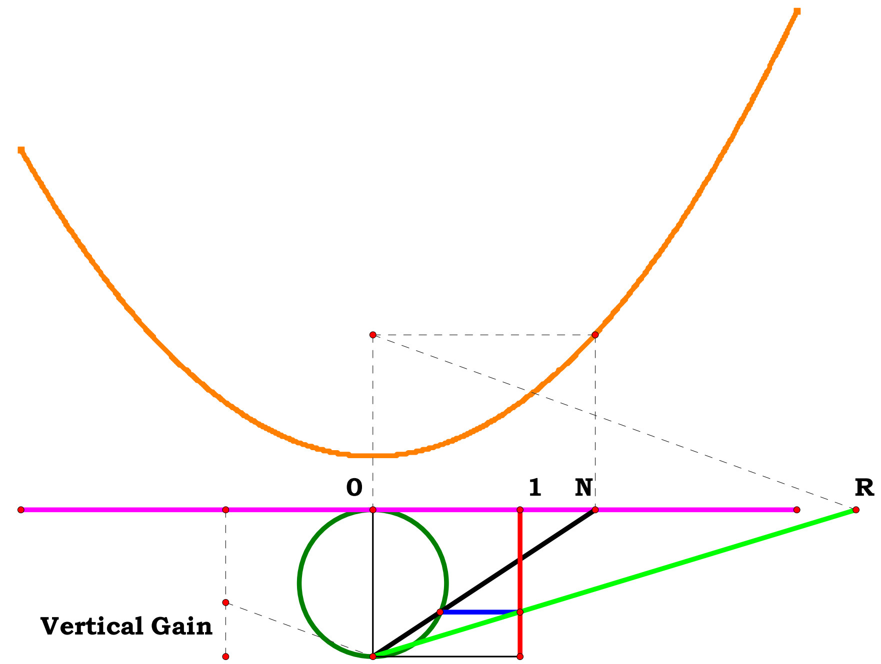


$\dfrac{N^2+1}{N^2}$-R = 0.00000

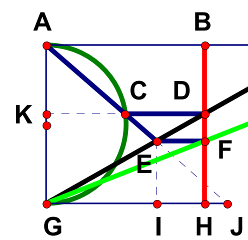


$$AB := 1 \qquad AN := 3 \qquad DF := \frac{1}{AN^2 + 1} \qquad CD := \frac{AN^2 - AN + 1}{AN^2 + 1}$$

$$NR := \frac{CD \cdot AB}{DF} \qquad AR := AN + NR \qquad AR - \left(AN^2 + 1\right) = 0$$



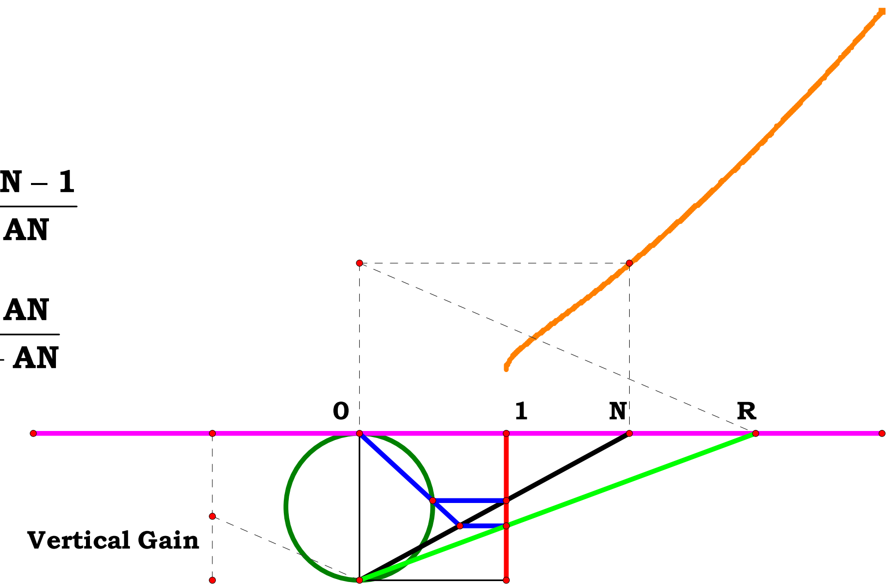
$(N^2+1)\text{-}R = 0.00000$



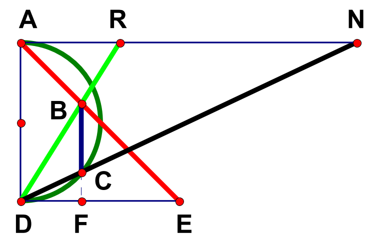
$$AB := 1$$
$$AN := 3$$
$$BD := \frac{AN - 1}{AN}$$

$$AK := BD \qquad KG := AB - AK \qquad CK := \sqrt{AK \cdot KG} \qquad GJ := \frac{CK \cdot AB}{AK} \qquad GI := \frac{GJ \cdot AN}{GJ + AN}$$

$$EI := \frac{AB \cdot GI}{AN} \qquad FH := EI \qquad AR := \frac{AB^2}{EI} \qquad AR - \left(AN \cdot \sqrt{AN - 1} + 1\right) = 0$$

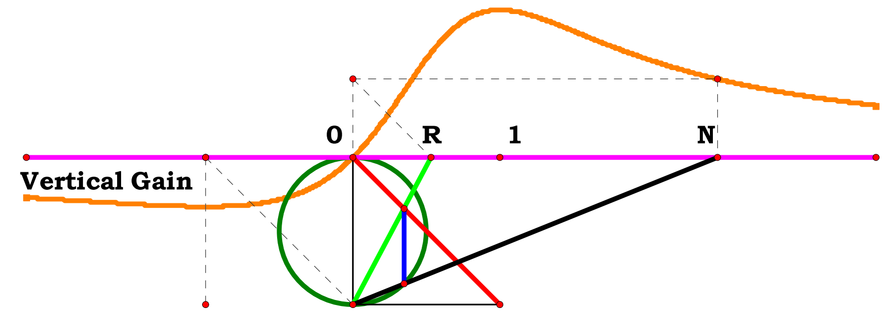


$(N \cdot \sqrt{N-1} + 1) - R = 0.00000$

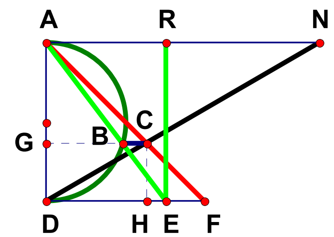


$AB := 1$

$AN := 3$



$DF := \dfrac{AN}{AN^2 + 1}$ $\quad EF := AB - DF$ $\quad AR := \dfrac{DF \cdot AB}{EF}$ $\quad AR - \dfrac{AN}{AN^2 - AN + 1} = 0$ $\quad \dfrac{N}{N^2+N+1} - R = -0.27283$ $\quad \dfrac{N}{(N^2-N)+1} - R = 0.00000$



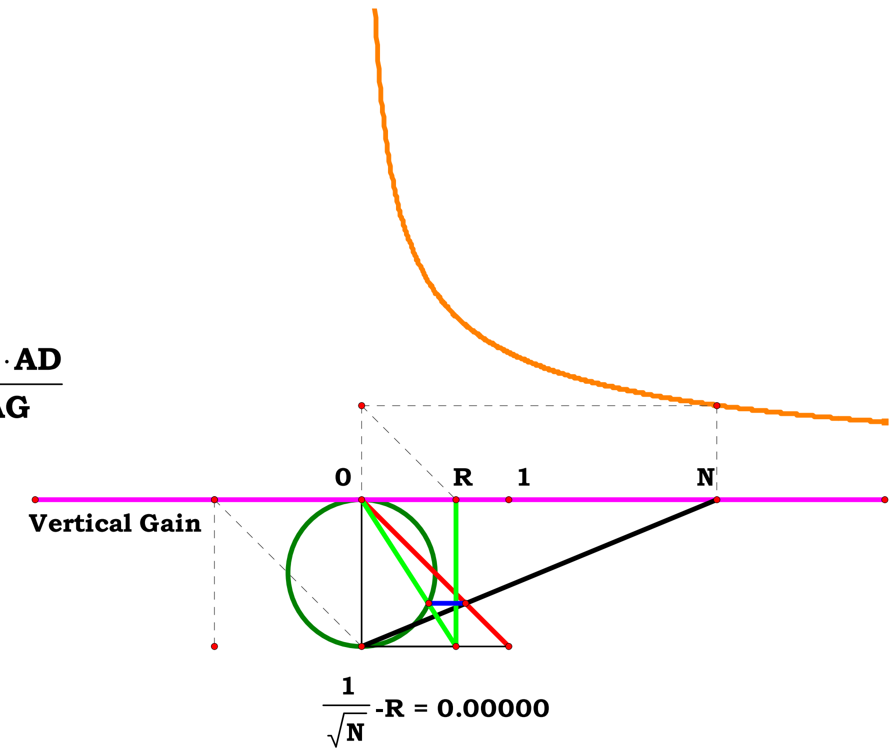
$$AD := 1$$

$$AN := 3$$

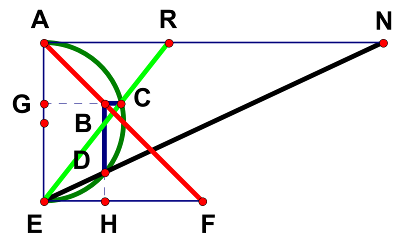
$$DH := \frac{AN}{AN + 1}$$

$$CH := \frac{AD \cdot DH}{AN} \qquad DG := CH \qquad AG := AD - DG \qquad BG := \sqrt{AG \cdot DG} \qquad DE := \frac{BG \cdot AD}{AG}$$

$$AR := DE \qquad AR - \frac{1}{\sqrt{AN}} = 0$$

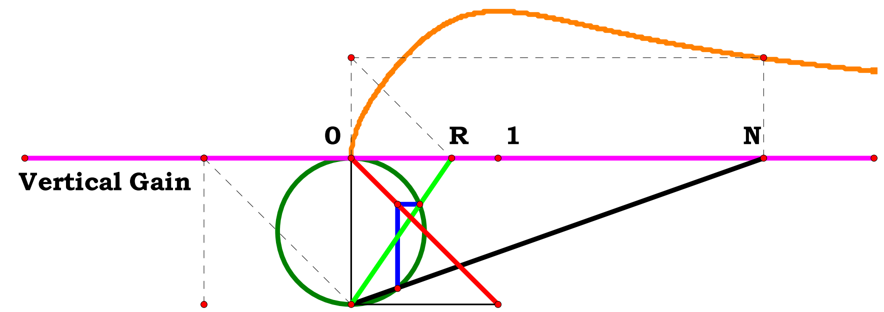


$$CH - \frac{1}{AN + 1} = 0$$



$AE := 1$

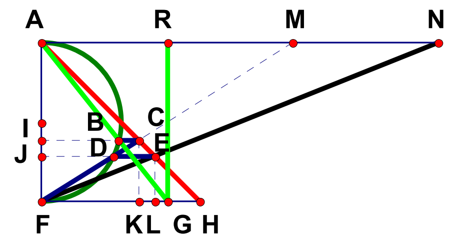
$AN := 3$



$$\sqrt{\frac{N}{(N^2 - N) + 1}} - R = 0.00000$$

$$EH := \frac{AN}{AN^2 + 1} \qquad AG := EH \qquad GE := AE - EH \qquad CG := \sqrt{AG \cdot GE}$$

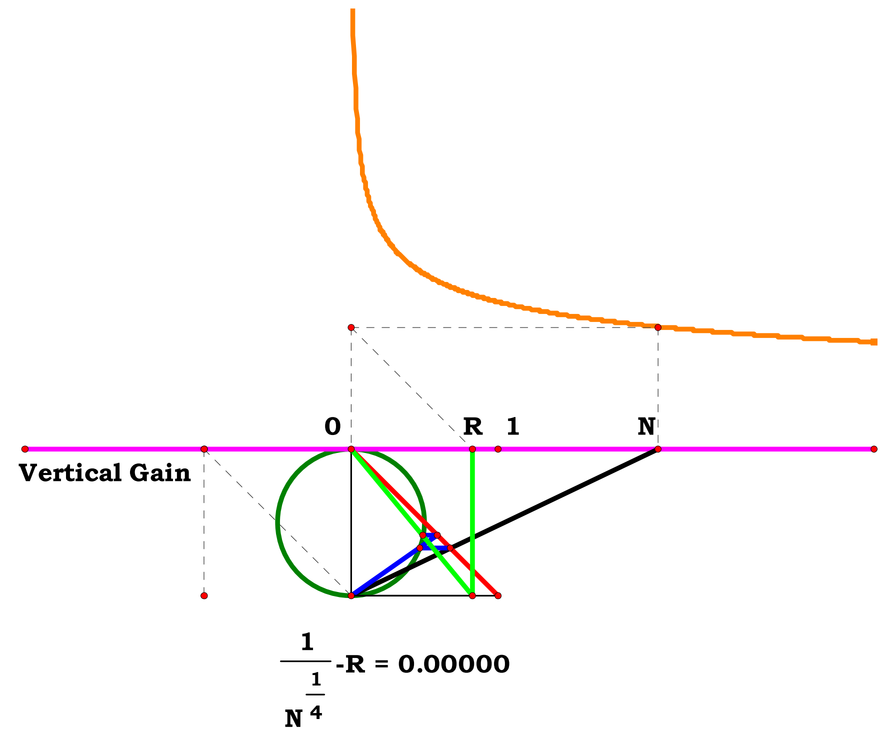
$$AR := \frac{CG \cdot AE}{GE} \qquad AR - \sqrt{\frac{AN}{AN^2 - AN + 1}} = 0$$



$AF := 1$

$AN := 3$

$FL := \dfrac{AN}{AN+1}$



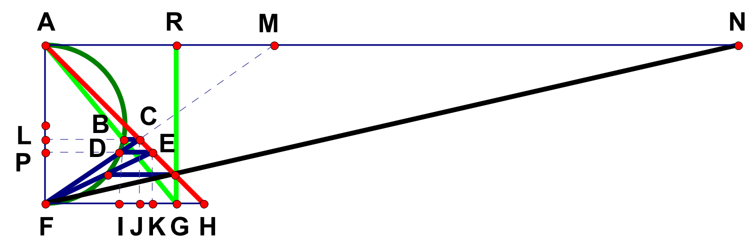
$EL := \dfrac{AF \cdot FL}{AN}$ $\quad AJ := AF - EL$ $\quad JF := EL$ $\quad DJ := \sqrt{AJ \cdot JF}$ $\quad AM := \dfrac{DJ \cdot AF}{JF}$

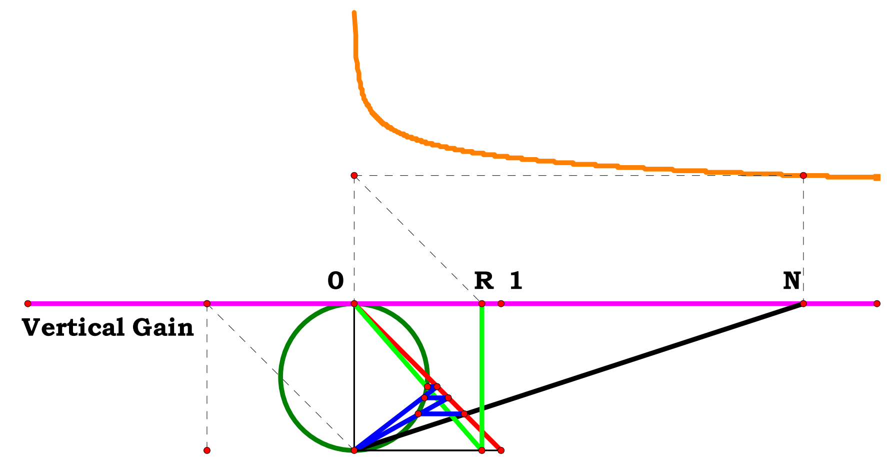
$FK := \dfrac{AM \cdot AF}{AM + AF}$ $\quad CK := \dfrac{AF \cdot FK}{AM}$ $\quad AI := AF - CK$ $\quad IF := CK$ $\quad BI := \sqrt{AI \cdot IF}$

$FG := \dfrac{BI \cdot AF}{AI}$ $\quad AR := FG$ $\quad AR - \dfrac{1}{AN^{\frac{1}{4}}} = 0$

$EL - \dfrac{1}{AN+1} = 0$ $\quad AJ - \dfrac{AN}{AN+1} = 0$ $\quad DJ - \dfrac{AN^{\frac{1}{2}}}{AN+1} = 0$ $\quad AM - AN^{\frac{1}{2}} = 0$

$FK - \dfrac{AN^{\frac{1}{2}}}{AN^{\frac{1}{2}}+1} = 0$ $\quad CK - \dfrac{1}{AN^{\frac{1}{2}}+1} = 0$ $\quad AI - \dfrac{AN^{\frac{1}{2}}}{AN^{\frac{1}{2}}+1} = 0$ $\quad BI - \dfrac{AN^{\frac{1}{4}}}{AN^{\frac{1}{2}}+1} = 0$





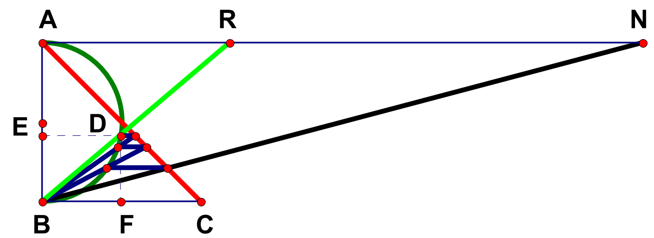
$\frac{1}{N^{\frac{1}{8}}}$-R = 0.00000

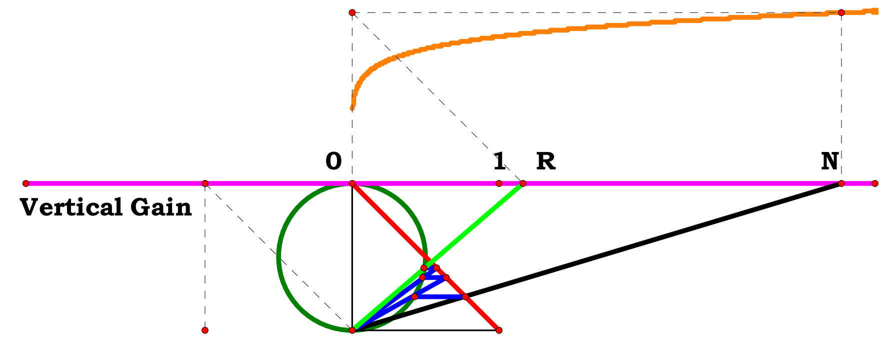
$AF := 1 \quad AN := 5 \quad EK := \dfrac{1}{AN^{\frac{1}{2}}+1} \quad DI := EK \quad DP := \dfrac{AN^{\frac{1}{4}}}{AN^{\frac{1}{2}}+1} \quad AM := \dfrac{DP \cdot AF}{DI}$

$FJ := \dfrac{AM \cdot AF}{AM + AF} \quad CJ := \dfrac{AF \cdot FJ}{AM} \quad FL := CJ \quad AL := AF - FL$

$BL := \sqrt{AL \cdot FL} \quad FG := \dfrac{BL \cdot AF}{AL} \quad AR := FG \quad AR - \dfrac{1}{AN^{\frac{1}{8}}} = 0$

$AL - \dfrac{AN^{\frac{1}{4}}}{AN^{\frac{1}{4}}+1} = 0 \qquad BL - \dfrac{AN^{\frac{1}{8}}}{AN^{\frac{1}{4}}+1} = 0$

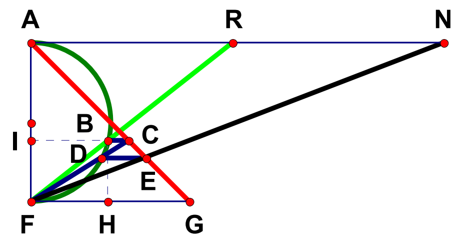




$N^{\frac{1}{8}}$-R = 0.00000

$$AB := 1 \qquad AN := 4 \qquad DE := \frac{AN^{\frac{1}{8}}}{AN^{\frac{1}{4}} + 1} \qquad AE := \frac{AN^{\frac{1}{4}}}{AN^{\frac{1}{4}} + 1} \qquad BE := AB - AE$$

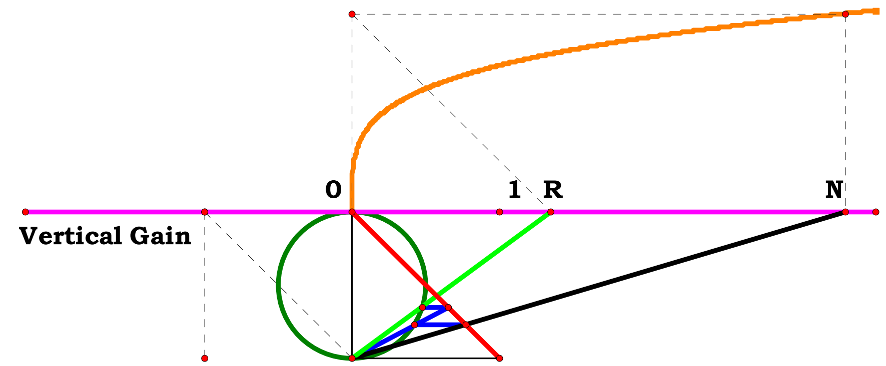
$$AR := \frac{DE \cdot AB}{BE} \qquad AR - AN^{\frac{1}{8}} = 0$$



$$AF := 1$$

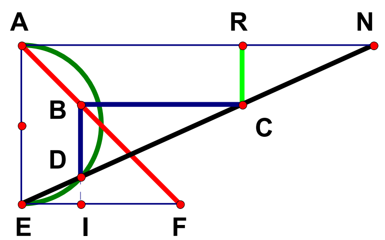
$$AN := 3$$

$$AI := \frac{AN^{\frac{1}{2}}}{AN^{\frac{1}{2}} + 1}$$



$N^{\frac{1}{4}}$-R = 0.00000

$$BI := \frac{AN^{\frac{1}{4}}}{AN^{\frac{1}{2}} + 1} \qquad FI := AF - AI \qquad AR := \frac{BI \cdot AF}{FI} \qquad AR - AN^{\frac{1}{4}} = 0$$

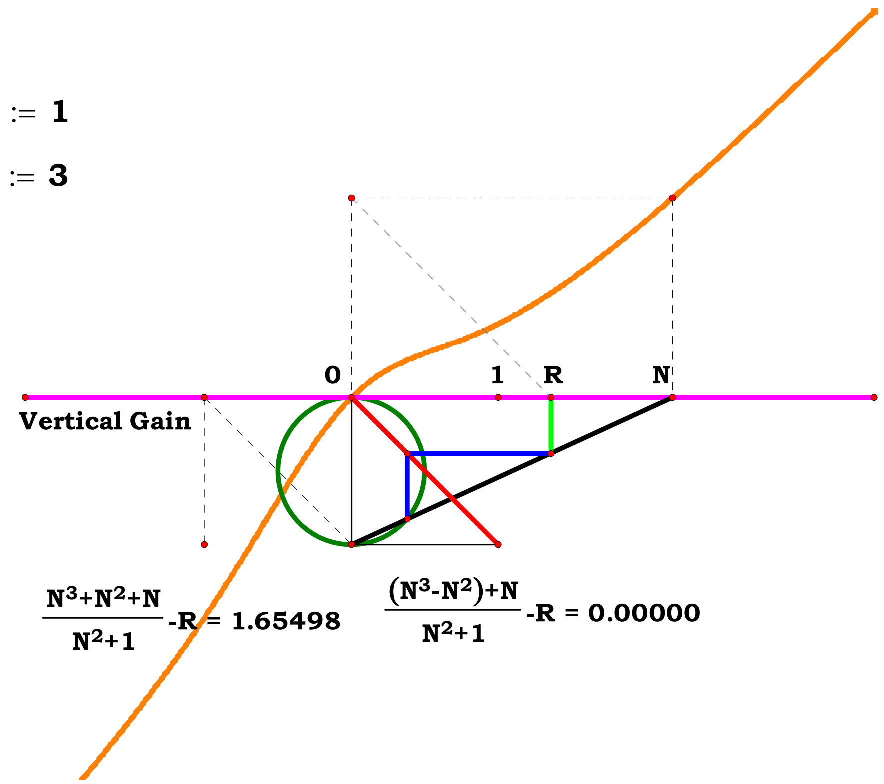


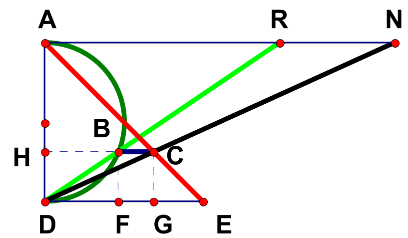
$$AE := 1$$

$$AN := 3$$

$$EI := \frac{AN}{AN^2 + 1} \quad CR := EI \quad NR := \frac{AN \cdot CR}{AE} \quad AR := AN - NR$$

$$AR - \frac{AN^3 - AN^2 + AN}{AN^2 + 1} = 0$$

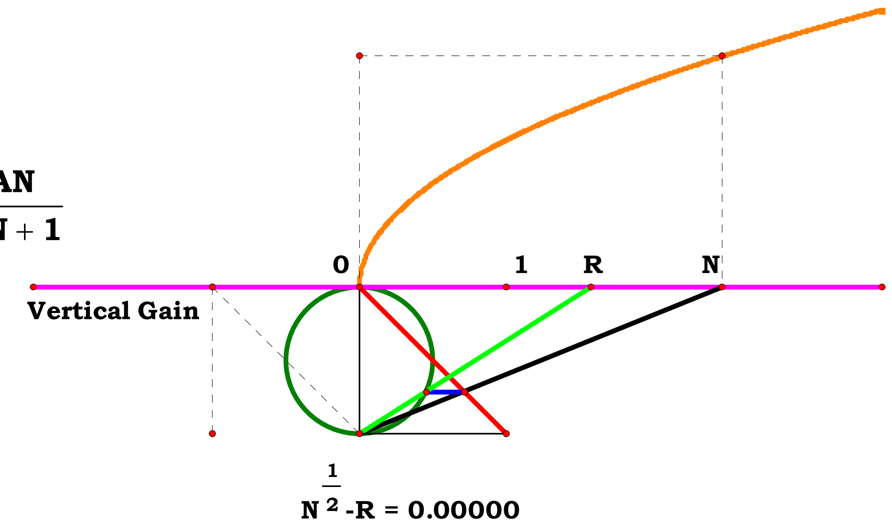




$$AD := 1$$

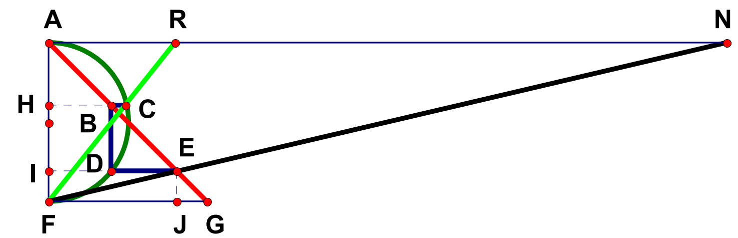
$$AN := 3$$

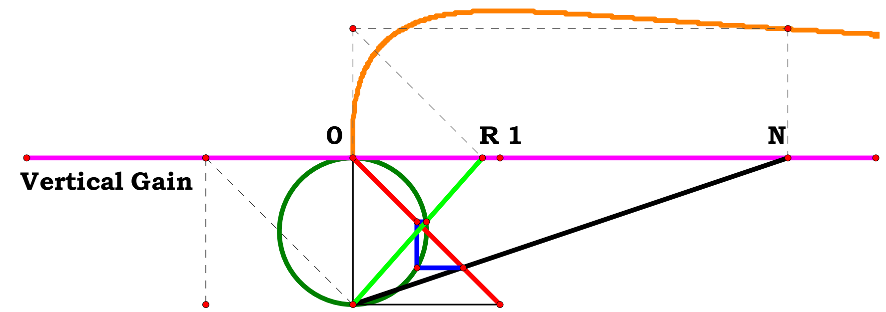
$$DG := \frac{AN}{AN+1}$$



$$CG := \frac{AD \cdot DG}{AN} \qquad AH := AD - CG \quad BH := \sqrt{AH \cdot CG} \qquad AR := \frac{BH \cdot AD}{CG}$$

$$AR - AN^{\frac{1}{2}} = 0$$





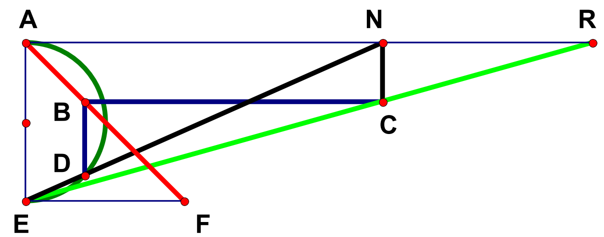
$$\left(\frac{N^{\frac{1}{2}}}{\left(N-N^{\frac{1}{2}}\right)+1}\right)^{\frac{1}{2}}-R = 0.00000$$

$$AF := 1 \quad AN := 4 \quad FJ := \frac{AN}{AN+1} \quad EJ := \frac{AF \cdot FJ}{AN} \quad FI := EJ \quad AI := AF - FI$$

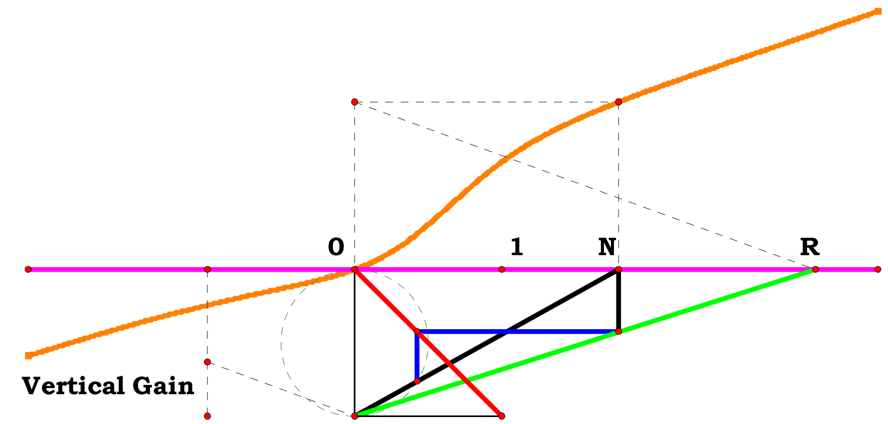
$$DI := \sqrt{AI \cdot FI} \quad AH := DI \quad FH := AF - AH \quad CH := \sqrt{AH \cdot FH} \quad AR := \frac{CH \cdot AF}{FH}$$

$$AR - \sqrt{\frac{\sqrt{AN}}{AN - \sqrt{AN} + 1}} = 0$$

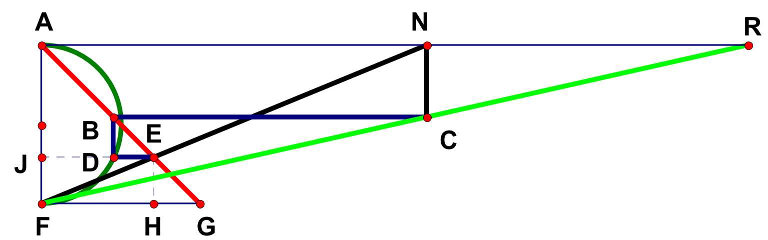
$$DI - \frac{AN^{\frac{1}{2}}}{AN+1} = 0$$



$$AE := 1 \qquad AN := 3 \qquad CN := \frac{AN}{AN^2 + 1} \qquad AR := \frac{AN \cdot AE}{AE - CN} \qquad AR - \frac{AN^3 + AN}{AN^2 - AN + 1} = 0$$

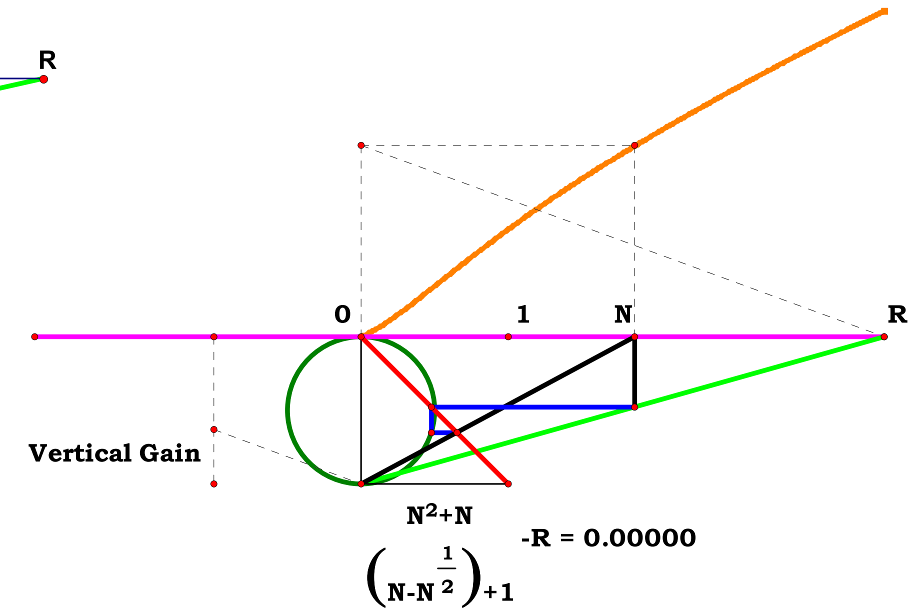


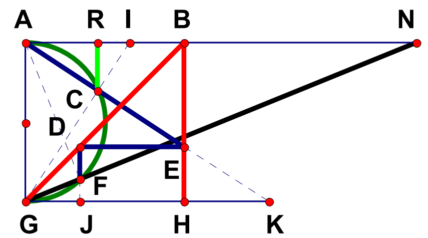
$$\frac{N^3+N}{N^2+N+1}-R = -1.86401 \qquad \frac{N^3+N}{(N^2-N)+1}-R = 0.00000$$



$$AF := 1 \qquad AN := 3 \qquad FH := \frac{AN}{AN+1} \qquad EH := \frac{AF \cdot FH}{AN} \qquad AJ := AF - EH$$

$$DJ := \sqrt{AJ \cdot EH} \qquad CN := DJ \qquad AR := \frac{AN \cdot AF}{AF - DJ} \qquad AR - \frac{AN^2 + AN}{AN - \sqrt{AN} + 1} = 0$$



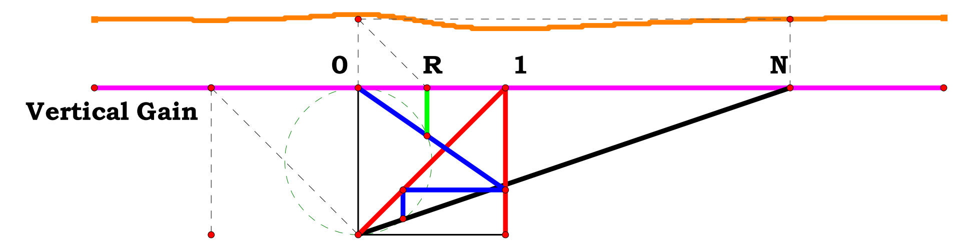


$$AB := 1$$

$$AN := 3$$

$$GN := \sqrt{AN^2 + AB^2}$$

$$\frac{N^4+N^3+2\cdot N^2+N+1}{2\cdot N^4+2\cdot N^3+5\cdot N^2+2\cdot N+2}\text{-}R = 0.01414$$



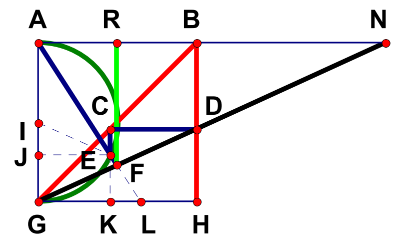
$$\frac{(((N^4\text{-}N^3)+2\cdot N^2)\text{-}N)+1}{(((2\cdot N^4\text{-}2\cdot N^3)+5\cdot N^2)\text{-}2\cdot N)+2}\text{-}R = 0.00000$$

$$FG := \frac{AB^2}{GN} \qquad GJ := \frac{AN \cdot FG}{GN} \qquad EH := GJ \qquad BE := AB - EH \qquad GK := \frac{AB^2}{BE}$$

$$AK := \sqrt{GK^2 + AB^2} \qquad AC := \frac{AB^2}{AK} \qquad AR := \frac{GK \cdot AC}{AK} \qquad AR - \frac{AN^4 - AN^3 + 2\cdot AN^2 - AN + 1}{2\cdot AN^4 + 5\cdot AN^2 - 2\cdot AN^3 - 2\cdot AN + 2} = 0$$

$$GN - \sqrt{AN^2 + 1} = 0 \qquad FG - \frac{1}{\left(AN^2 + 1\right)^{\frac{1}{2}}} = 0 \qquad GJ - \frac{AN}{AN^2 + 1} = 0 \qquad BE - \frac{AN^2 - AN + 1}{AN^2 + 1} = 0$$

$$GK - \frac{AN^2 + 1}{AN^2 - AN + 1} = 0 \qquad AK - \frac{\left(2\cdot AN^4 + 5\cdot AN^2 - 2\cdot AN^3 - 2\cdot AN + 2\right)^{\frac{1}{2}}}{AN^2 - AN + 1} = 0 \qquad AC - \frac{AN^2 - AN + 1}{\left(2\cdot AN^4 + 5\cdot AN^2 - 2\cdot AN^3 - 2\cdot AN + 2\right)^{\frac{1}{2}}} = 0$$



$$AB := 1$$

$$AN := 3$$

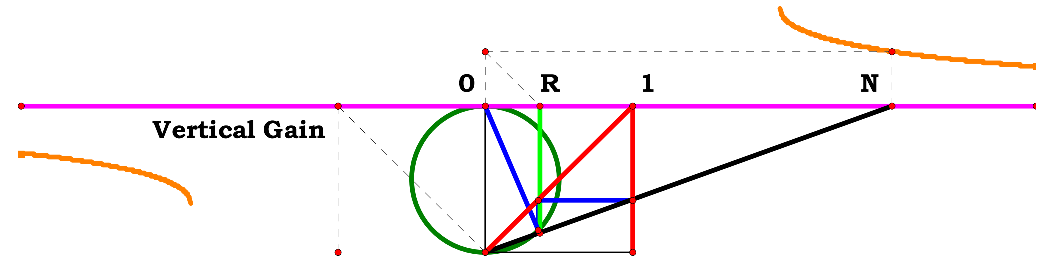
$$DH := \frac{1}{AN}$$

$$EJ := DH \quad EI := \frac{AB}{2} \quad IJ := \sqrt{EI^2 - EJ^2} \quad AJ := \frac{AB}{2} + IJ \quad GL := \frac{EJ \cdot AB}{AJ}$$

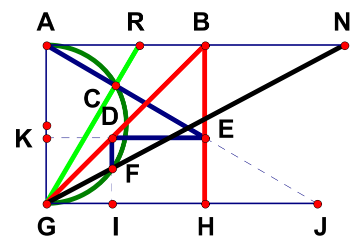
$$AR := \frac{GL \cdot AN}{GL + AN} \qquad AR - \frac{2 \cdot AN}{AN^2 + AN \cdot \sqrt{AN^2 - 4} + 2} = 0$$

$$AR - \frac{AN^2 - AN \cdot \sqrt{AN^2 - 4}}{3AN - \sqrt{AN^2 - 4}} = 0$$

$$\frac{2 \cdot N}{N^2 + N \cdot \sqrt{N^2 - 4} + 2} - R = 0.00000$$

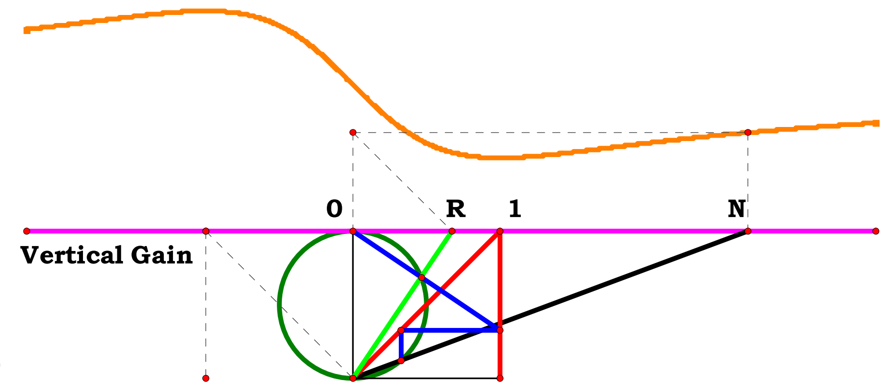


$$\frac{N^2 - N \cdot (N^2 - 4)^{\frac{1}{2}}}{3 \cdot N - (N^2 - 4)^{\frac{1}{2}}} - R = 0.00000$$



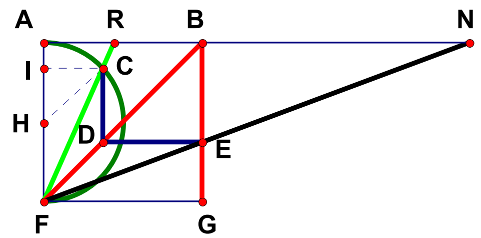
$AB := 1$

$AN := 3$



$$GI := \frac{AN}{AN^2 + 1} \quad AK := AB - GI \quad GJ := \frac{AB^2}{AK} \quad AR := \frac{1}{GJ} \quad AR - \frac{AN^2 - AN + 1}{AN^2 + 1} = 0$$

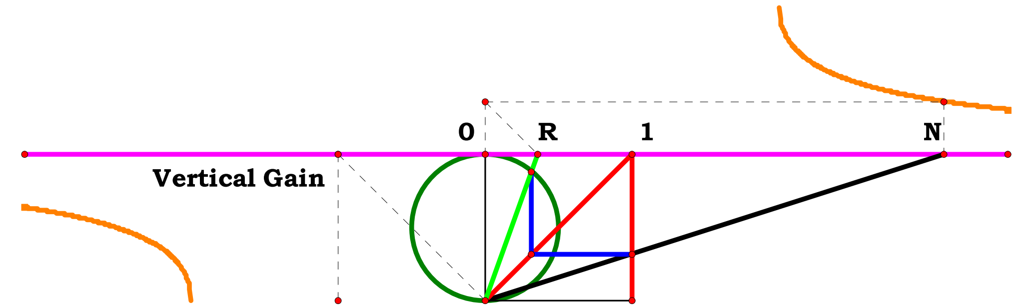
$\frac{N^2+N+1}{N^2+1}$-R = 0.65319 $\quad$ $\frac{(N^2-N)+1}{N^2+1}$-R = 0.00000



$$AB := 1$$

$$AN := 3$$

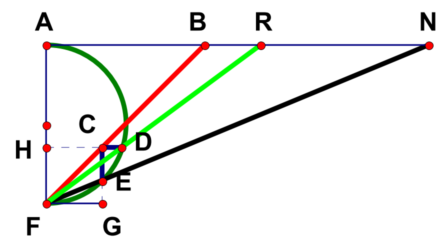
$$EG := \frac{1}{AN}$$



$$\frac{N-(N^2-4)^{\frac{1}{2}}}{2}-R = 0.00000$$

$$CH := \frac{AB}{2} \qquad CI := EG \qquad HI := \sqrt{CH^2 - CI^2} \qquad FI := \frac{AB}{2} + HI \qquad AR := \frac{CI \cdot AB}{FI}$$

$$AR - \frac{2}{AN + \left(AN^2 - 4\right)^{\frac{1}{2}}} = 0 \qquad AR - \frac{AN - \sqrt{AN^2 - 4}}{2} = 0$$

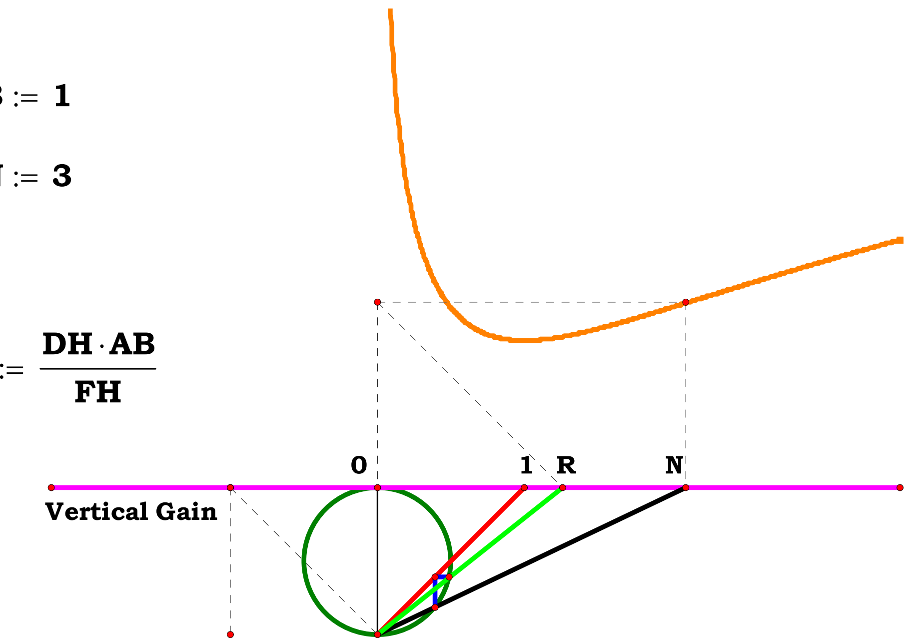


$$AB := 1$$

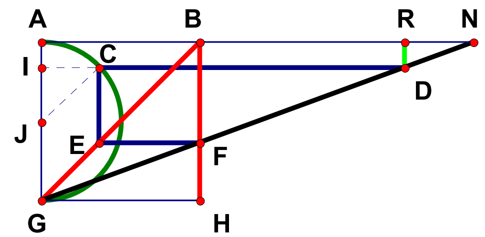
$$AN := 3$$

$$FG := \frac{AN}{AN^2 + 1} \qquad FH := FG \qquad AH := AB - FH \qquad DH := \sqrt{FH \cdot AH} \qquad AR := \frac{DH \cdot AB}{FH}$$

$$AR - \sqrt{\frac{AN^2 - AN + 1}{AN}} = 0$$



$$\frac{(N^2\text{-}N)+1}{N}^{\frac{1}{2}}\text{-}R = 0.00000$$



$$AB := 1$$

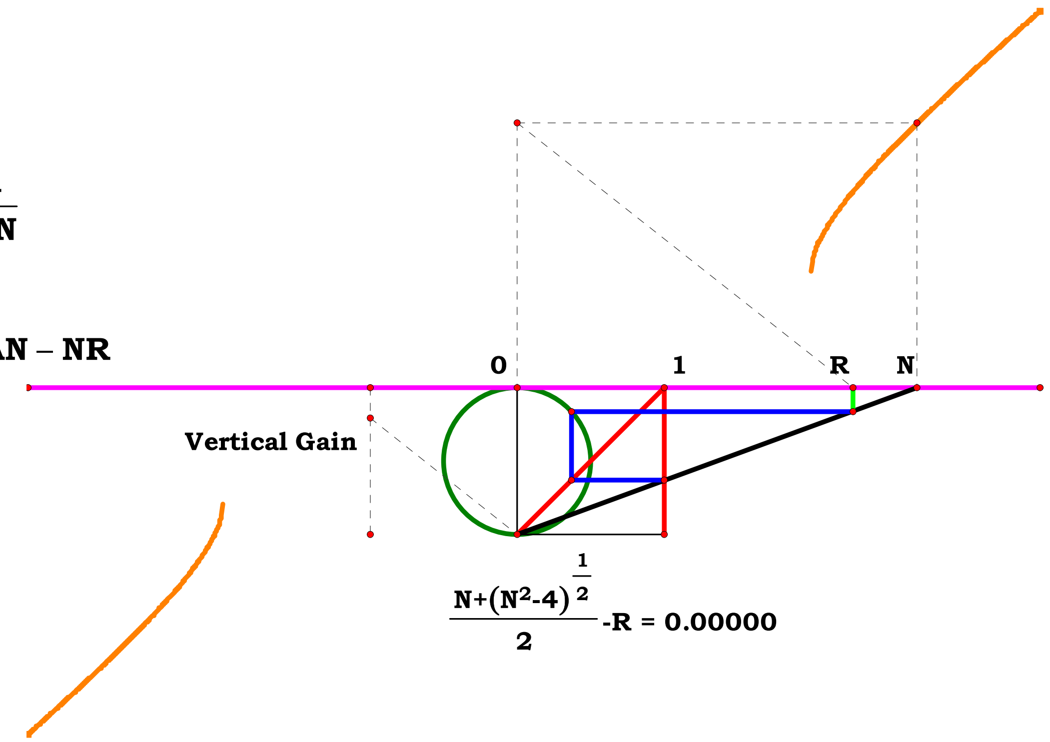
$$AN := 3$$

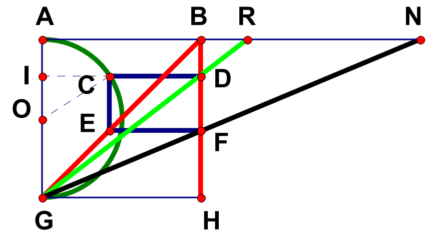
$$FH := \frac{1}{AN}$$

$$CJ := \frac{AB}{2} \qquad JI := \sqrt{CJ^2 - FH^2} \qquad AI := \frac{AB}{2} - JI \qquad NR := \frac{AN \cdot AI}{AB} \qquad AR := AN - NR$$

$$AR - \frac{AN + \sqrt{AN^2 - 4}}{2} = 0$$

$$AI - \frac{AN - \left(AN^2 - 4\right)^{\frac{1}{2}}}{2 \cdot AN} = 0$$





$$AB := 1$$

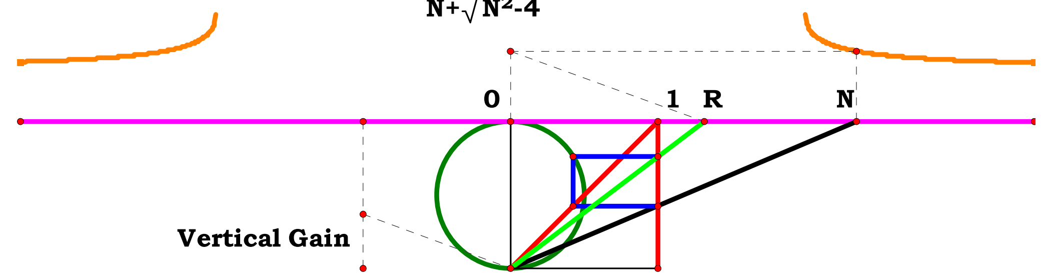
$$AN := 5$$

$$FH := \frac{1}{AN}$$

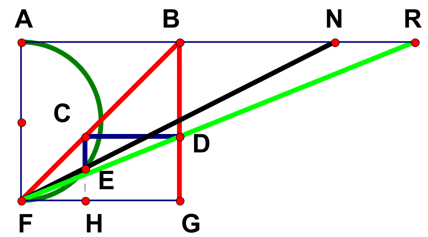
$$CO := \frac{AB}{2} \qquad CI := FH \qquad OI := \sqrt{CO^2 - CI^2} \qquad GI := \frac{AB}{2} + OI \qquad AR := \frac{AB^2}{GI}$$

$$AR - \frac{2 \cdot AN}{AN + \left(AN^2 - 4\right)^{\frac{1}{2}}} = 0 \qquad AR - \frac{AN^2 - AN \cdot \sqrt{AN^2 - 4}}{2} = 0$$

$$\frac{2 \cdot N}{N + \sqrt{N^2 - 4}} - R = 0.00000$$



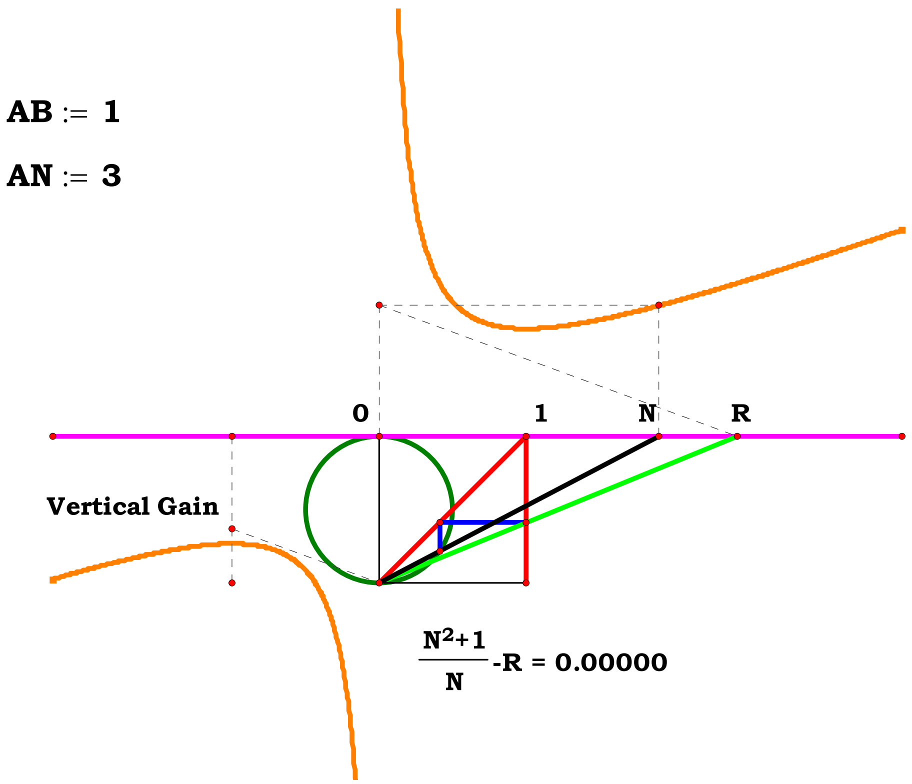
$$\frac{N^2 - N \cdot \left(N^2 - 4\right)^{\frac{1}{2}}}{2} - R = 0.00000$$

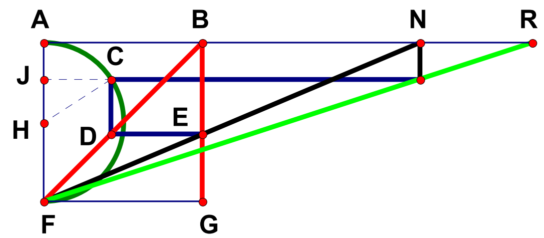


$\mathbf{AB} := 1$

$\mathbf{AN} := 3$

$$\mathbf{FH} := \frac{\mathbf{AN}}{\mathbf{AN}^2 + 1} \qquad \mathbf{DG} := \mathbf{FH} \qquad \mathbf{AR} := \frac{\mathbf{AB}^2}{\mathbf{DG}} \qquad \mathbf{AR} - \frac{\mathbf{AN}^2 + 1}{\mathbf{AN}} = 0$$

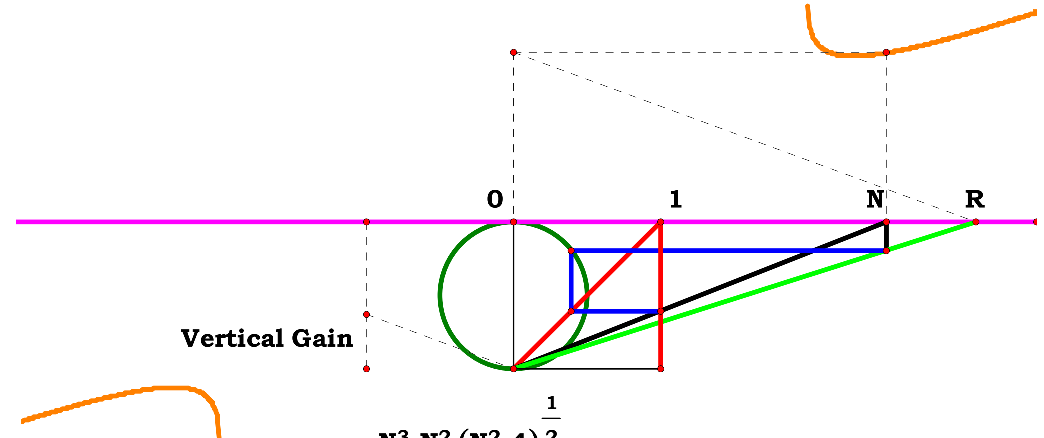




$AB := 1$

$AN := 5$

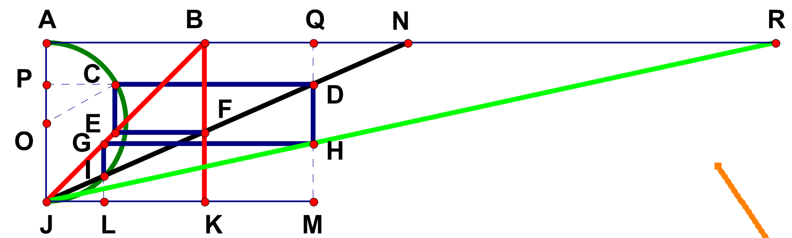
$EG := \dfrac{1}{AN}$



$CJ := EG \quad CH := \dfrac{AB}{2} \quad JH := \sqrt{CH^2 - CJ^2} \quad AJ := \dfrac{AB}{2} - JH$

$$AR := \frac{AN \cdot AB}{AB - AJ} \qquad AR - \frac{2 \cdot AN^2}{AN + \left(AN^2 - 4\right)^{\frac{1}{2}}} = 0 \qquad AR - \frac{AN^3 - AN^2 \cdot \sqrt{AN^2 - 4}}{2} = 0$$

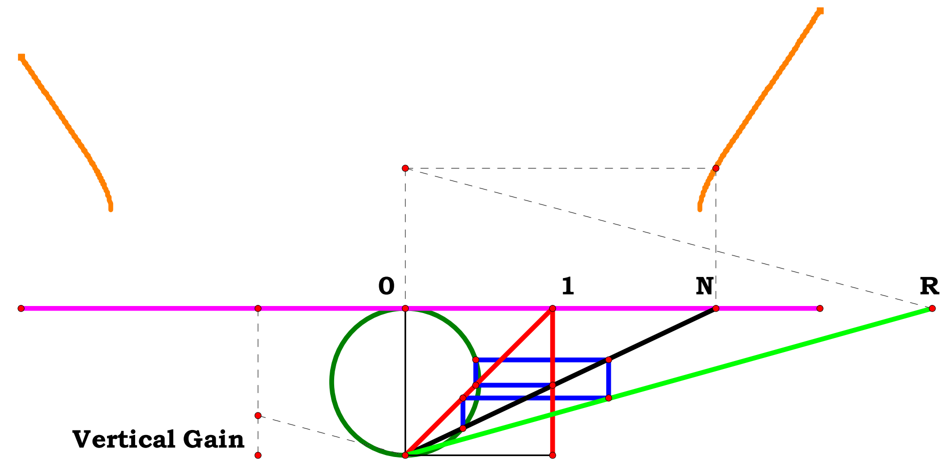
$$\frac{N^3 - N^2 \cdot (N^2 - 4)^{\frac{1}{2}}}{2} - R = 0.00000$$



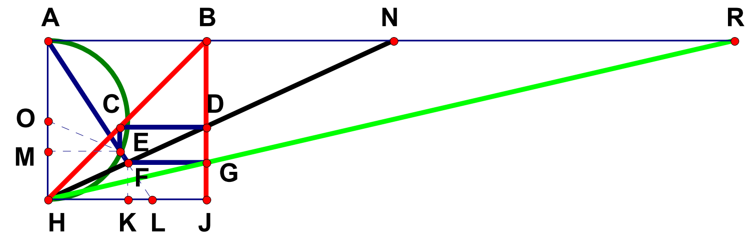
$$AB := 1 \quad AN := 3 \quad JL := \frac{AN}{AN^2+1} \quad AP := \frac{AN - \left(AN^2-4\right)^{\frac{1}{2}}}{2 \cdot AN} \quad NQ := \frac{AN \cdot AP}{AB}$$

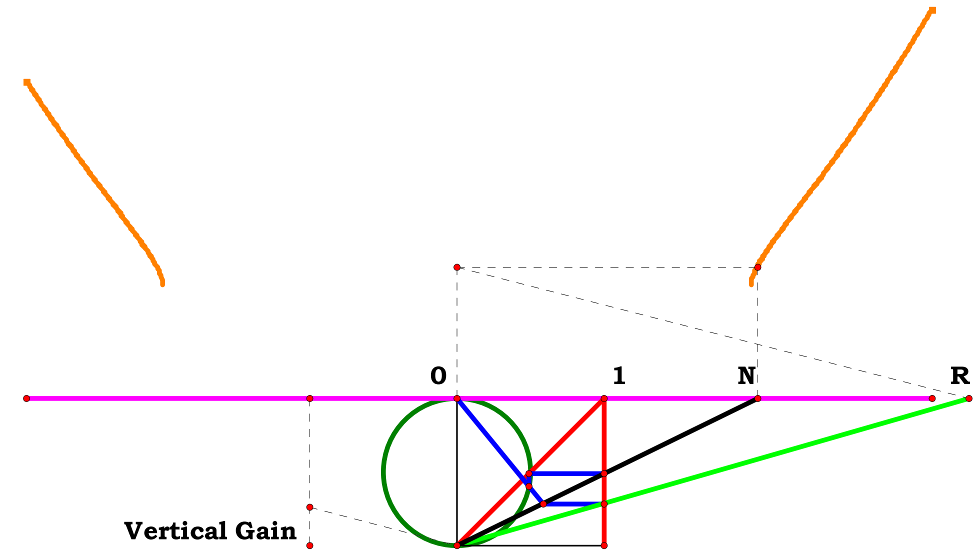
$$AQ := AN - NQ \quad JM := AQ \quad AR := \frac{JM \cdot AB}{JL}$$

$$AR - \frac{AN^3 + AN^2 \cdot \sqrt{AN^2-4} + AN + \sqrt{AN^2-4}}{2 \cdot AN} = 0$$



$$\frac{N^3+N^2\cdot(N^2-4)^{\frac{1}{2}}+N+(N^2-4)^{\frac{1}{2}}}{2\cdot N}-R = 0.00000$$





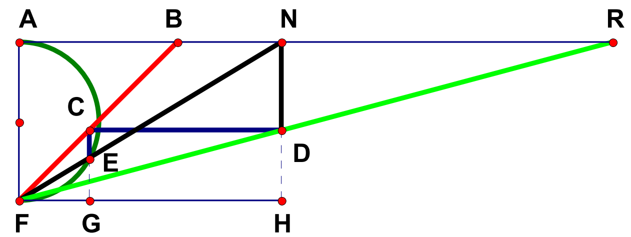
$$\frac{3\cdot N-(N^2-4)^{\frac{1}{2}}}{N-(N^2-4)^{\frac{1}{2}}}-R = 0.00000$$

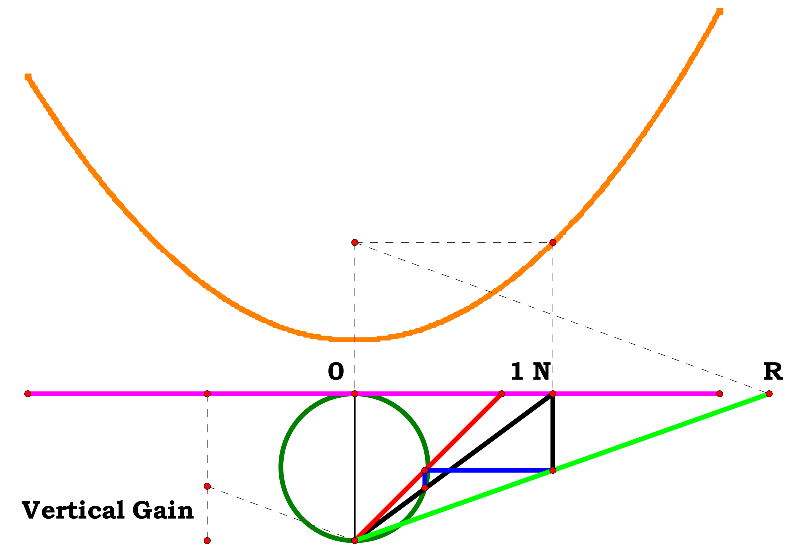
$$AB := 1 \quad AN := 5 \qquad DJ := \frac{1}{AN} \quad EO := \frac{AB}{2} \quad EM := DJ$$

$$MO := \sqrt{EO^2 - EM^2} \quad AM := \frac{AB}{2} + MO \quad HL := \frac{EM\cdot AB}{AM} \quad HK := \frac{HL\cdot AN}{HL + AN}$$

$$FK := \frac{AB\cdot HK}{AN} \quad GJ := FK \quad AR := \frac{AB^2}{GJ} \quad AR - \frac{3\cdot AN - \sqrt{AN^2-4}}{AN - \sqrt{AN^2-4}} = 0$$

$$AR - \left[\frac{AN^2 + AN\cdot\left(AN^2-4\right)^{\frac{1}{2}} + 2}{2}\right] = 0 \qquad HK - \frac{2\cdot AN}{AN^2 + AN\cdot\left(AN^2-4\right)^{\frac{1}{2}} + 2} = 0 \qquad FK - \frac{2}{AN^2 + AN\cdot\left(AN^2-4\right)^{\frac{1}{2}} + 2} = 0$$

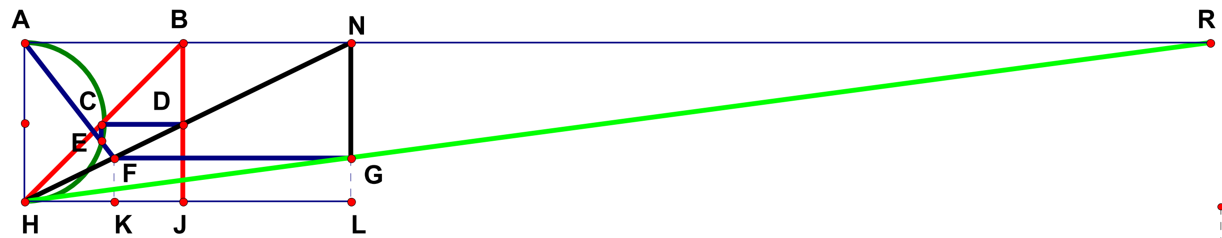


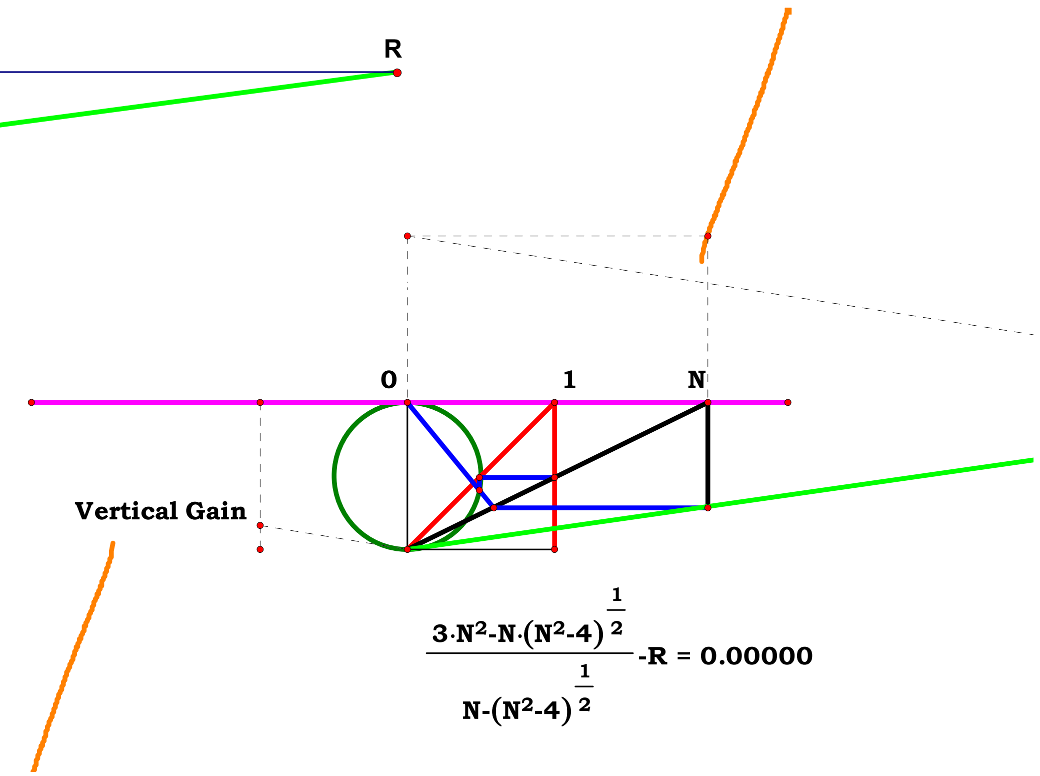


$$\mathbf{AB} := 1 \qquad \mathbf{AN} := 5 \qquad \mathbf{FG} := \frac{\mathbf{AN}}{\mathbf{AN}^2 + 1} \qquad \mathbf{DH} := \mathbf{FG}$$

$$\mathbf{AR} := \frac{\mathbf{AN} \cdot \mathbf{AB}}{\mathbf{DH}} \qquad \mathbf{AR} - \left(\mathbf{AN}^2 + 1\right) = 0$$

$(N^2+1)-R = 0.00000$



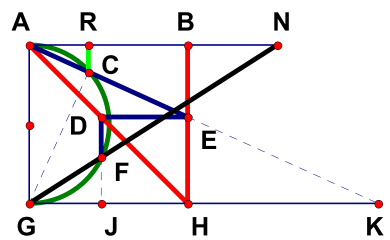


$$\frac{3\cdot N^2-N\cdot(N^2-4)^{\frac{1}{2}}}{N-(N^2-4)^{\frac{1}{2}}}-R = 0.00000$$

$AB := 1 \quad AN := 3 \quad FK := \dfrac{2}{AN^2 + AN\cdot\left(AN^2-4\right)^{\frac{1}{2}} + 2} \quad GL := FK \quad AR := \dfrac{AN\cdot AB}{GL}$

$$AR - \frac{AN^3 + AN^2\cdot\left(AN^2-4\right)^{\frac{1}{2}} + 2\cdot AN}{2} = 0 \qquad AR - \frac{3\cdot AN^2 - AN\cdot\sqrt{AN^2-4}}{AN-\sqrt{AN^2-4}} = 0$$

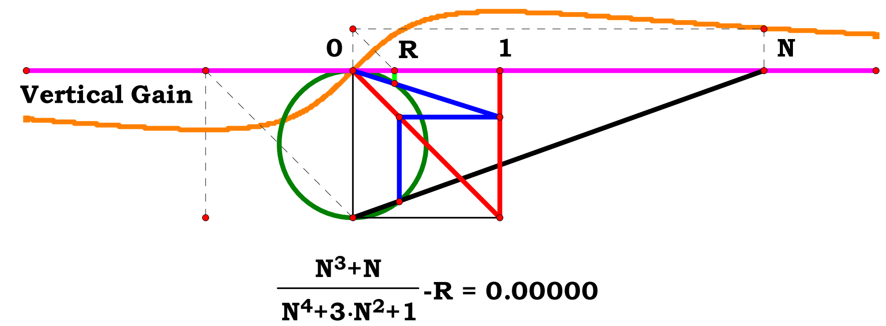
$$\frac{3\cdot AN^2 - AN\cdot\sqrt{AN^2-4}}{AN-\sqrt{AN^2-4}} - \frac{AN^3 + AN^2\cdot\left(AN^2-4\right)^{\frac{1}{2}} + 2\cdot AN}{2} = 0$$



$AB := 1$

$AN := 3$

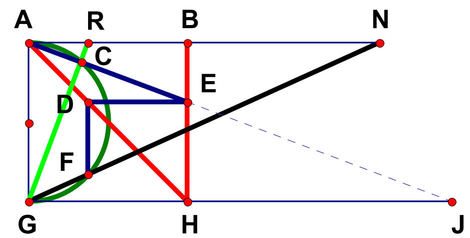
$$GJ := \frac{AN}{AN^2 + 1}$$



$$BE := GJ \quad GK := \frac{AB^2}{BE} \quad AK := \sqrt{AB^2 + GK^2} \quad AC := \frac{AB^2}{AK} \quad AR := \frac{GK \cdot AC}{AK}$$

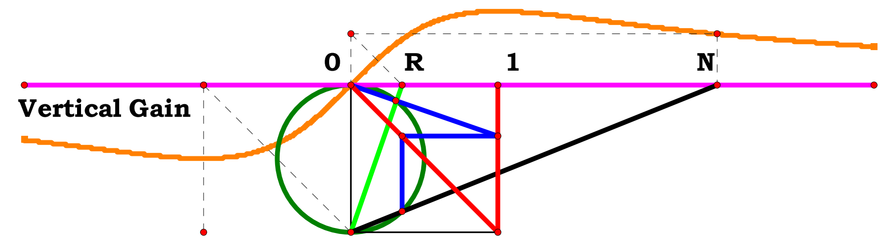
$$AR - \frac{AN^3 + AN}{AN^4 + 3 \cdot AN^2 + 1} = 0$$

$$GK - \frac{AN^2 + 1}{AN} = 0 \qquad AK - \frac{\left(AN^4 + 3 \cdot AN^2 + 1\right)^{\frac{1}{2}}}{AN} = 0$$



$\mathbf{AB} := 1$

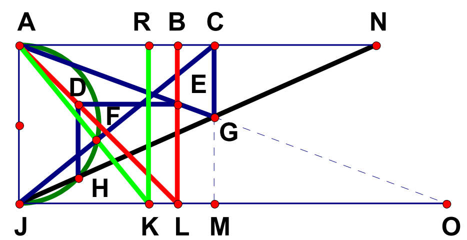
$\mathbf{AN} := 3$



$\frac{N}{N^2+1}$-R = 0.00000

$$GJ := \frac{AN^2+1}{AN} \quad AJ := \frac{\left(AN^4+3\cdot AN^2+1\right)^{\frac{1}{2}}}{AN} \quad AC := \frac{AB^2}{AJ} \quad AR := \frac{AJ\cdot AC}{GJ}$$

$$AR - \frac{AN}{AN^2+1} = 0$$



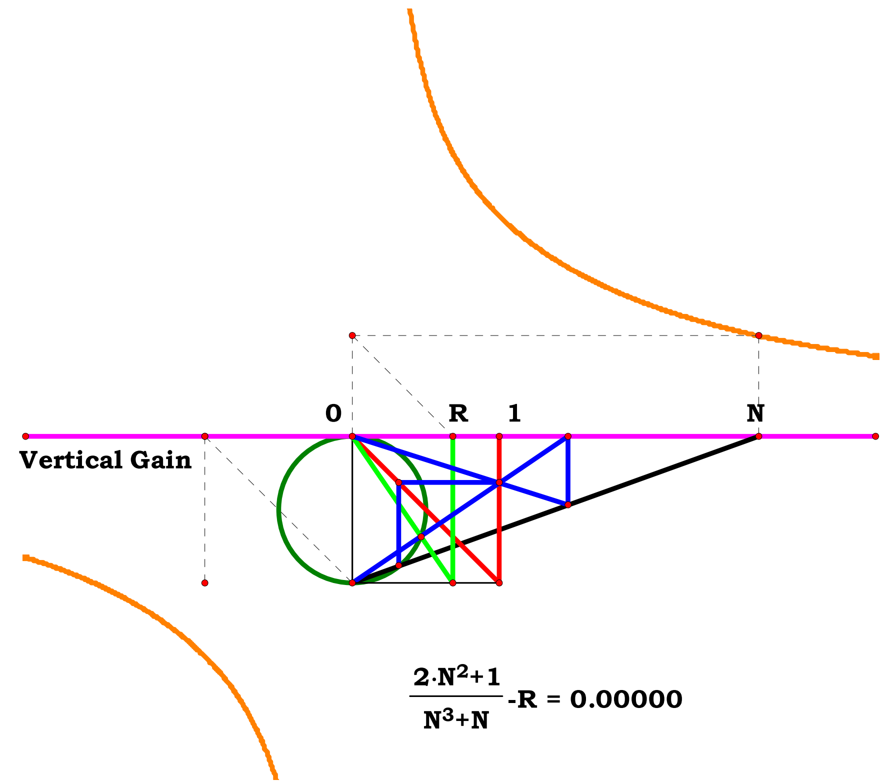
$$AB := 1$$

$$AN := 3$$

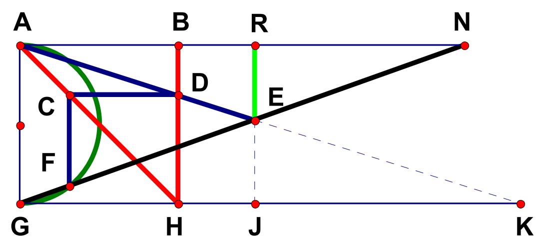
$$JO := \frac{AN^2 + 1}{AN} \qquad JM := \frac{JO \cdot AN}{JO + AN} \qquad AC := JM \quad JK := \frac{1}{AC} \qquad AR := JK$$

$$AR - \frac{2 \cdot AN^2 + 1}{AN^3 + AN} = 0$$

$$JM - \frac{AN^3 + AN}{2 \cdot AN^2 + 1} = 0$$

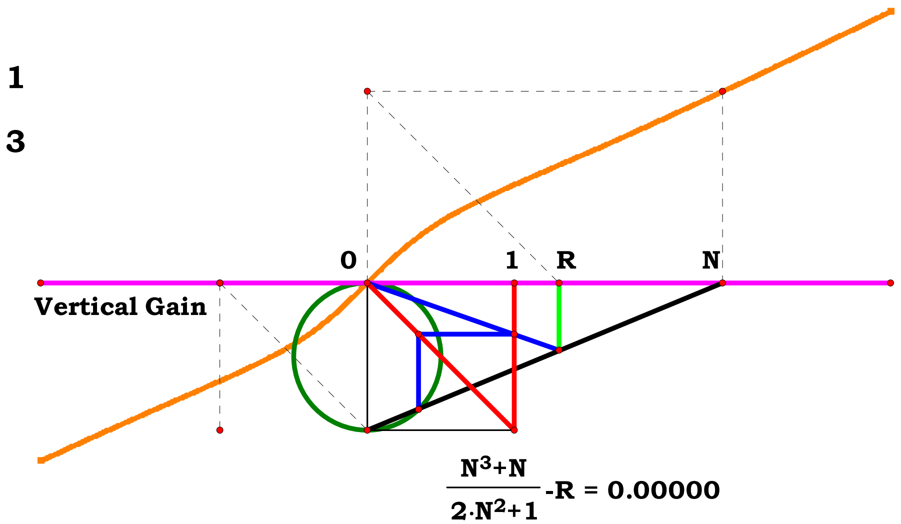


$$\frac{2 \cdot N^2 + 1}{N^3 + N} - R = 0.00000$$

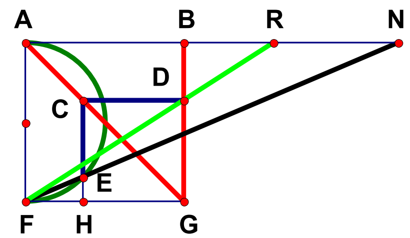


$$AB := 1$$

$$AN := 3$$



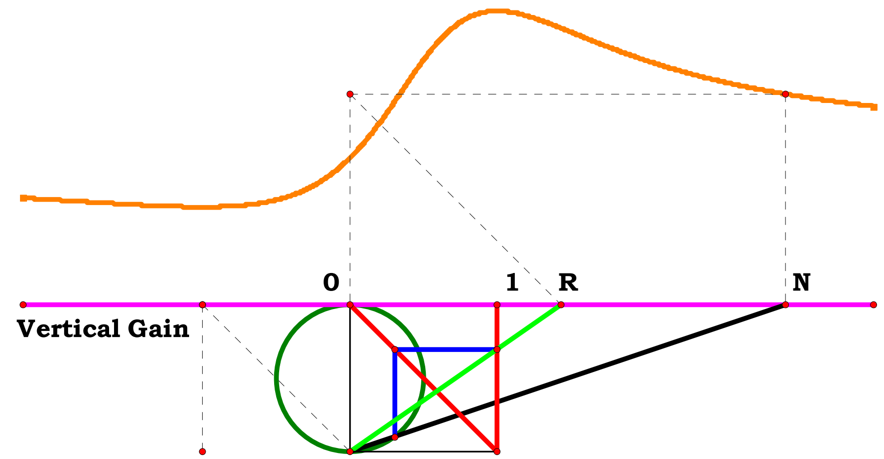
$$GJ := \frac{AN^3 + AN}{2 \cdot AN^2 + 1} \qquad AR := GJ \qquad AR - \frac{AN^3 + AN}{2 \cdot AN^2 + 1} = 0$$



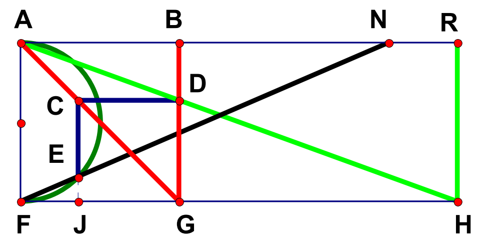
$AB := 1$

$AN := 5$

$$FH := \frac{AN}{AN^2 + 1} \quad DG := AB - FH \quad AR := \frac{AB^2}{DG} \quad AR - \frac{AN^2 + 1}{AN^2 - AN + 1} = 0$$



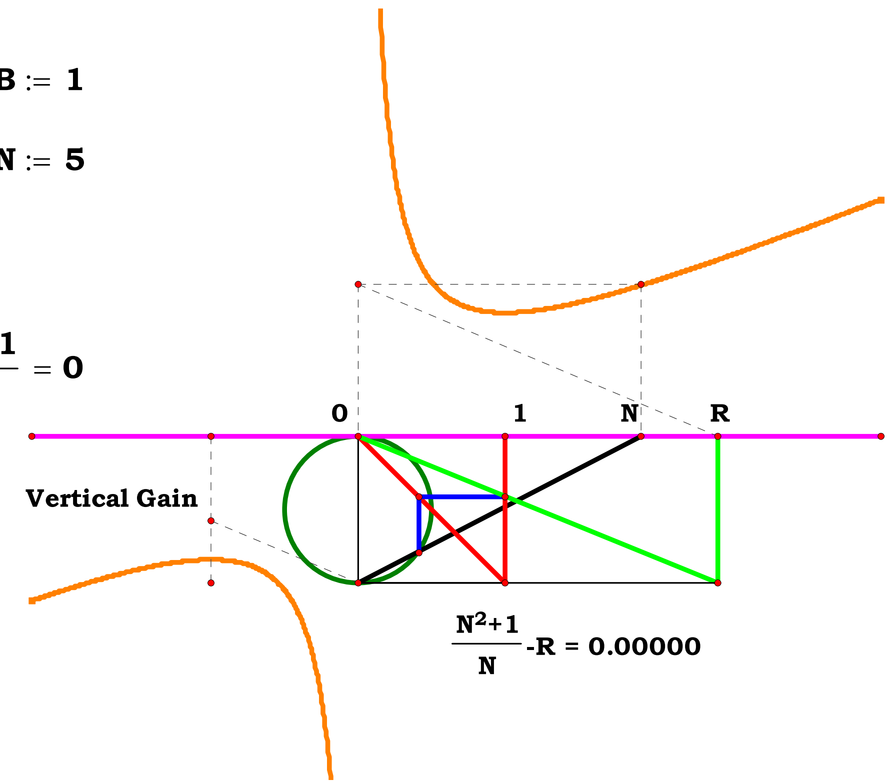
$$\frac{N^2+1}{N^2+N+1} - R = -0.66746 \qquad \frac{N^2+1}{(N^2-N)+1} - R = 0.00000$$

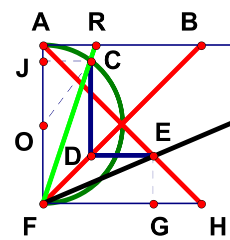


$\mathbf{AB} := \mathbf{1}$

$\mathbf{AN} := \mathbf{5}$

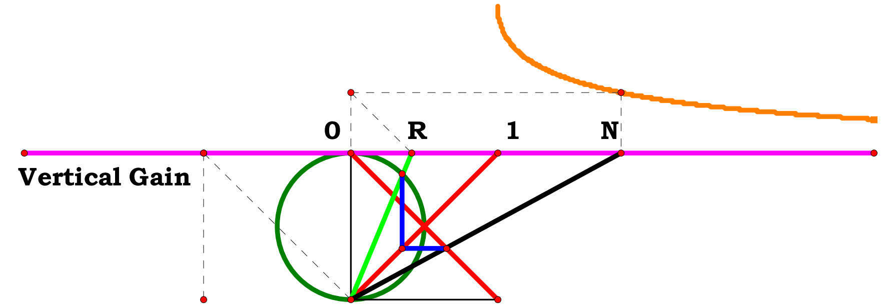
$$\mathbf{FJ} := \frac{\mathbf{AN}}{\mathbf{AN}^2 + 1} \qquad \mathbf{BD} := \mathbf{FJ} \qquad \mathbf{FH} := \frac{\mathbf{AB}^2}{\mathbf{BD}} \qquad \mathbf{AR} := \mathbf{FH} \qquad \mathbf{AR} - \frac{\mathbf{AN}^2 + 1}{\mathbf{AN}} = 0$$





$$AB := 1$$

$$AN := 3$$

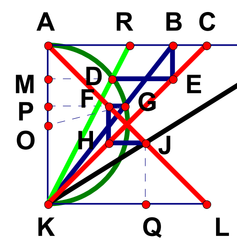


$$\frac{(N+1)-((N^2+2\cdot N)-3)^{\frac{1}{2}}}{2}-R = 0.00000$$

$$GH := \frac{1}{AN+1} \qquad CJ := GH \qquad CO := \frac{AB}{2} \qquad JO := \sqrt{CO^2 - CJ^2} \qquad AJ := \frac{AB}{2} - JO$$

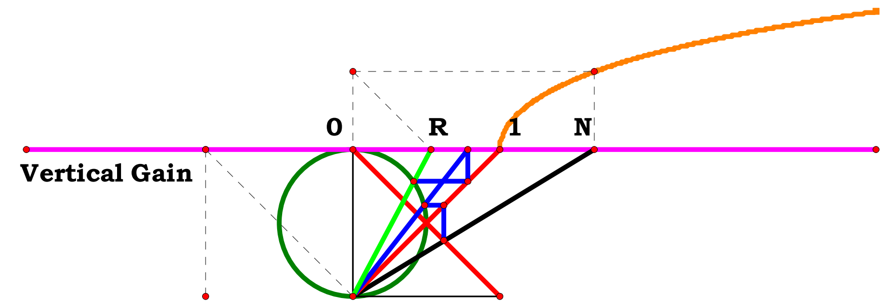
$$FJ := AB - AJ \qquad AR := \frac{CJ \cdot AB}{FJ}$$

$$AR - \frac{2}{AN + 1 + \left(AN^2 + 2\cdot AN - 3\right)^{\frac{1}{2}}} = 0 \qquad AR - \frac{AN + 1 - \left(AN^2 + 2\cdot AN - 3\right)^{\frac{1}{2}}}{2} = 0$$



$AC := 1$

$AN := 5$



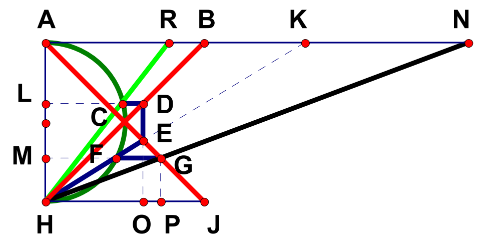
$$\left(N^{\frac{1}{2}}-1\right)^{\frac{1}{2}}-R = 0.00000$$

$$LQ := \frac{1}{AN+1} \qquad AP := LQ \quad KP := AC - AP \quad GP := \sqrt{AP \cdot KP} \quad AB := \frac{GP \cdot AC}{KP}$$

$$BE := AC - AB \qquad AM := BE \qquad KM := AC - AM \qquad DM := \sqrt{AM \cdot KM} \quad AR := \frac{DM \cdot AC}{KM}$$

$$AR - \left(AN^{\frac{1}{2}} - 1\right)^{\frac{1}{2}} = 0$$

$$AB - \frac{1}{AN^{\frac{1}{2}}} = 0 \qquad BE - \frac{AN^{\frac{1}{2}}-1}{AN^{\frac{1}{2}}} = 0 \qquad \frac{AN^{\frac{1}{2}}-1}{AN^{\frac{1}{2}}}$$



$$AB := 1$$

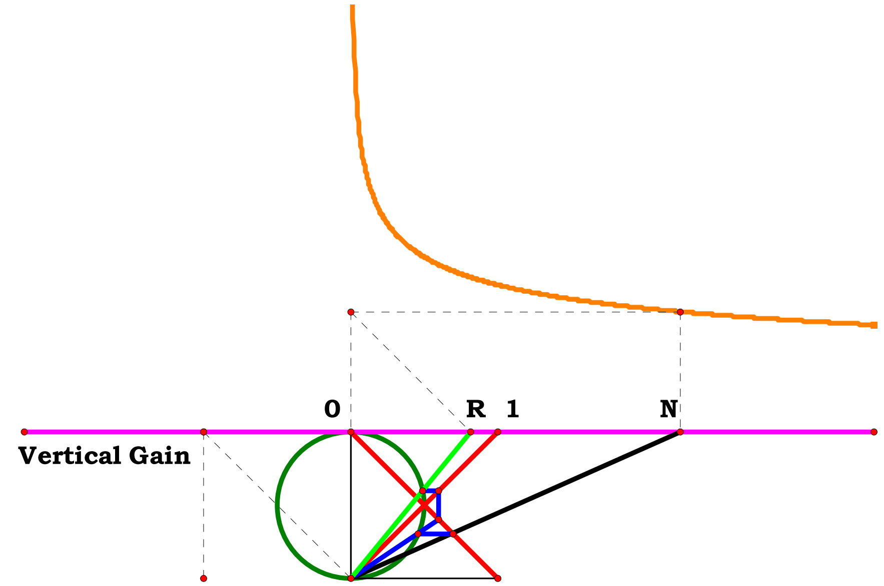
$$AN := 3$$

$$HM := \frac{1}{AN + 1}$$

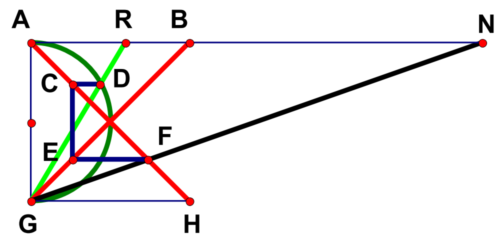
$$AM := AB - HM \quad FM := \sqrt{AM \cdot HM} \quad AK := \frac{FM \cdot AB}{HM} \quad HO := \frac{AB \cdot AK}{AB + AK}$$

$$JO := AB - HO \quad AL := JO \quad HL := AB - AL \quad CL := \sqrt{AL \cdot HL} \quad AR := \frac{CL \cdot AB}{HL}$$

$$AR - \frac{1}{AN^{\frac{1}{4}}} = 0$$



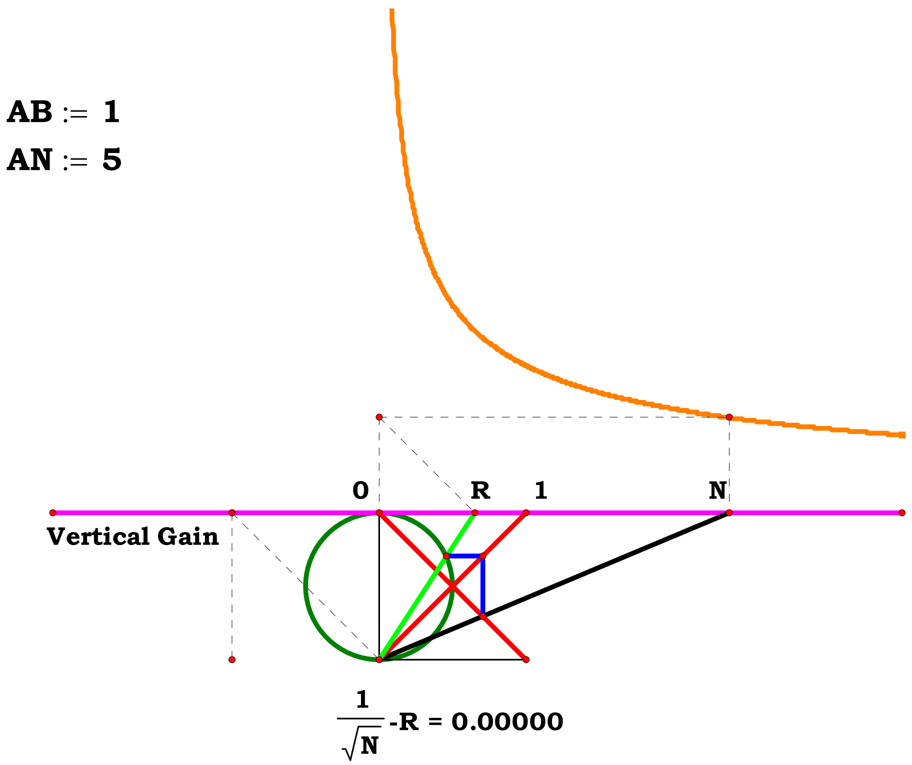
$$\frac{1}{N^{\frac{1}{4}}} - R = 0.00000$$

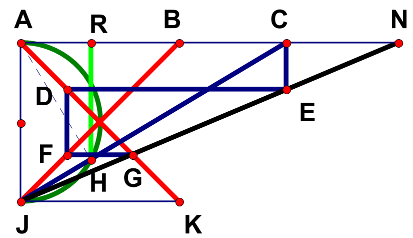


$$AB := 1$$

$$AN := 5$$

$$AR := \frac{1}{AN^{\frac{1}{2}}}$$

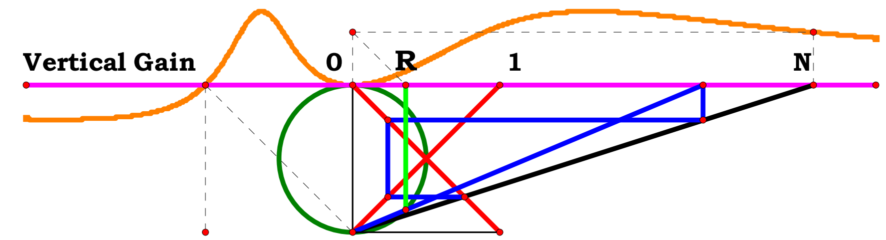




$$AB := 1$$

$$AN := 5$$

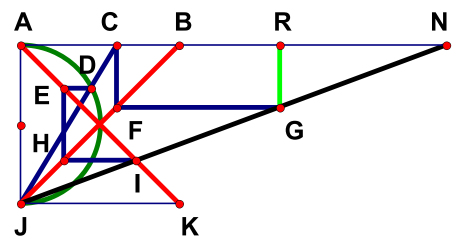
$$CE := \frac{1}{AN + 1}$$



$$\left|\frac{N^3-N^2}{((N^4+N^2)-2\cdot N)+1}\right|-R = -0.14996 \qquad \frac{N^3+N^2}{N^4+N^2+2\cdot N+1}-R = 0.00000$$

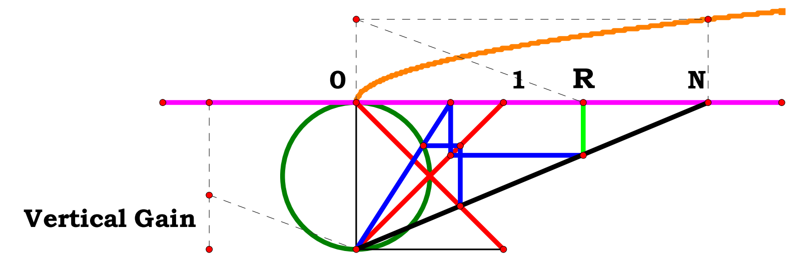
$$CN := \frac{AN \cdot CE}{AB} \qquad AC := AN - CN \qquad CJ := \sqrt{AC^2 + AB^2} \qquad AH := \frac{AC \cdot AB}{CJ}$$

$$AR := \frac{AB \cdot AH}{CJ} \qquad AR - \frac{AN^3 + AN^2}{AN^4 + AN^2 + 2 \cdot AN + 1} = 0$$



$\mathbf{AB} := 1$

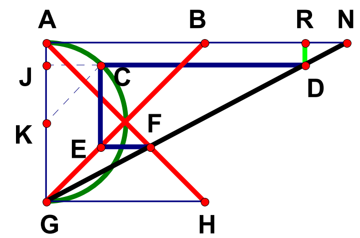
$\mathbf{AN} := 3$



$\sqrt{N} - R = 0.00000$

$$\mathbf{AC} := \frac{1}{\mathbf{AN}^{\frac{1}{2}}} \qquad \mathbf{BC} := \mathbf{AB} - \mathbf{AC} \qquad \mathbf{RG} := \mathbf{BC} \qquad \mathbf{NR} := \frac{\mathbf{AN}\cdot\mathbf{RG}}{\mathbf{AB}} \qquad \mathbf{AR} := \mathbf{AN} - \mathbf{NR}$$

$$\mathbf{AR} - \sqrt{\mathbf{AN}} = 0$$



$AB := 1$

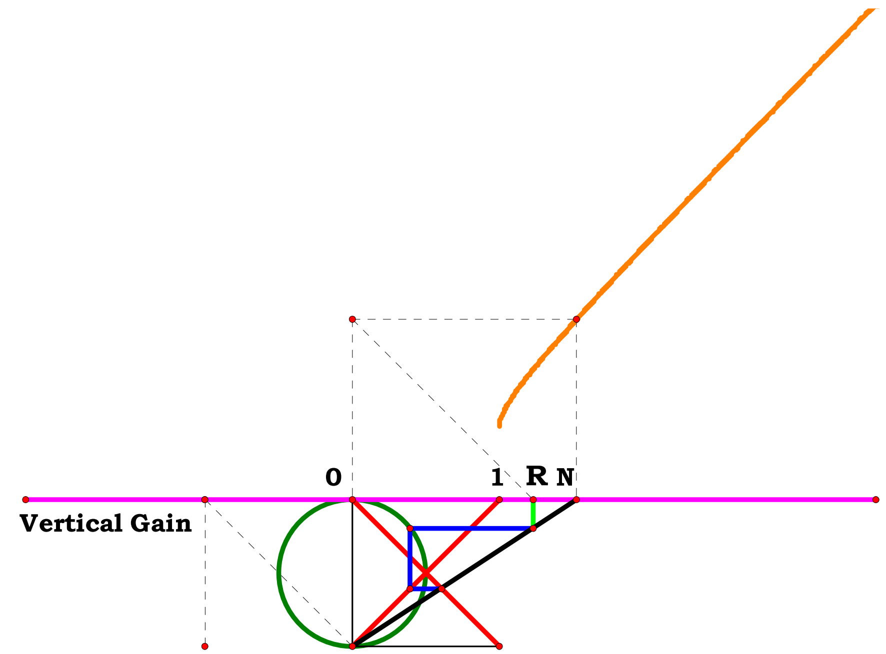
$AN := 3$

$$CJ := \frac{1}{AN + 1} \quad CK := \frac{AB}{2} \quad JK := \sqrt{CK^2 - CJ^2} \quad AJ := \frac{AB}{2} - JK \quad DR := AJ$$

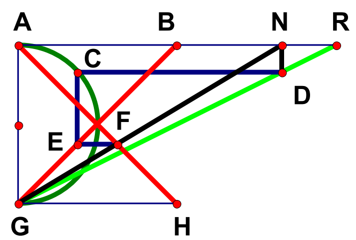
$$NR := \frac{AN \cdot DR}{AB} \quad AR := AN - NR \quad AR - \frac{AN^2 + AN + AN \cdot \left(AN^2 + 2 \cdot AN - 3\right)^{\frac{1}{2}}}{2 \cdot AN + 2} = 0$$

$$AR - \frac{2AN}{AN^2 + 2AN - AN \cdot \sqrt{AN^2 + 2AN - 3} - \sqrt{AN^2 + 2AN - 3} + 1} = 0$$

$$DR - \frac{AN + 1 - \left(AN^2 + 2 \cdot AN - 3\right)^{\frac{1}{2}}}{2AN + 2} = 0$$



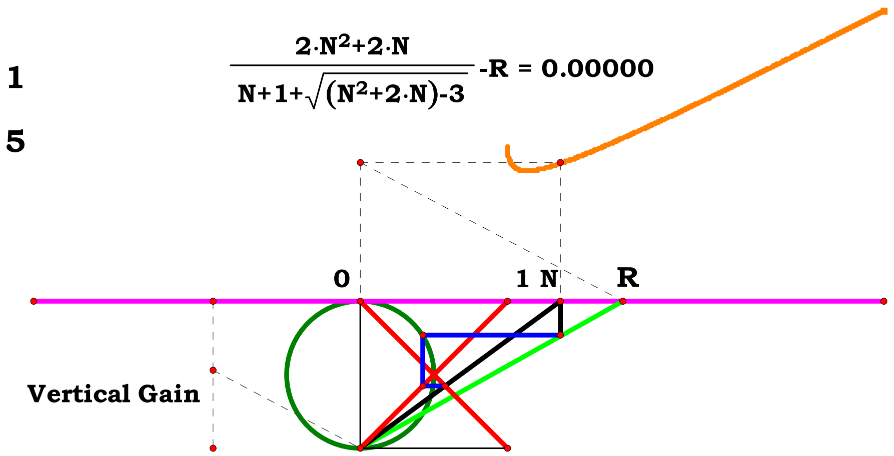
$$\frac{2 \cdot N}{(((N^2+2 \cdot N)-N \cdot \sqrt{(N^2+2 \cdot N)-3})+1)-\sqrt{(N^2+2 \cdot N)-3}} - R = 0.00000$$



$$AB := 1$$

$$AN := 5$$

$$\frac{2\cdot N^2+2\cdot N}{N+1+\sqrt{(N^2+2\cdot N)-3}}-R = 0.00000$$

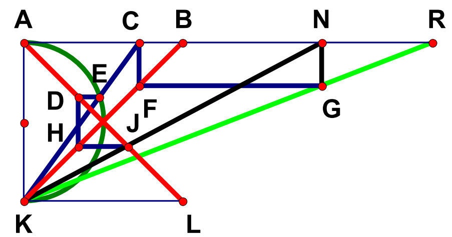


$$\frac{(((N^3+2\cdot N^2)-N^2\cdot\sqrt{(N^2+2\cdot N)-3})+N)-N\cdot\sqrt{(N^2+2\cdot N)-3}}{2}-R = 0.00000$$

$$DN := \frac{AN+1-\left(AN^2+2\cdot AN-3\right)^{\frac{1}{2}}}{2AN+2} \qquad AR := \frac{AN\cdot AB}{AB-DN}$$

$$AR-\frac{2\cdot AN^2+2\cdot AN}{AN+1+\left(AN^2+2\cdot AN-3\right)^{\frac{1}{2}}} = 0$$

$$AR-\frac{AN^3+2\cdot AN^2-AN^2\cdot\left(AN^2+2\cdot AN-3\right)^{\frac{1}{2}}+AN-AN\cdot\left(AN^2+2\cdot AN-3\right)^{\frac{1}{2}}}{2} = 0$$

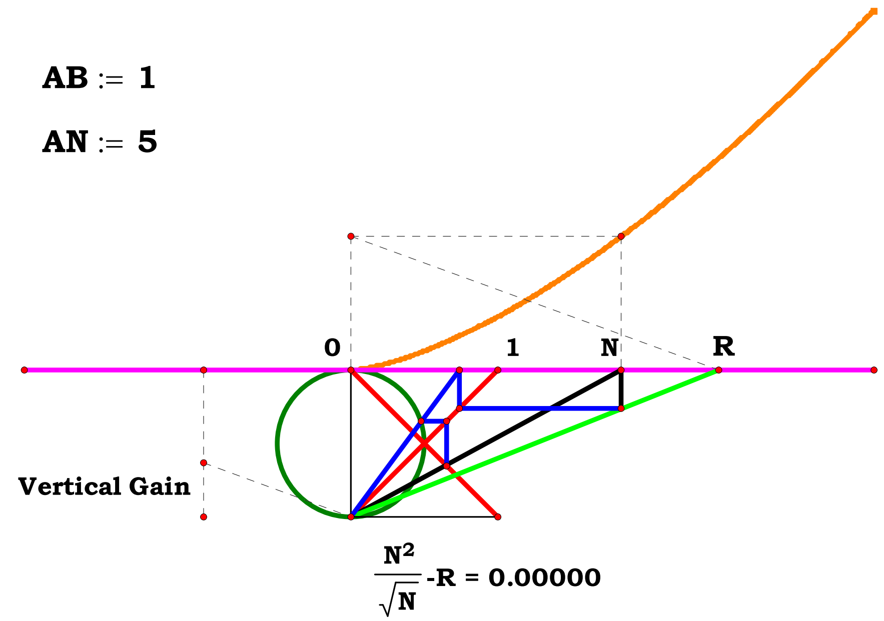


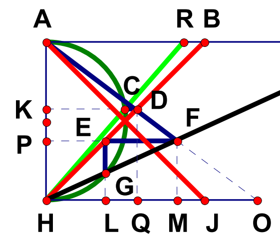
$$AB := 1$$

$$AN := 5$$

$$GN := \frac{AN^{\frac{1}{2}} - 1}{AN^{\frac{1}{2}}} \qquad AR := \frac{AN \cdot AB}{AB - GN} \qquad AR - AN^{\frac{3}{2}} = 0$$

$$AR - \frac{AN^2}{\sqrt{AN}} = 0$$





$$AB := 1$$

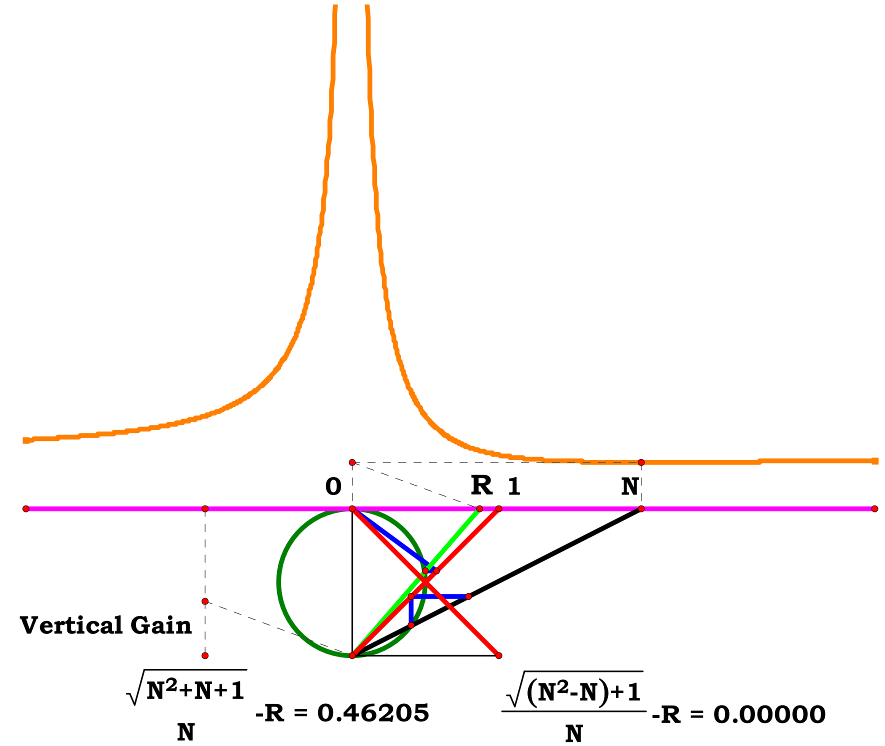
$$AN := 3$$

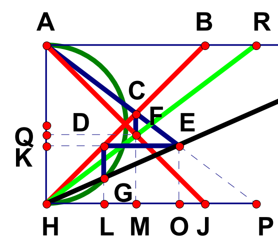
$$HL := \frac{AN}{AN^2 + 1} \qquad HM := \frac{AN \cdot HL}{AB} \qquad HO := \frac{HM \cdot AB}{AB - HL}$$

$$HQ := \frac{HO \cdot AB}{HO + AB} \qquad HK := HQ \qquad AK := AB - HK \qquad CK := \sqrt{AK \cdot HK}$$

$$AR := \frac{CK \cdot AB}{HK} \qquad AR - \frac{\left(AN^2 - AN + 1\right)^{\frac{1}{2}}}{AN} = 0$$

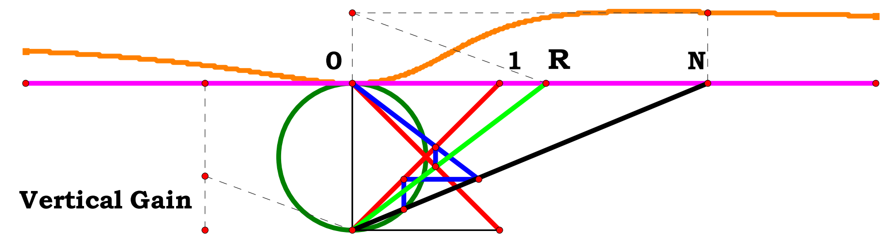
$$HQ - \frac{AN^2}{2 \cdot AN^2 - AN + 1} = 0$$





$\mathbf{AB} := 1$

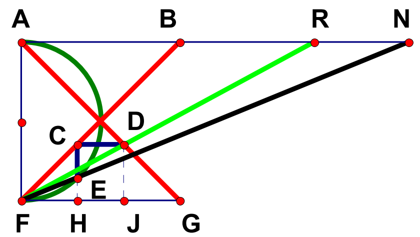
$\mathbf{AN} := 3$



$\frac{N^2}{N^2+N+1}$-R = -0.68904 $\frac{N^2}{(N^2-N)+1}$-R = 0.00000

$$HL := \frac{AN}{AN^2+1} \quad HO := \frac{AN \cdot HL}{AB} \quad HP := \frac{HO \cdot AB}{AB - HL} \quad HM := \frac{HP \cdot AB}{HP + AB}$$

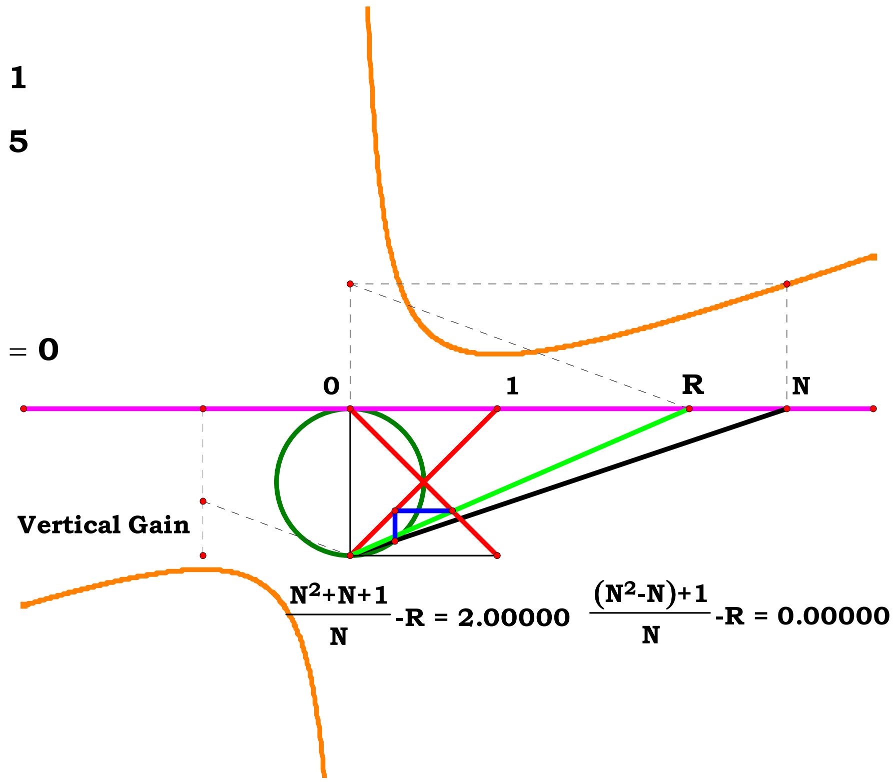
$$JM := AB - HM \quad AR := \frac{HM \cdot AB}{JM} \quad AR - HP = 0 \quad AR - \frac{AN^2}{AN^2 - AN + 1} = 0$$



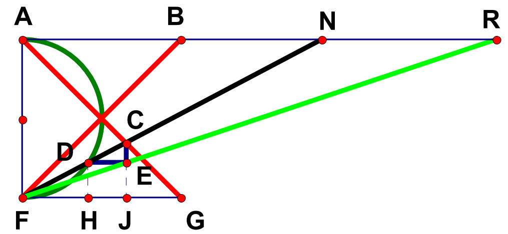
$AB := 1$

$AN := 5$

$$FH := \frac{AN}{AN^2 + 1} \quad FJ := AB - FH \quad AR := \frac{FJ \cdot AB}{FH} \quad AR - \frac{AN^2 - AN + 1}{AN} = 0$$



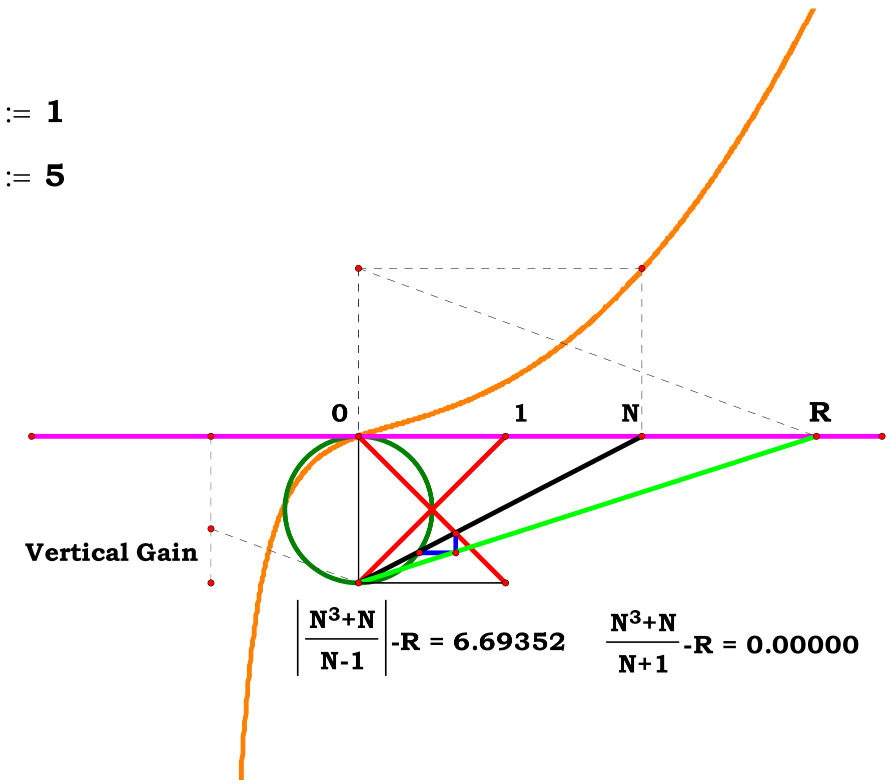
$$\frac{N^2+N+1}{N}\text{-R} = 2.00000 \qquad \frac{(N^2\text{-}N)+1}{N}\text{-R} = 0.00000$$

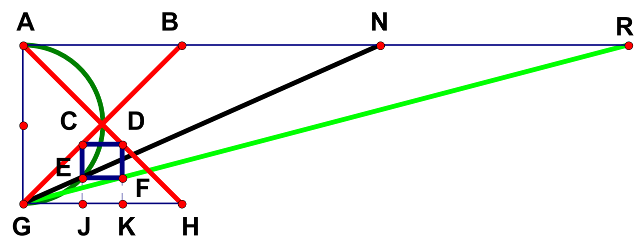


$\mathbf{AB} := 1$

$\mathbf{AN} := 5$

$$\mathbf{FJ} := \frac{\mathbf{AN}}{\mathbf{AN}+1} \qquad \mathbf{DH} := \frac{1}{\mathbf{AN}^2+1} \qquad \mathbf{AR} := \frac{\mathbf{FJ}\cdot\mathbf{AB}}{\mathbf{DH}} \qquad \mathbf{AR} - \frac{\mathbf{AN}^3+\mathbf{AN}}{\mathbf{AN}+1}$$



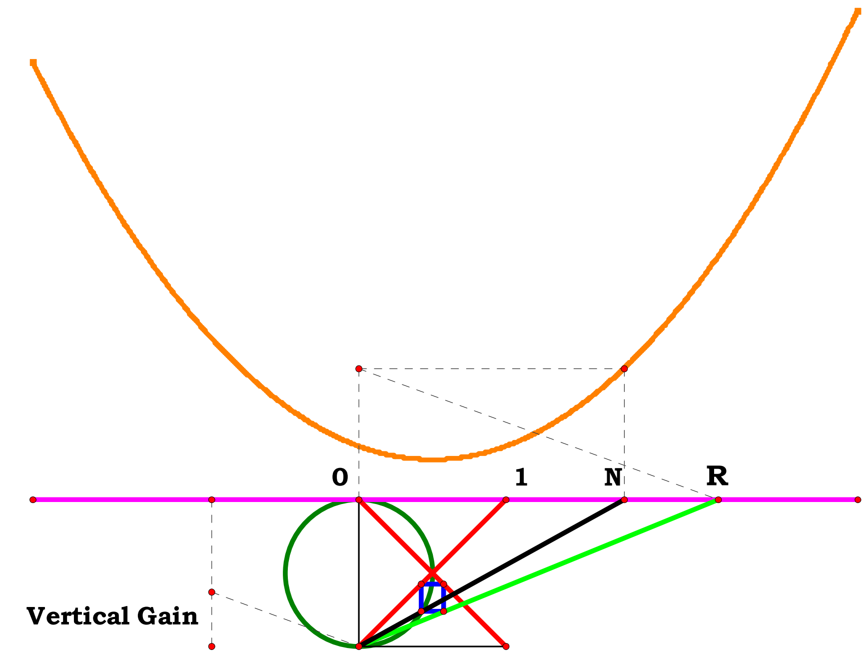


$AB := 1$

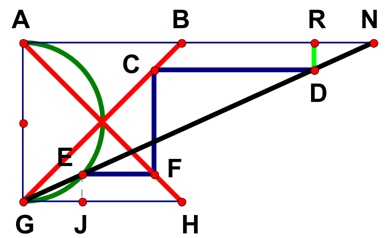
$AN := 5$

$$GJ := \frac{AN}{AN^2 + 1} \qquad EJ := \frac{1}{AN^2 + 1} \qquad GK := AB - GJ \qquad AR := \frac{GK \cdot AB}{EJ}$$

$$AR - \left(AN^2 - AN + 1\right) = 0$$

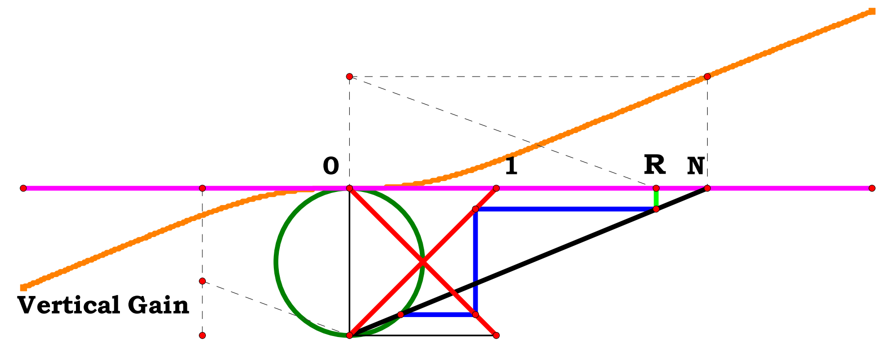


$(N^2+N+1)-R = 3.60317$ $((N^2-N)+1)-R = 0.00000$



$AB := 1$

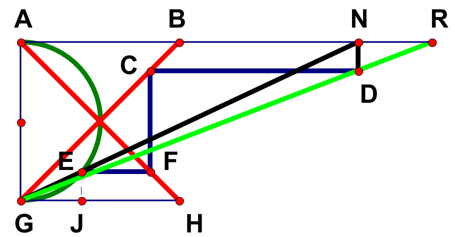
$AN := 5$



$$\frac{N^3}{N^2+1} - R = 0.00000$$

$$EJ := \frac{1}{AN^2 + 1} \qquad DR := EJ \qquad NR := \frac{AN \cdot DR}{AB} \qquad AR := AN - NR$$

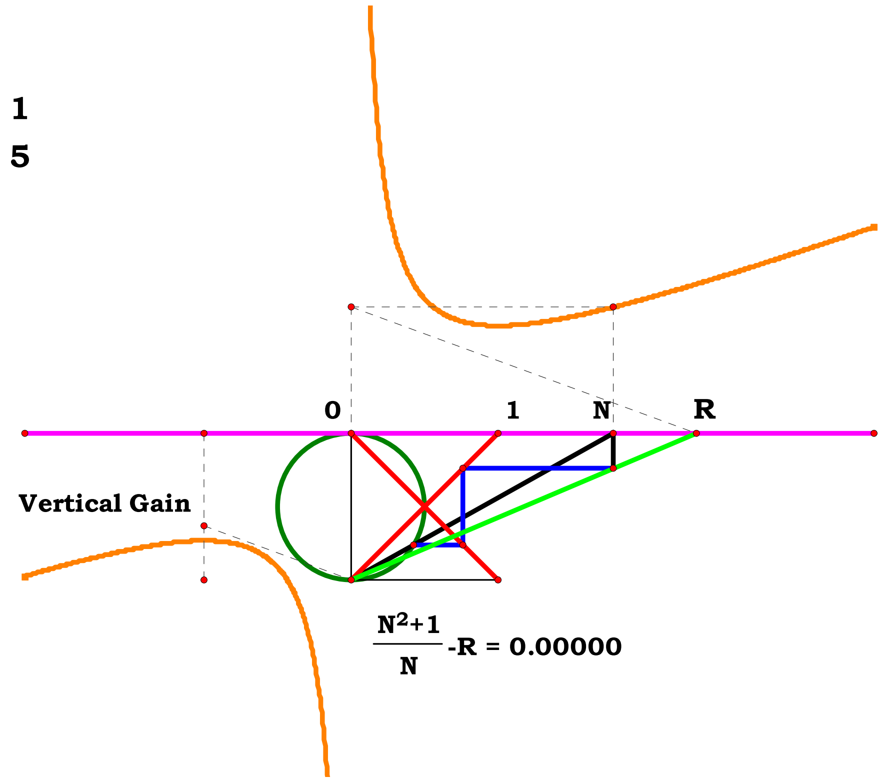
$$AR - \frac{AN^3}{AN^2 + 1} = 0$$



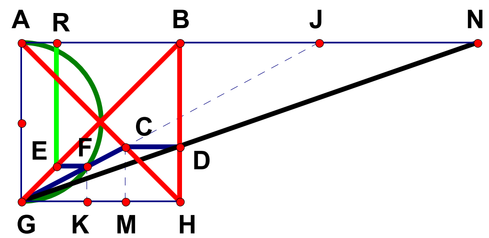
$AB := 1$

$AN := 5$

$EJ := \frac{1}{AN^2 + 1}$ $\quad DN := EJ$ $\quad AR := \frac{AN \cdot AB}{AB - EJ}$ $\quad AR - \frac{AN^2 + 1}{AN} = 0$

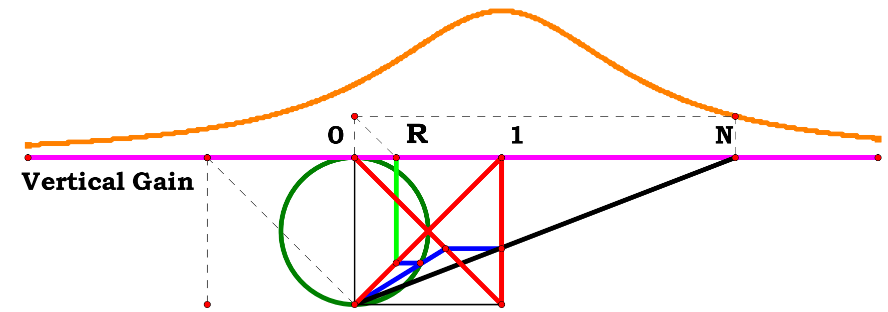


$\frac{N^2+1}{N}$-R = 0.00000



$AB := 1$

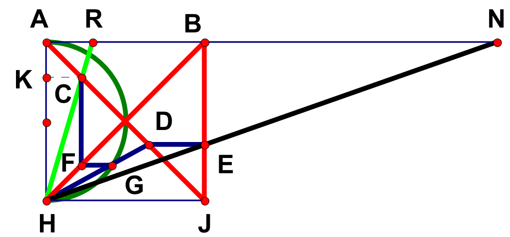
$AN := 5$



$\dfrac{1}{N^2+2\cdot N+2}$-R = -0.21093 $\quad$ $\dfrac{1}{(N^2-2\cdot N)+2}$-R = 0.00000

$DH := \dfrac{1}{AN} \quad GM := AB - DH \quad AJ := \dfrac{GM \cdot AB}{DH} \quad FK := \dfrac{1}{AJ^2+1}$

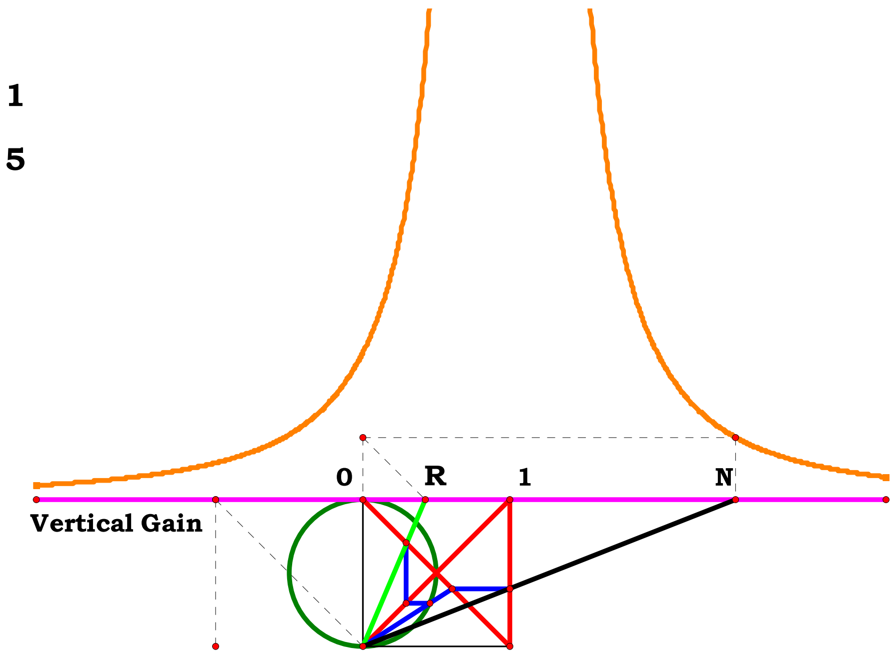
$AR := FK \quad AR - \dfrac{1}{AN^2 - 2\cdot AN + 2} = 0$



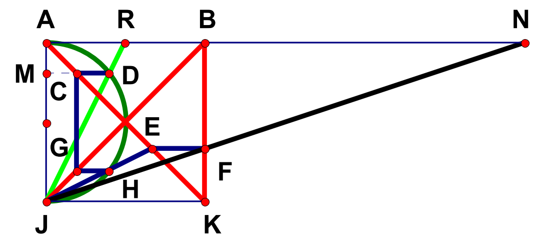
$AB := 1$

$AN := 5$

$$CK := \frac{1}{AN^2 - 2 \cdot AN + 2} \qquad AR := \frac{CK \cdot AB}{AB - CK} \qquad AR - \frac{1}{AN^2 - 2 \cdot AN + 1} = 0$$



$$\frac{1}{N^2+2 \cdot N+1} - R = -0.34287 \qquad \frac{1}{(N^2-2 \cdot N)+1} - R = 0.00000$$

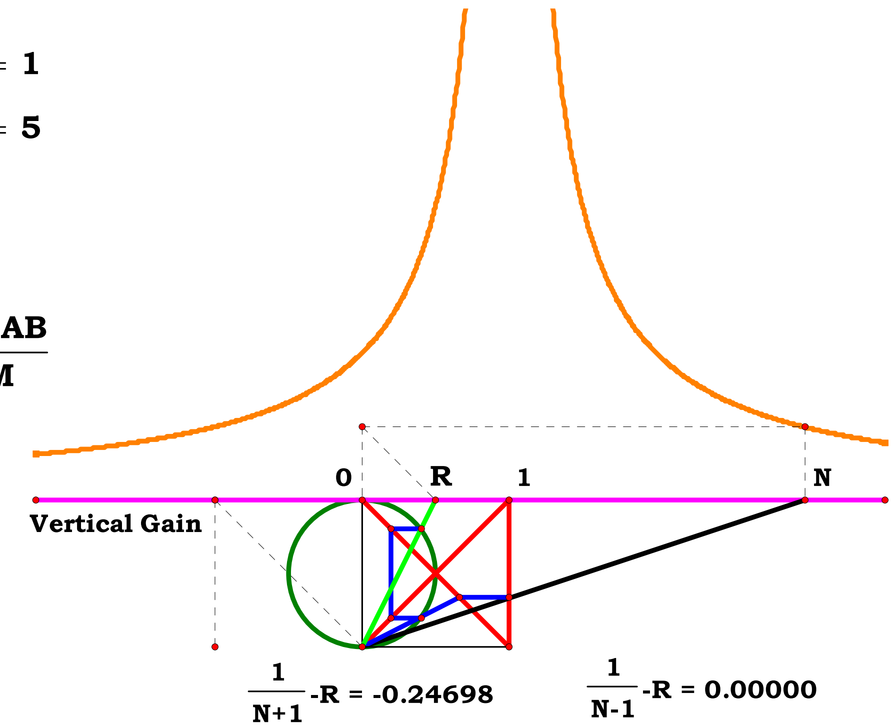


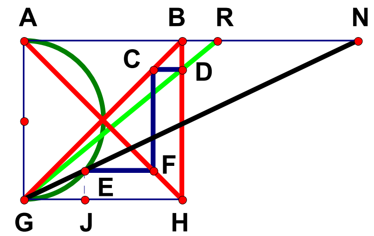
$AB := 1$

$AN := 5$

$$AM := \frac{1}{AN^2 - 2 \cdot AN + 2} \qquad JM := AB - AM \qquad DM := \sqrt{AM \cdot JM} \qquad AR := \frac{DM \cdot AB}{JM}$$

$$AR - \frac{1}{AN - 1} = 0$$



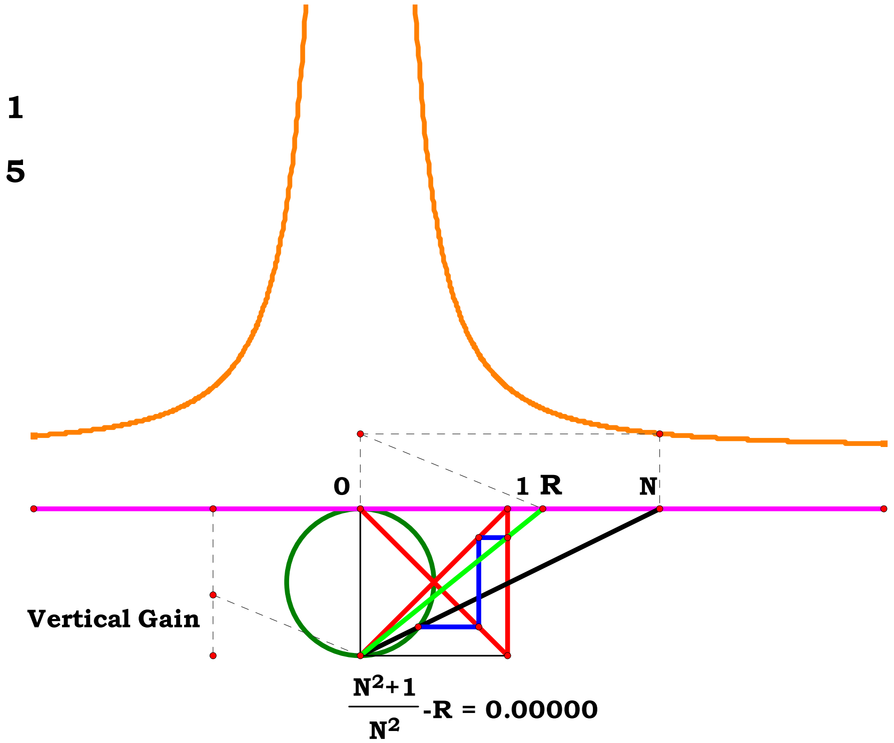


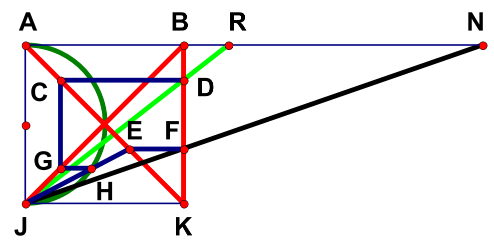
$AB := 1$

$AN := 5$

$EJ := \dfrac{1}{AN^2 + 1}$ $\quad BD := EJ$ $\quad DH := AB - BD$ $\quad AR := \dfrac{AB^2}{DH}$

$AR - \dfrac{AN^2 + 1}{AN^2} = 0$



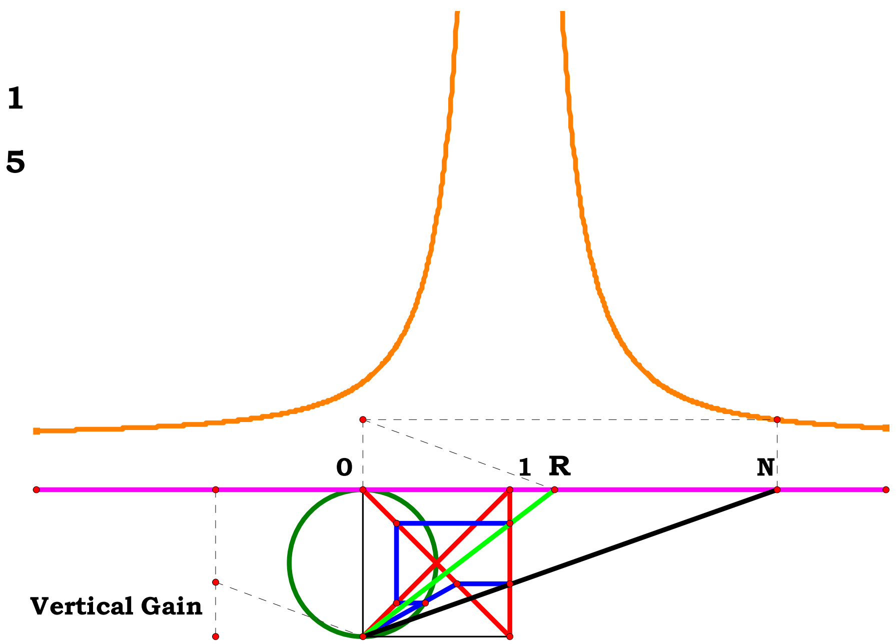


$$\mathbf{AB} := 1$$

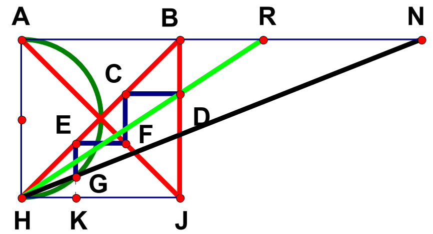
$$\mathbf{AN} := 5$$

$$\mathbf{BD} := \frac{1}{\mathbf{AN}^2 - 2\cdot\mathbf{AN} + 2} \qquad \mathbf{DK} := \mathbf{AB} - \mathbf{BD} \qquad \mathbf{AR} := \frac{\mathbf{AB}^2}{\mathbf{DK}}$$

$$\mathbf{AR} - \frac{\mathbf{AN}^2 - 2\cdot\mathbf{AN} + 2}{\mathbf{AN}^2 - 2\cdot\mathbf{AN} + 1} = 0$$

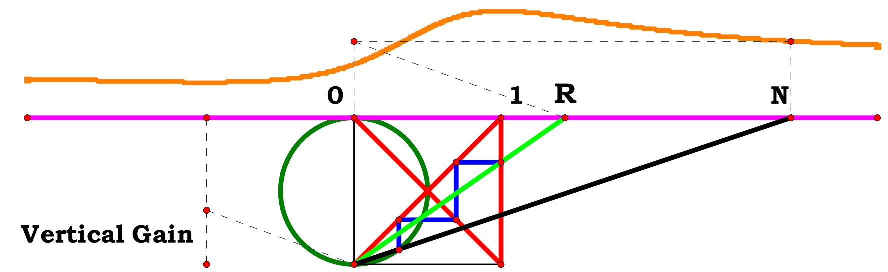


$$\frac{N^2+2\cdot N+2}{N^2+2\cdot N+1}\text{-R} = -0.23319 \qquad \frac{(N^2\text{-}2\cdot N)+2}{(N^2\text{-}2\cdot N)+1}\text{-R} = 0.00000$$



$AB := 1$

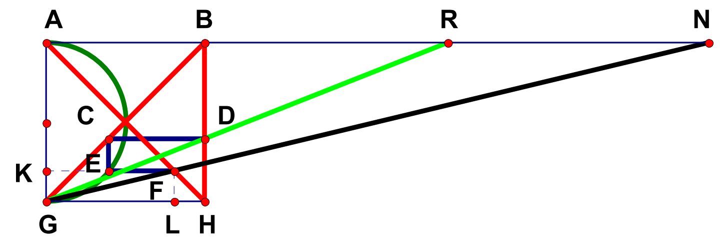
$AN := 5$



$$\frac{N^2+1}{N^2+N+1}\text{-}R = -0.66512 \qquad \frac{N^2+1}{(N^2\text{-}N)+1}\text{-}R = 0.00000$$

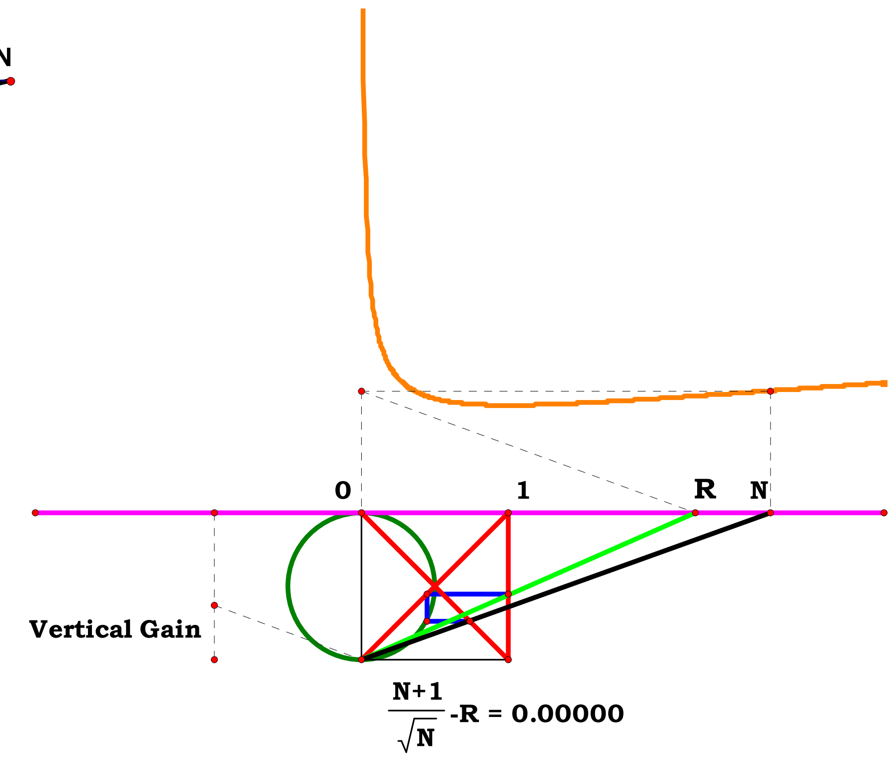
$$HK := \frac{AN}{AN^2+1} \qquad BD := HK \qquad DJ := AB - BD \qquad AR := \frac{AB^2}{DJ}$$

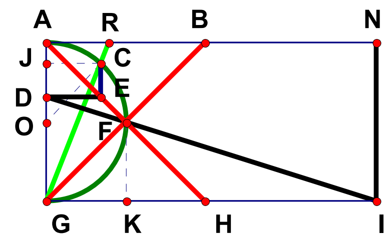
$$AR - \frac{AN^2+1}{AN^2-AN+1} = 0$$



$$AB := 1 \quad AN := 5 \quad FL := \frac{1}{AN+1} \quad GK := FL \quad AK := AB - GK$$

$$EK := \sqrt{AK \cdot GK} \quad DH := EK \quad AR := \frac{AB^2}{DH} \quad AR - \frac{AN+1}{\sqrt{AN}} = 0$$

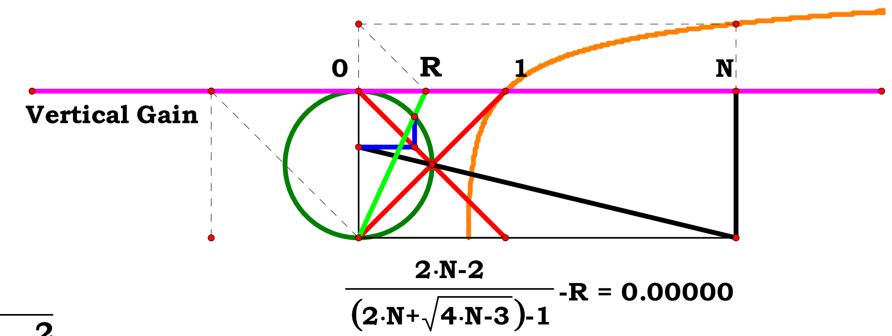




$AB := 1$

$AN := 5$

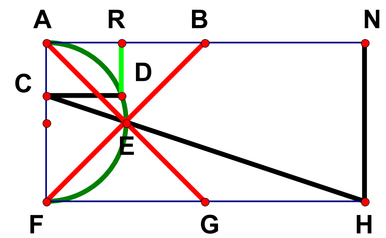
$GK := \dfrac{AB}{2}$



$DG := \dfrac{GK \cdot AN}{AN - GK}$ $\quad AD := AB - DG \quad CJ := AD \quad CO := \dfrac{AB}{2} \quad JO := \sqrt{CO^2 - CJ^2}$

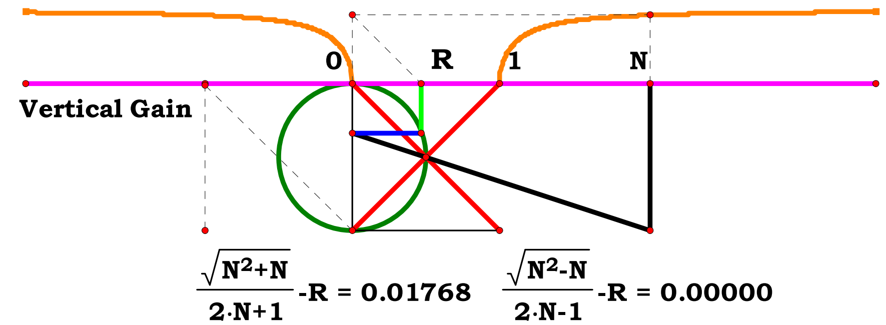
$GJ := GK + JO \quad AR := \dfrac{CJ \cdot AB}{GJ} \quad AR - \dfrac{2 \cdot AN - 2}{2AN + \sqrt{4AN - 3} - 1} = 0$

$DG - \dfrac{AN}{2 \cdot AN - 1} = 0$



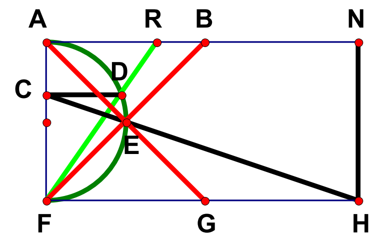
$AB := 1$

$AN := 3$



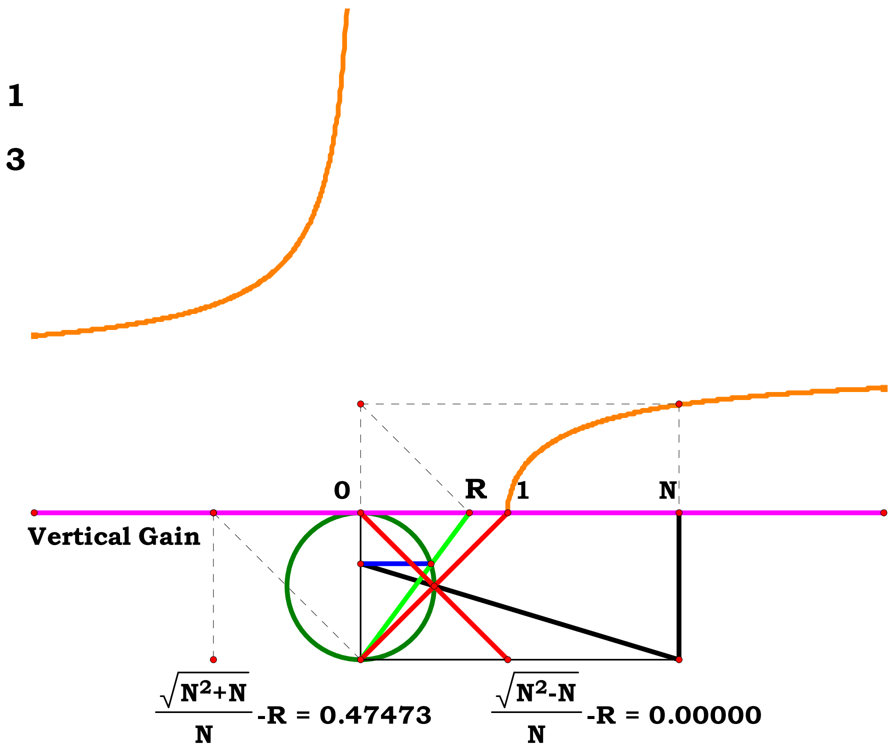
$CF := \frac{AN}{2 \cdot AN - 1}$ $\quad AC := AB - CF$ $\quad CD := \sqrt{CF \cdot AC}$ $\quad AR := CD$

$$AR - \frac{\left(AN^2 - AN\right)^{\frac{1}{2}}}{2 \cdot AN - 1} = 0$$



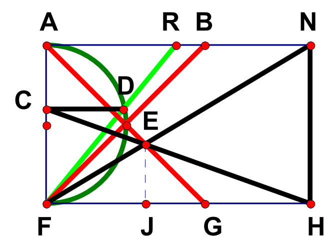
$$AB := 1$$

$$AN := 3$$



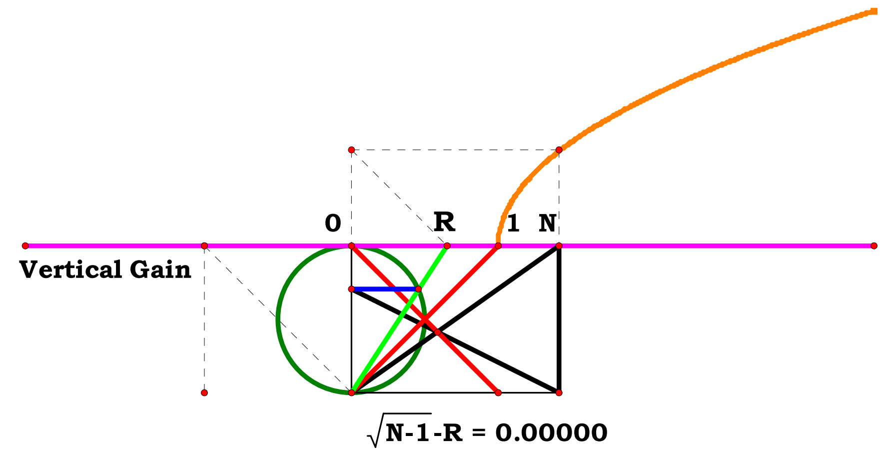
$$CF := \frac{AN}{2 \cdot AN - 1} \qquad CD := \frac{\left(AN^2 - AN\right)^{\frac{1}{2}}}{2 \cdot AN - 1} \qquad AR := \frac{CD \cdot AB}{CF}$$

$$AR - \frac{\left(AN^2 - AN\right)^{\frac{1}{2}}}{AN} = 0$$



$$AB := 1$$

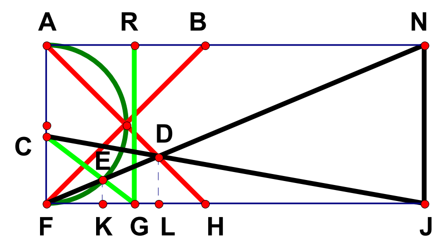
$$AN := 5$$



$$GJ := \frac{1}{AN + 1} \quad FJ := AB - GJ \quad HJ := AN - FJ \quad CF := \frac{GJ \cdot AN}{HJ}$$

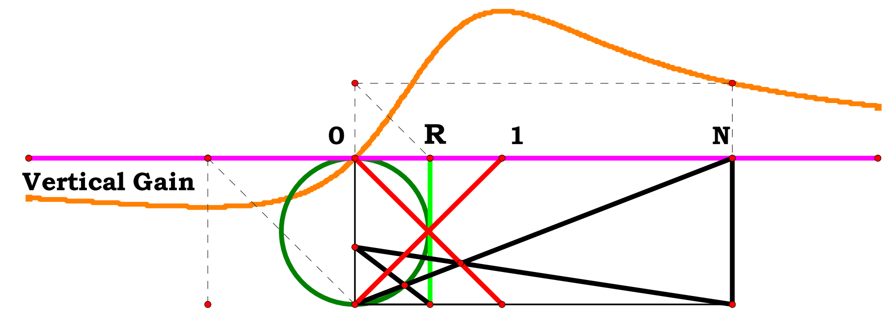
$$AC := AB - CF \quad CD := \sqrt{AC \cdot CF} \quad AR := \frac{CD \cdot AB}{CF} \quad AR - \sqrt{AN - 1} = 0$$

$$CF - \frac{1}{AN} = 0$$



$AB := 1$

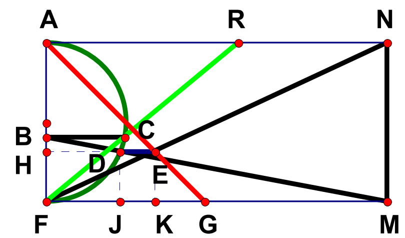
$AN := 5$



$\frac{N}{N^2+N+1}-R = -0.25776$ $\frac{N}{(N^2-N)+1}-R = 0.00000$

$CF := \frac{1}{AN}$ $EK := \frac{1}{AN^2+1}$ $FK := \frac{AN}{AN^2+1}$ $FG := \frac{FK \cdot CF}{CF - EK}$ $AR := FG$

$$AR - \frac{AN}{AN^2 - AN + 1} = 0$$



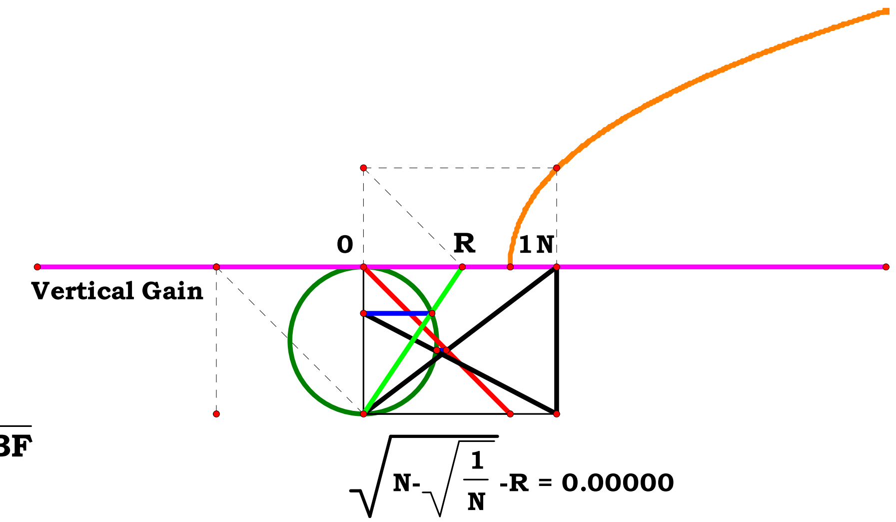
$$AF := 1$$

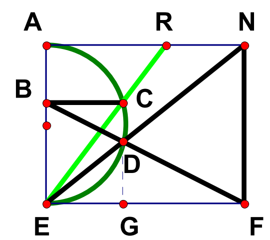
$$AN := 5$$

$$GK := \frac{1}{AN+1} \qquad FH := GK \qquad AH := AF - FH \qquad DH := \sqrt{AH \cdot FH}$$

$$FJ := DH \qquad JM := AN - FJ \qquad BF := \frac{GK \cdot AN}{JM} \qquad AB := AF - BF \qquad BC := \sqrt{AB \cdot BF}$$

$$AR := \frac{BC \cdot AF}{BF} \qquad AR - \sqrt{AN - \frac{1}{\sqrt{AN}}} = 0$$



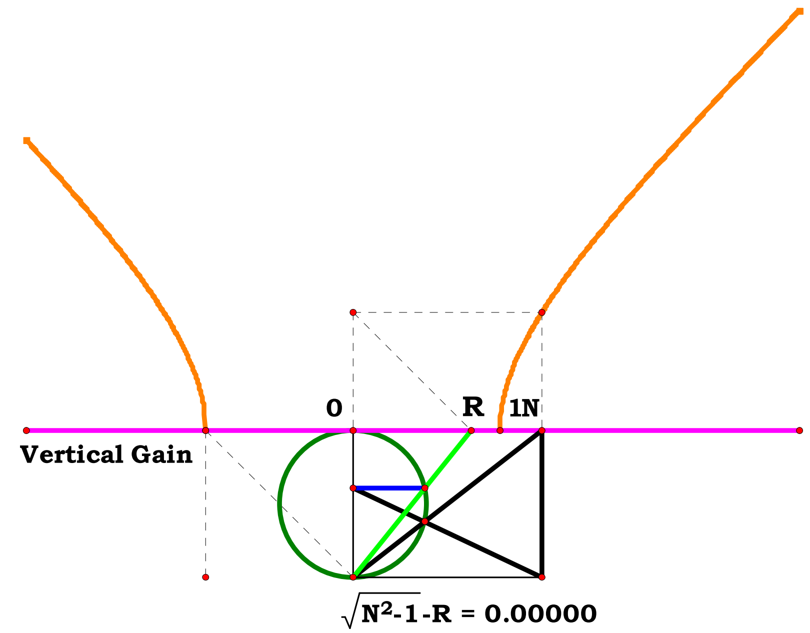


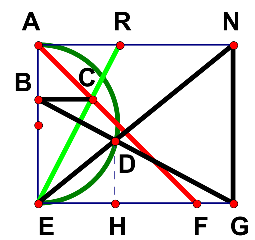
$\mathbf{AE} := 1$

$\mathbf{AN} := 3$

$\mathbf{DG} := \dfrac{1}{\mathbf{AN}^2 + 1}$ $\quad \mathbf{EG} := \dfrac{\mathbf{AN}}{\mathbf{AN}^2 + 1}$ $\quad \mathbf{FG} := \mathbf{AN} - \mathbf{EG}$ $\quad \mathbf{BE} := \dfrac{\mathbf{DG} \cdot \mathbf{AN}}{\mathbf{FG}}$

$\mathbf{AB} := \mathbf{AE} - \mathbf{BE}$ $\quad \mathbf{BC} := \sqrt{\mathbf{AB} \cdot \mathbf{BE}}$ $\quad \mathbf{AR} := \dfrac{\mathbf{BC} \cdot \mathbf{AE}}{\mathbf{BE}}$ $\quad \mathbf{AR} - \sqrt{\mathbf{AN}^2 - 1} = 0$



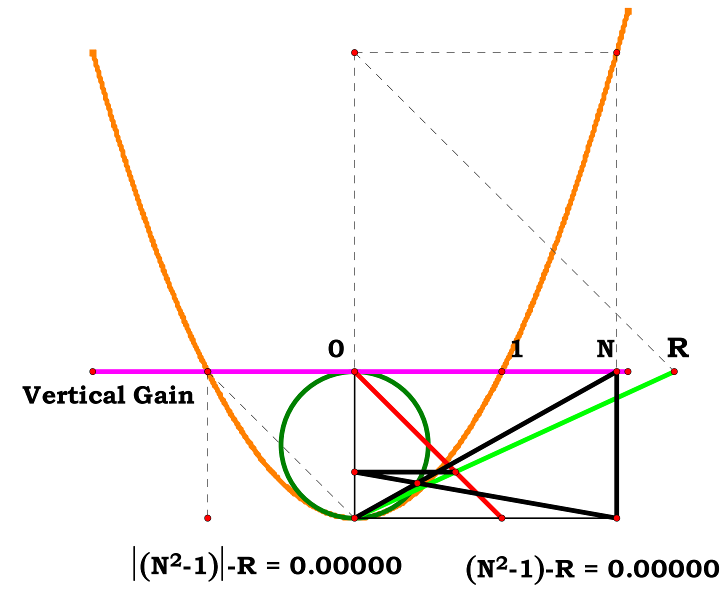


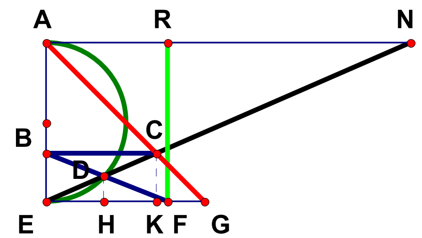
$AE := 1$

$AN := 5$

$$DH := \frac{1}{AN^2 + 1} \qquad EH := \frac{AN}{AN^2 + 1} \qquad GH := AN - EH$$

$$BE := \frac{DH \cdot AN}{GH} \qquad AB := AE - BE \qquad AR := \frac{AB \cdot AE}{BE} \qquad AR - \left(AN^2 - 1\right) = 0$$



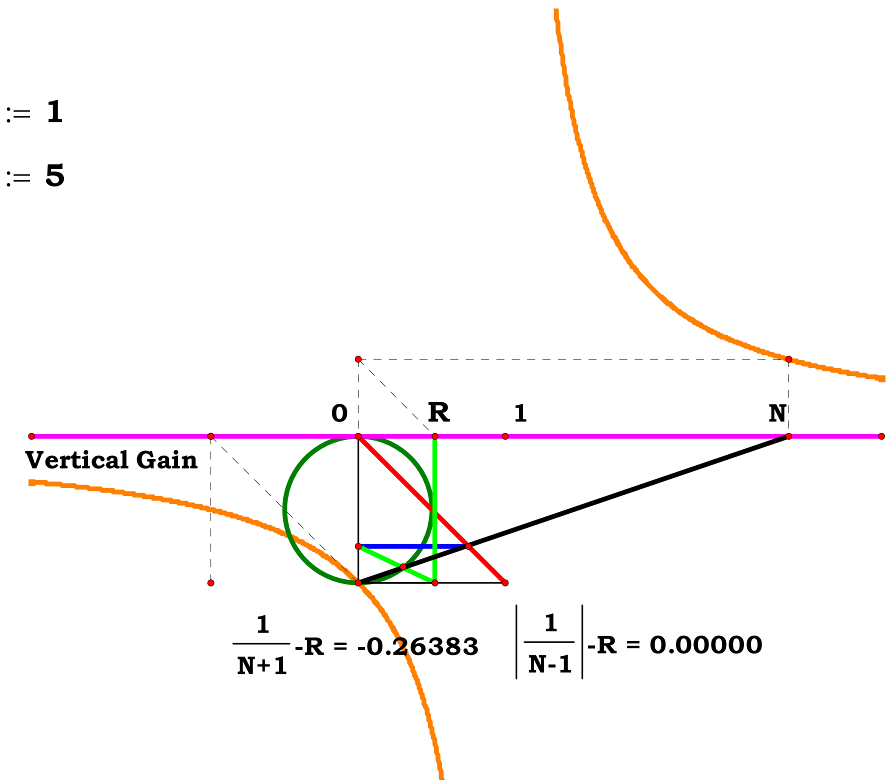


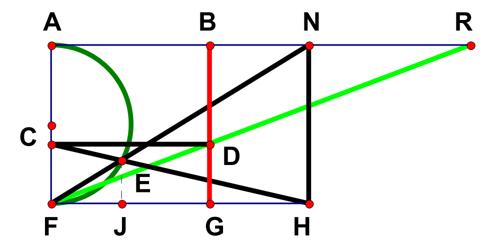
$AB := 1$

$AN := 5$

$DH := \dfrac{1}{AN^2 + 1} \qquad EH := \dfrac{AN}{AN^2 + 1} \qquad GK := \dfrac{1}{AN + 1} \qquad BE := GK$

$EF := \dfrac{EH \cdot BE}{BE - DH} \qquad AR := EF \qquad AR - \dfrac{1}{AN - 1} = 0$



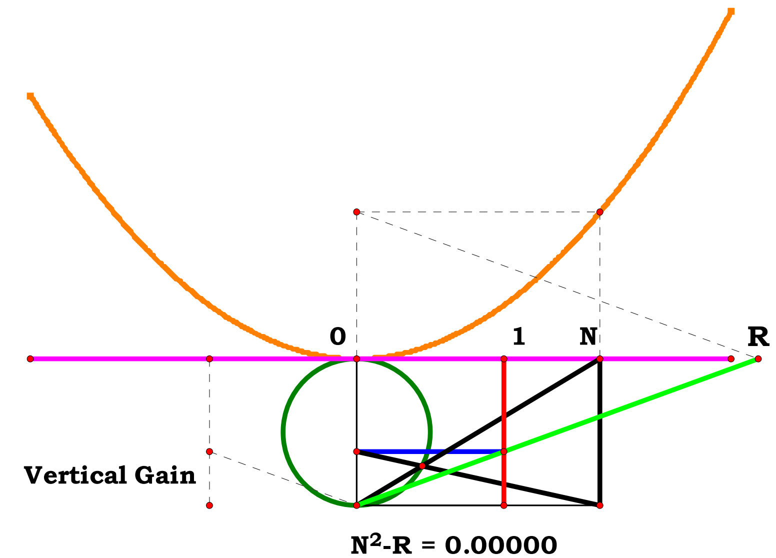


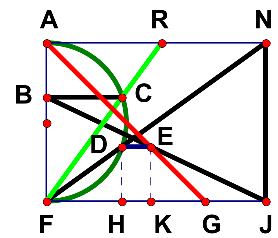
$AB := 1$

$AN := 3$

$EJ := \dfrac{1}{AN^2 + 1}$ $\quad FJ := \dfrac{AN}{AN^2 + 1}$ $\quad HJ := AN - FJ$ $\quad CF := \dfrac{EJ \cdot AN}{HJ}$

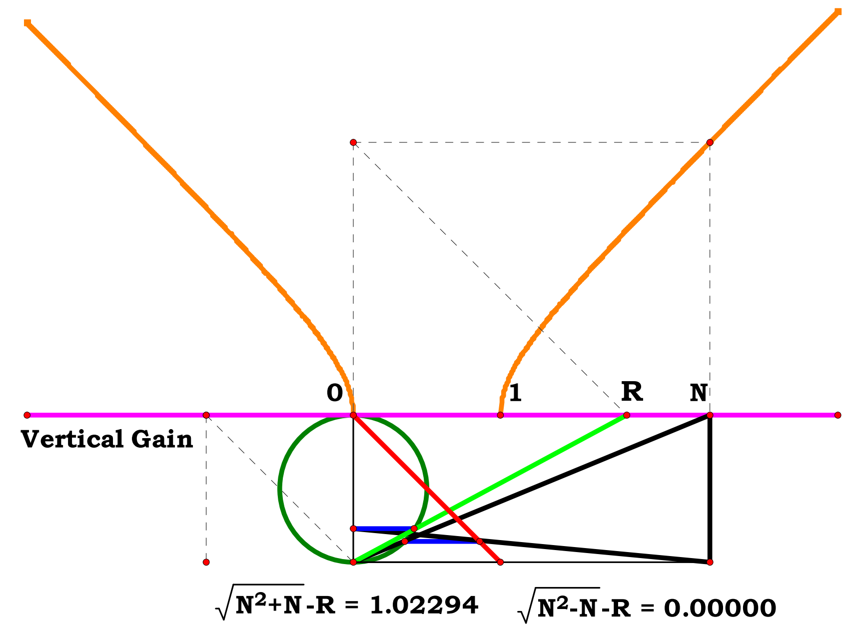
$DG := CF$ $\quad AR := \dfrac{AB^2}{DG}$ $\quad AR - AN^2 = 0$





$AF := 1$

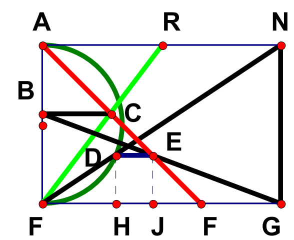
$AN := 3$



$$DH := \frac{1}{AN^2+1} \quad JK := AN - AF + DH \quad EK := DH \quad BF := \frac{EK \cdot AN}{JK}$$

$$AB := AF - BF \quad BC := \sqrt{AB \cdot BF} \quad AR := \frac{BC \cdot AF}{BF} \quad AR - \sqrt{AN^2 - AN} = 0$$

$$BF - \frac{1}{AN^2 - AN + 1} = 0$$

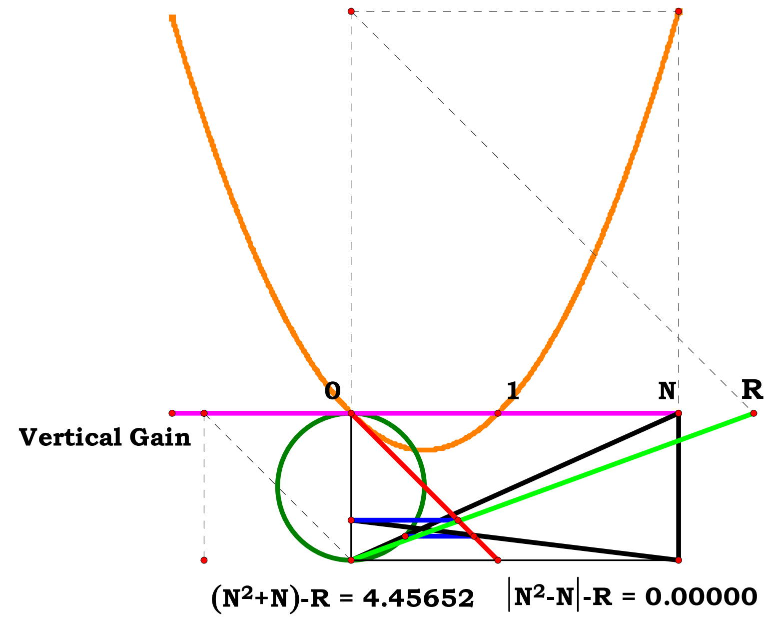


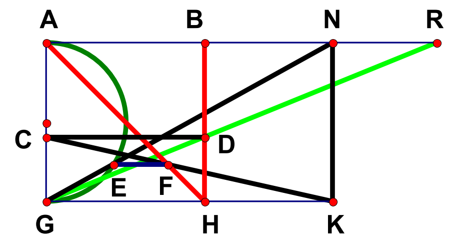
$$AF := 1$$

$$AN := 3$$

$$BF := \frac{1}{AN^2 - AN + 1} \qquad BC := AF - BF \qquad AR := \frac{BC \cdot AF}{BF}$$

$$AR - \left(AN^2 - AN\right) = 0$$

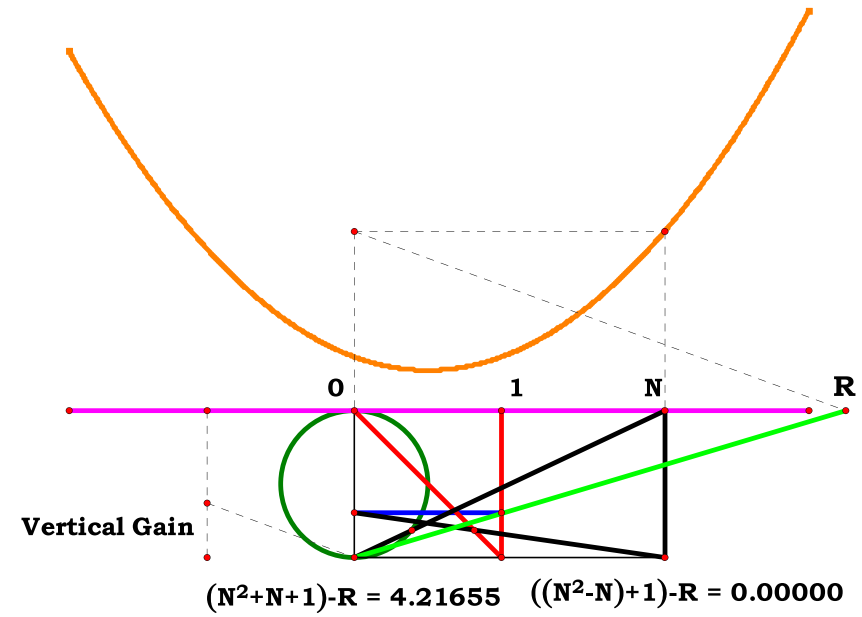


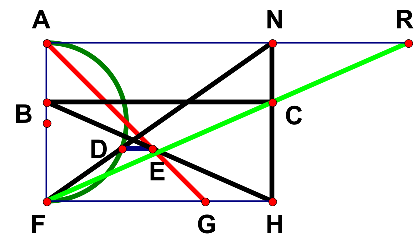


$AB := 1$

$AN := 5$

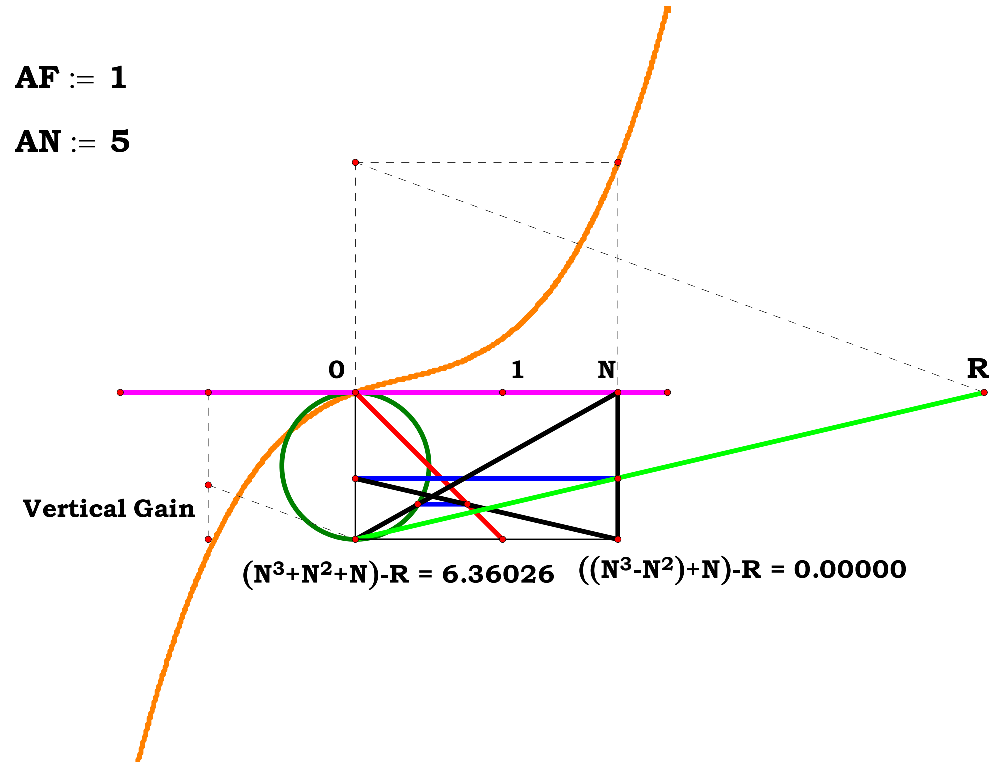
$$CG := \frac{1}{AN^2 - AN + 1} \qquad DH := CG \qquad AR := \frac{AB^2}{CG} \qquad AR - \left(AN^2 - AN + 1\right) = 0$$





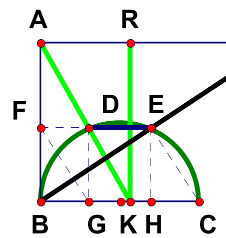
$$AF := 1$$

$$AN := 5$$



$$BF := \frac{1}{AN^2 - AN + 1} \qquad CH := BF \qquad AR := \frac{AN \cdot AF}{CH}$$

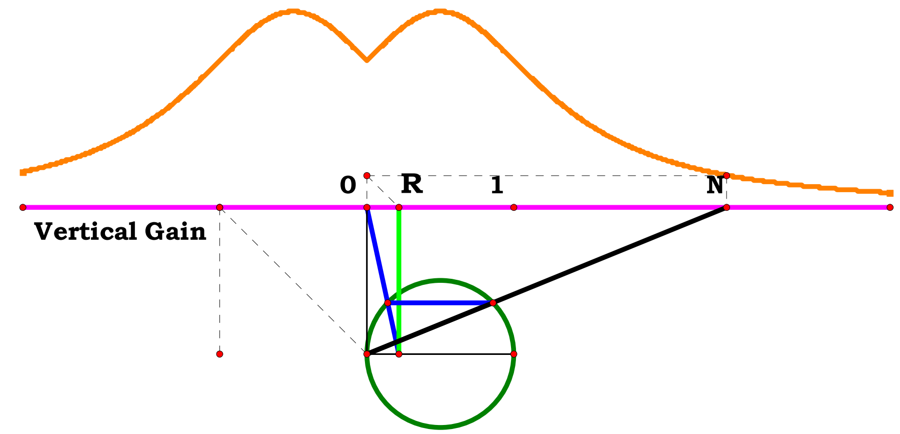
$$AR - \left(AN^3 - AN^2 + AN\right) = 0$$



$$AB := 1$$

$$AN := 3$$

$$BN := \sqrt{AN^2 + AB^2}$$

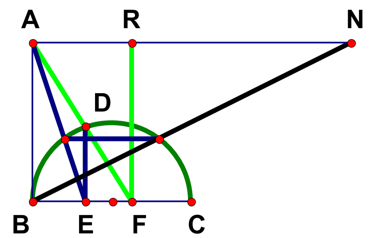


$$\frac{1}{(N^2 - N) + 1} - R = 0.00000$$

$$BE := \frac{AN \cdot AB}{BN} \quad EH := \frac{AB \cdot BE}{BN} \quad CE := \frac{AB^2}{BN} \quad CH := \sqrt{CE^2 - EH^2} \quad BG := CH$$

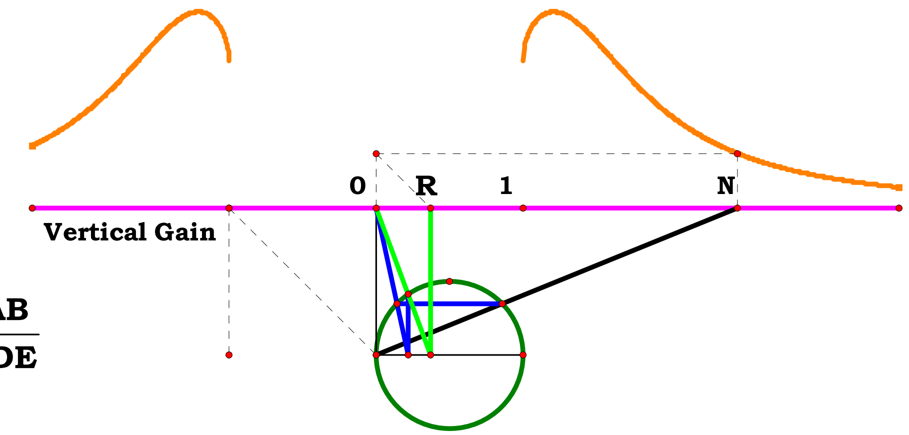
$$BK := \frac{BG \cdot AB}{AB - EH} \quad AR := BK \quad AR - \frac{1}{AN^2 - AN + 1} = 0$$

$$CH - \frac{1}{AN^2 + 1} = 0 \qquad EH - \frac{AN}{AN^2 + 1} = 0$$



$$AB := 1$$

$$AN := 3$$

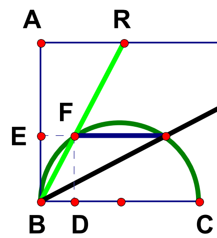


$$\frac{1}{\left(N^2-N-\sqrt{N^2-N}\right)+1}-R = 0.00000$$

$$BE := \frac{1}{AN^2 - AN + 1} \qquad CE := AB - BE \qquad DE := \sqrt{BE \cdot CE} \qquad BF := \frac{BE \cdot AB}{AB - DE}$$

$$AR := BF \qquad AR - \frac{1}{AN^2 - AN - \left(AN^2 - AN\right)^{\frac{1}{2}} + 1} = 0$$

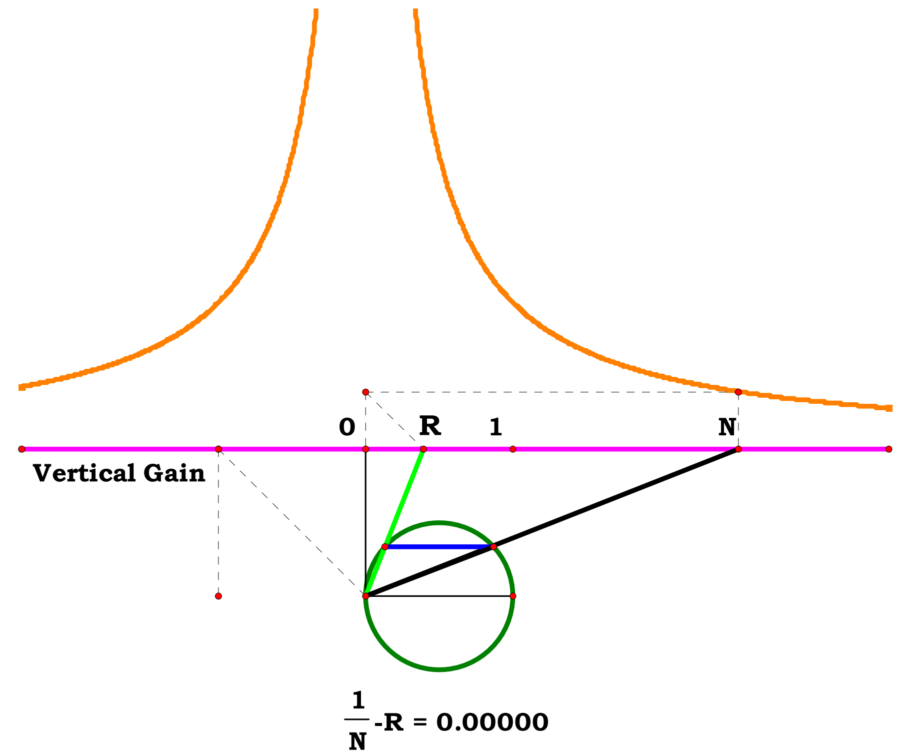
$$DE - \frac{\left(AN^2 - AN\right)^{\frac{1}{2}}}{AN^2 - AN + 1} = 0$$

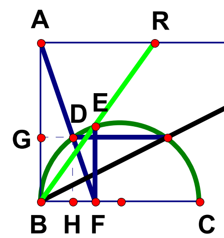


$$AB := 1$$

$$AN := 3$$

$$BD := \frac{1}{AN^2 + 1} \qquad DF := \frac{AN}{AN^2 + 1} \qquad AR := \frac{BD \cdot AB}{DF} \qquad AR - \frac{1}{AN} = 0$$



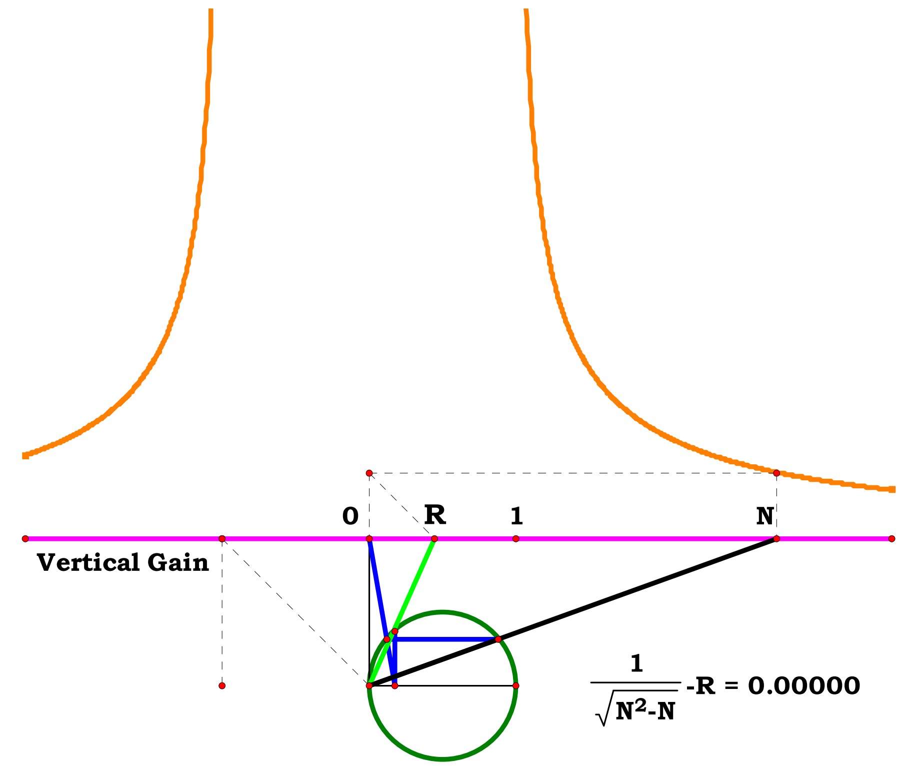


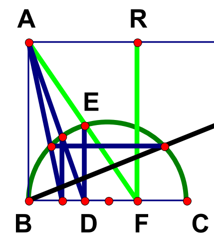
$$AB := 1$$

$$AN := 3$$

$$DG := \frac{1}{AN^2 + 1} \qquad AG := AB - \frac{AN}{AN^2 + 1} \qquad BF := \frac{DG \cdot AB}{AG} \qquad CF := AB - BF$$

$$EF := \sqrt{BF \cdot CF} \qquad AR := \frac{BF \cdot AB}{EF} \qquad AR - \frac{1}{\sqrt{AN^2 - AN}} = 0$$

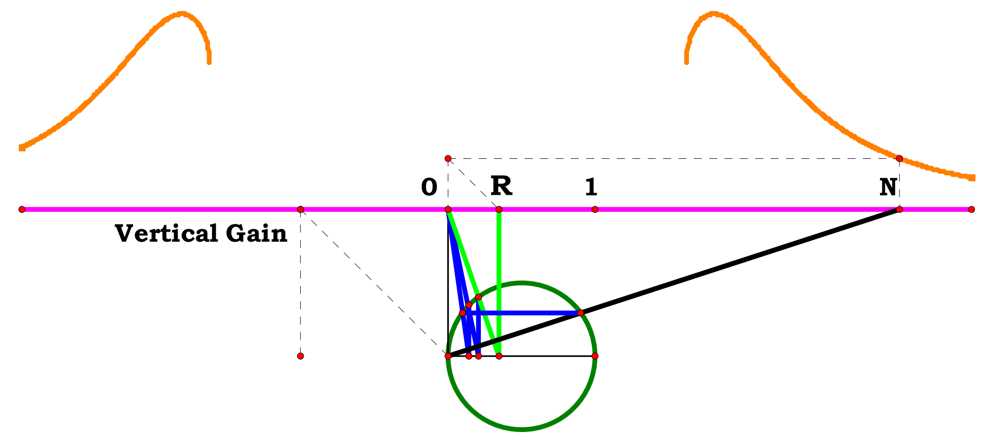




$$AB := 1$$

$$AN := 5$$

$$\frac{1}{\left(N^2 - N - \sqrt{N^2 - N} - \sqrt{N^2 - N - \sqrt{N^2 - N}}\right) + 1} - R = 0.00000$$

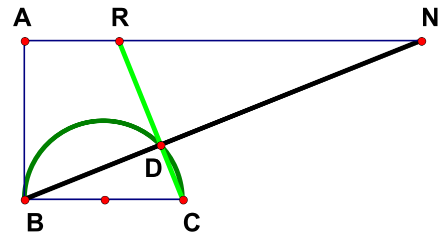


$$BD := \frac{1}{AN^2 - AN - \left(AN^2 - AN\right)^{\frac{1}{2}} + 1} \qquad CD := AB - BD \qquad DE := \sqrt{BD \cdot CD}$$

$$BF := \frac{BD \cdot AB}{AB - DE} \qquad AR := BF$$

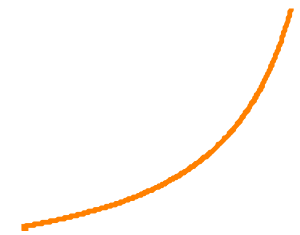
$$AR - \frac{1}{AN^2 - AN + 1 - \left(AN^2 - AN\right)^{\frac{1}{2}} - \left[AN^2 - AN - \left(AN^2 - AN\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}} = 0$$

$$DE - \frac{\left[AN^2 - AN - \left(AN^2 - AN\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}}{AN^2 - AN - \left(AN^2 - AN\right)^{\frac{1}{2}} + 1} = 0$$



$$AB := 1$$

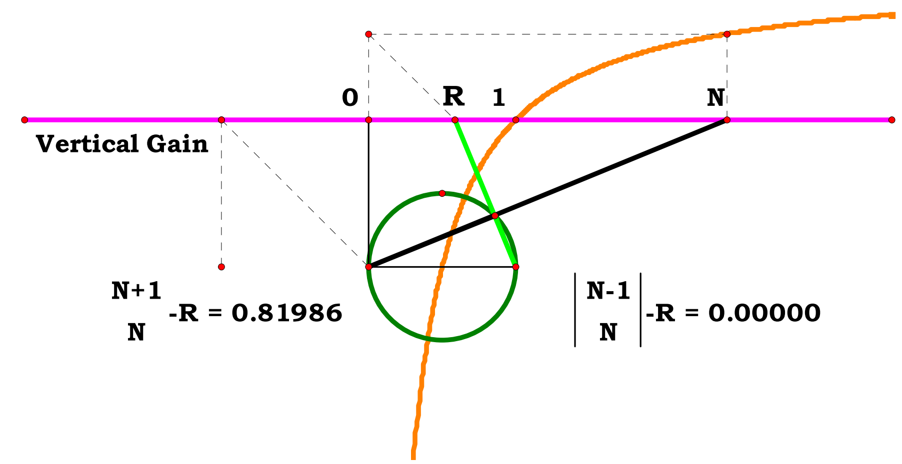
$$AN := 3$$

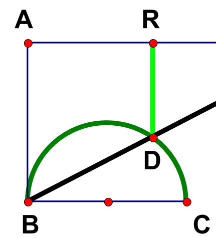


$$BN := \sqrt{AN^2 + AB^2} \qquad BD := \frac{AN \cdot AB}{BN} \qquad DN := BN - BD \qquad NR := \frac{BN \cdot DN}{AN}$$

$$AR := AN - NR \qquad AR - \frac{AN - 1}{AN} = 0$$

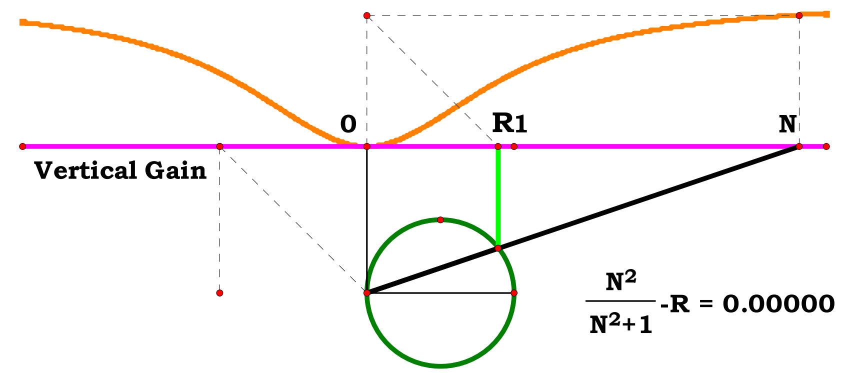
$$BD - \frac{AN}{\left(AN^2 + 1\right)^{\frac{1}{2}}} = 0$$





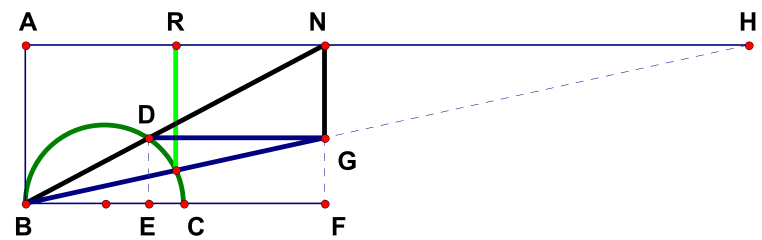
$AB := 1$

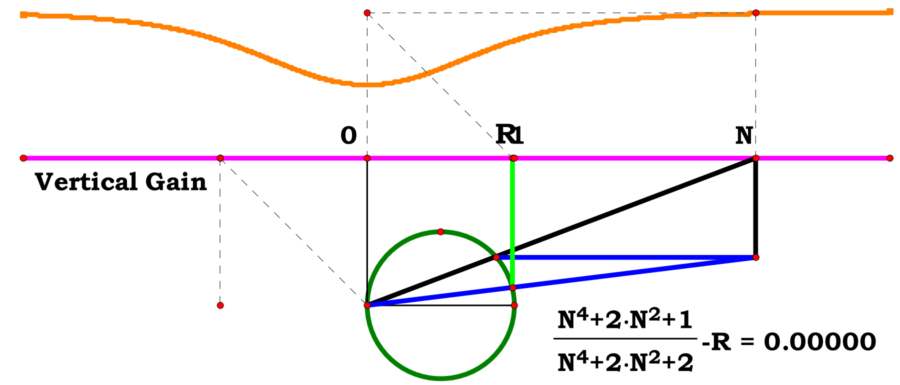
$AN := 3$



$BD := \dfrac{AN}{\left(AN^2+1\right)^{\frac{1}{2}}}$ $\quad BN := \sqrt{AN^2+AB^2}$ $\quad AR := \dfrac{AN \cdot BD}{BN}$

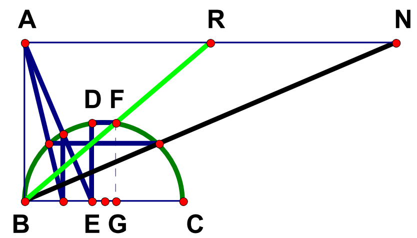
$$AR - \frac{AN^2}{AN^2+1} = 0$$





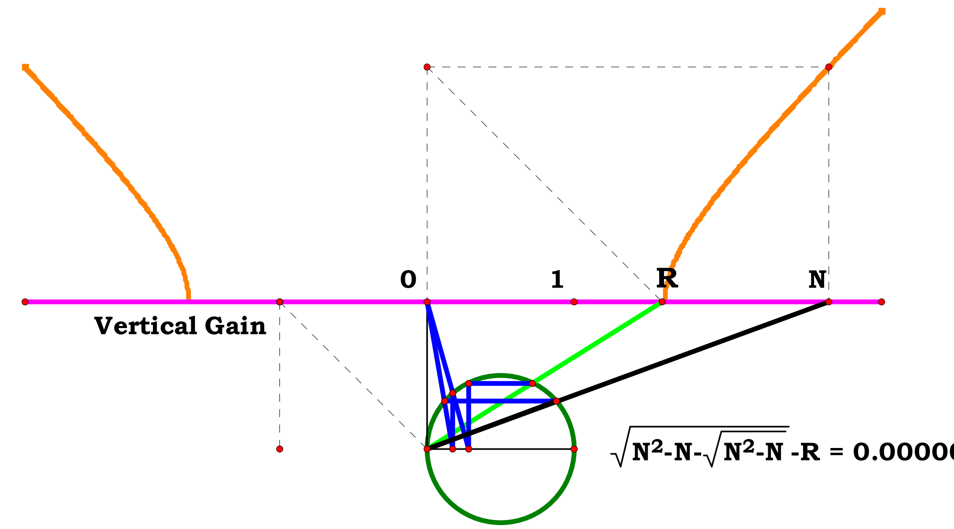
$$AB := 1 \qquad AN := 3 \qquad FG := \frac{AN}{AN^2+1} \qquad AH := \frac{AN\cdot AB}{FG} \qquad AR := \frac{AH^2}{AH^2+1}$$

$$AR - \frac{AN^4+2\cdot AN^2+1}{AN^4+2\cdot AN^2+2} = 0$$



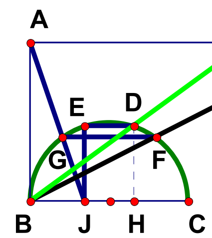
$$AB := 1$$

$$AN := 5$$



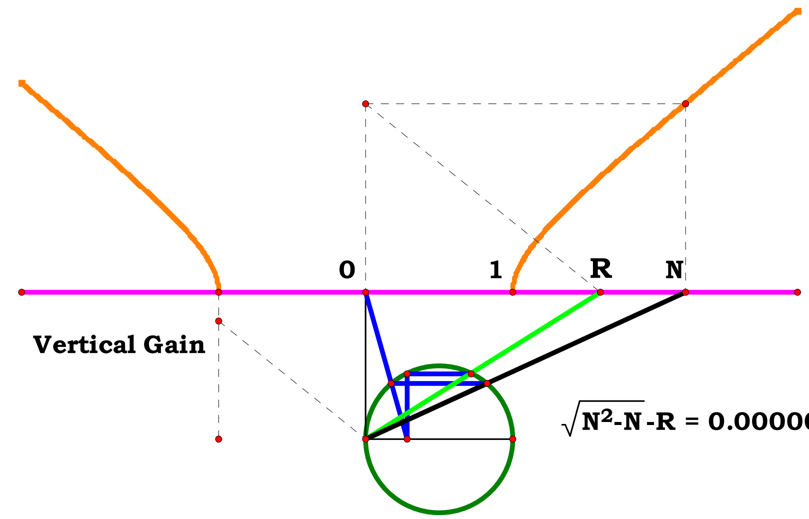
$$BE := \frac{1}{AN^2 - AN - \left(AN^2 - AN\right)^{\frac{1}{2}} + 1} \qquad BG := AB - BE$$

$$DE := \frac{\left[AN^2 - AN - \left(AN^2 - AN\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}}{AN^2 - AN - \left(AN^2 - AN\right)^{\frac{1}{2}} + 1} \qquad AR := \frac{BG \cdot AB}{DE} \qquad AR - \left[AN^2 - AN - \left(AN^2 - AN\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} = 0$$



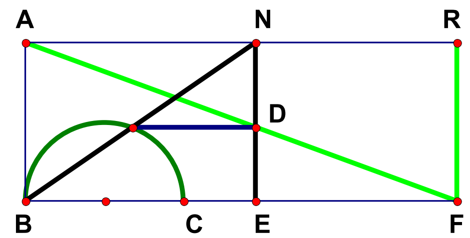
$$AB := 1$$

$$AN := 3$$



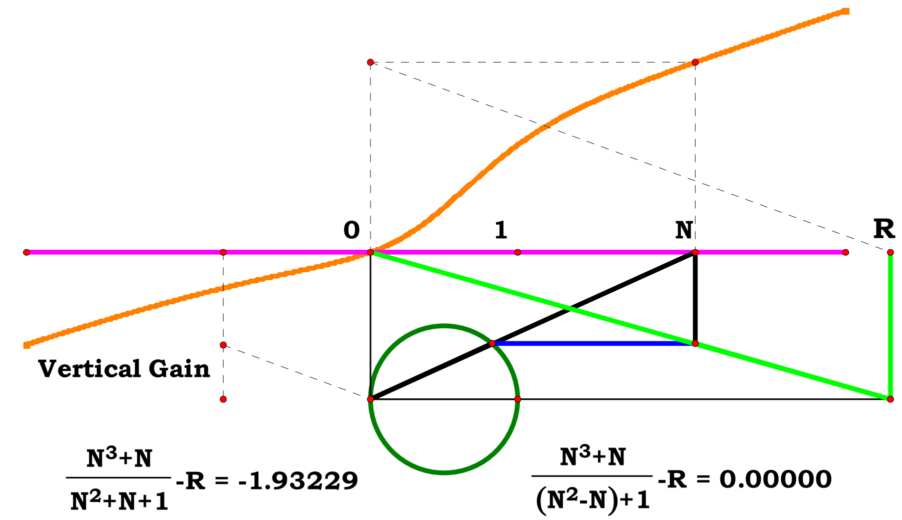
$$BJ := \frac{1}{AN^2 - AN + 1} \qquad BH := AB - BJ \qquad EJ := \frac{\left(AN^2 - AN\right)^{\frac{1}{2}}}{AN^2 - AN + 1}$$

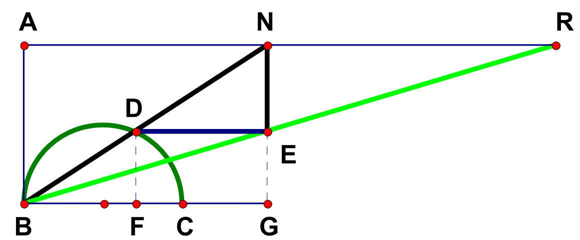
$$AR := \frac{BH \cdot AB}{EJ} \qquad AR - \left(AN^2 - AN\right)^{\frac{1}{2}} = 0$$



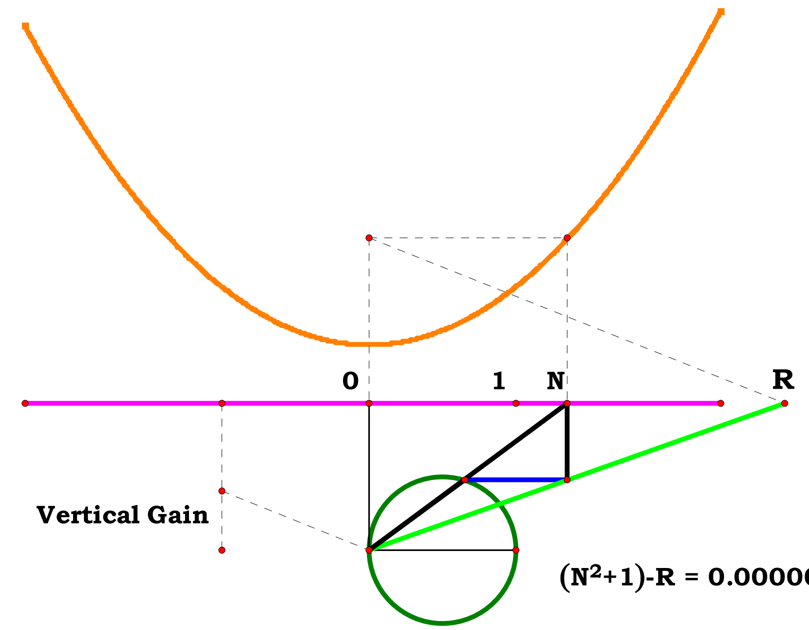
$$AB := 1$$
$$AN := 5$$

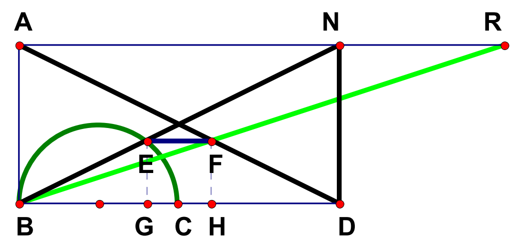
$$DE := \frac{AN}{AN^2 + 1} \qquad BF := \frac{AN \cdot AB}{AB - DE} \qquad AR := BF \qquad AR - \frac{AN^3 + AN}{AN^2 - AN + 1} = 0$$





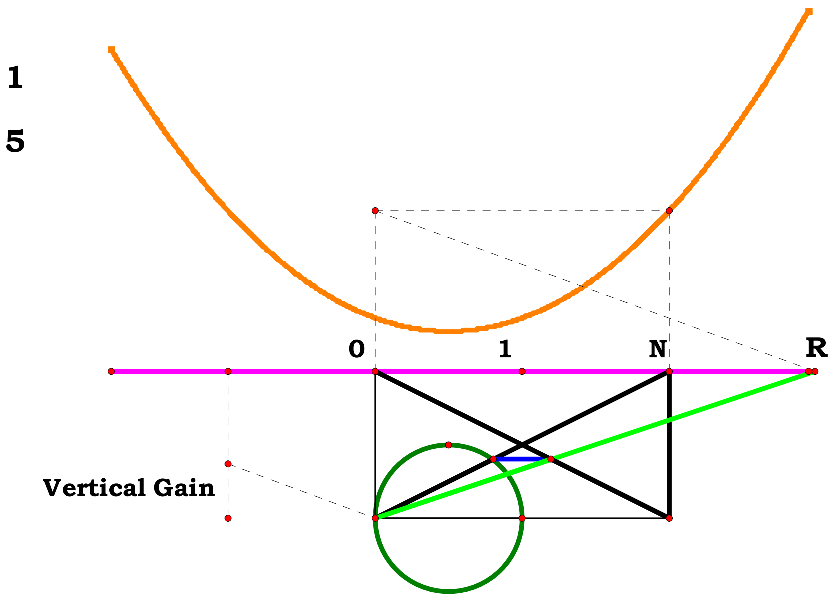
$AB := 1 \quad AN := 3 \qquad DF := \dfrac{AN}{AN^2+1} \qquad AR := \dfrac{AN \cdot AB}{DF} \qquad AR - \left(AN^2 + 1\right) = 0$





$AB := 1$

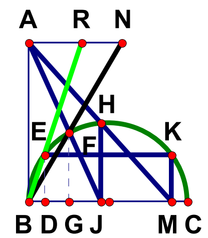
$AN := 5$



$$BN := \sqrt{AN^2 + AB^2} \qquad BE := \frac{AN}{\left(AN^2+1\right)^{\frac{1}{2}}} \qquad EG := \frac{AN}{AN^2+1}$$

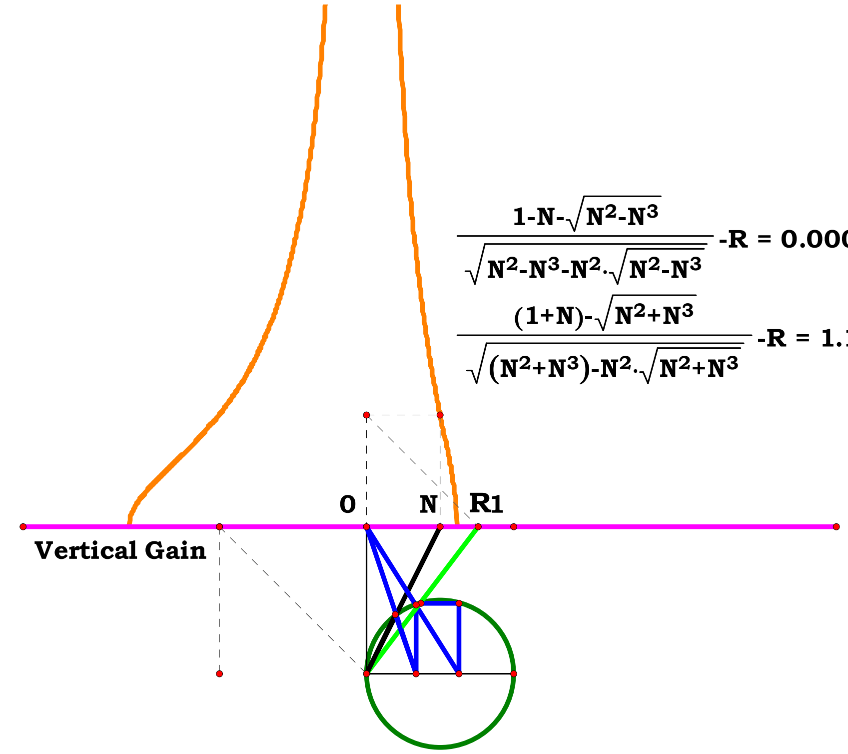
$$DH := \frac{AN \cdot BE}{BN} \qquad BH := AN - DH \qquad AR := \frac{BH \cdot AB}{EG} \qquad AR - \left(AN^2 - AN + 1\right) = 0$$

$(N^2+N+1)-R = 3.99344$ $((N^2-N)+1)-R = 0.00000$



$AB := 1$

$AN := .2$



$$\frac{1-N-\sqrt{N^2-N^3}}{\sqrt{N^2-N^3-N^2\cdot\sqrt{N^2-N^3}}}-R = 0.00000$$

$$\frac{(1+N)-\sqrt{N^2+N^3}}{\sqrt{(N^2+N^3)-N^2\cdot\sqrt{N^2+N^3}}}-R = 1.11890$$

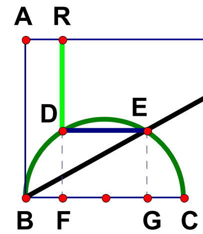
$$FG := \frac{AN}{AN^2+1} \qquad BG := \frac{AN\cdot FG}{AB} \qquad BJ := \frac{BG\cdot AB}{AB-FG} \qquad CJ := AB-BJ$$

$$HJ := \sqrt{BJ\cdot CJ} \qquad BM := \frac{BJ\cdot AB}{AB-HJ} \qquad CM := AB-BM \qquad KM := \sqrt{BM\cdot CM}$$

$$BD := CM \qquad DE := KM \qquad AR := \frac{BD\cdot AB}{DE} \qquad AR - \frac{\left[1-AN-AN\cdot(1-AN)^{\frac{1}{2}}\right]^{\frac{1}{2}}}{AN} = 0$$

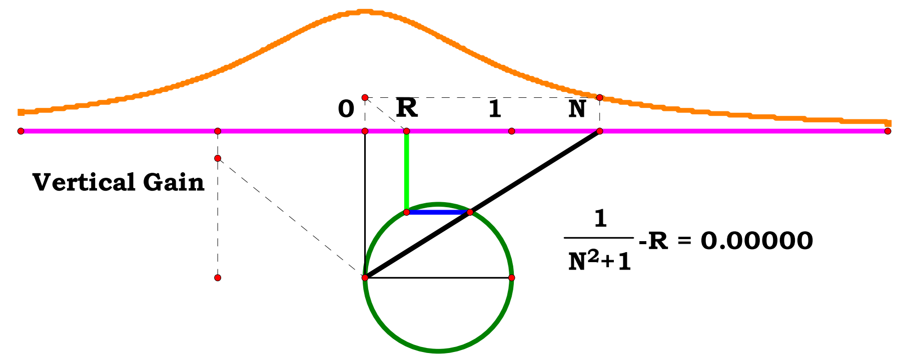
$$AR - \frac{1-AN-\sqrt{AN^2-AN^3}}{\sqrt{AN^2-AN^3-AN^2\cdot\sqrt{AN^2-AN^3}}} = 0 \qquad BG - \frac{AN^2}{AN^2+1} = 0 \qquad BJ - \frac{AN^2}{AN^2-AN+1} = 0 \qquad CJ - \frac{1-AN}{AN^2-AN+1} = 0$$

$$HJ - \frac{AN\,(1-AN)^{\frac{1}{2}}}{AN^2-AN+1} = 0 \qquad BM - \frac{AN^2}{AN^2+1-AN-AN\cdot(1-AN)^{\frac{1}{2}}} = 0 \qquad CM - \frac{1-AN-AN\cdot(1-AN)^{\frac{1}{2}}}{AN^2+1-AN-AN\cdot(1-AN)^{\frac{1}{2}}} = 0 \qquad KM - \frac{AN\sqrt{1-AN-AN\cdot(1-AN)^{\frac{1}{2}}}}{AN^2+1-AN-AN\cdot(1-AN)^{\frac{1}{2}}} = 0$$

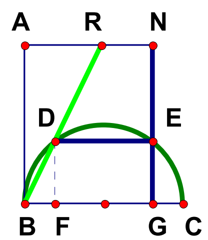


$AB := 1$

$AN := 3$



$CG := \dfrac{1}{AN^2 + 1}$ $\quad BF := CG \quad AR := BF$

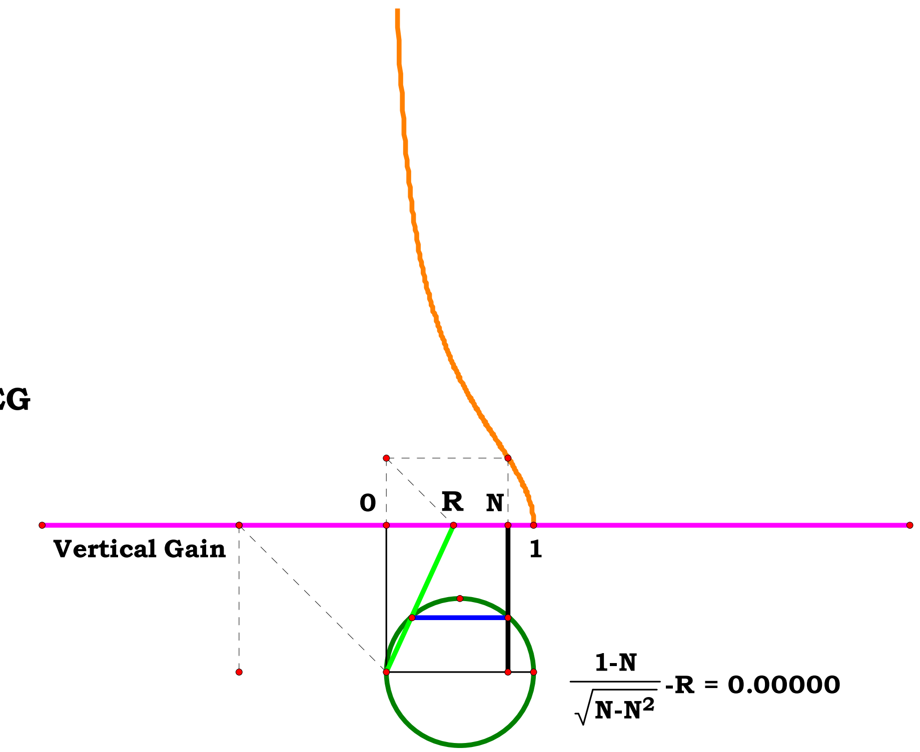


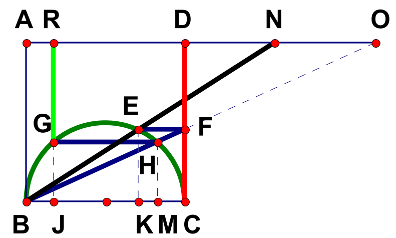
$$AB := 1$$

$$AN := .8$$

$$BG := AN \qquad CG := AB - AN \qquad EG := \sqrt{BG \cdot CG} \qquad BF := CG \qquad DF := EG$$

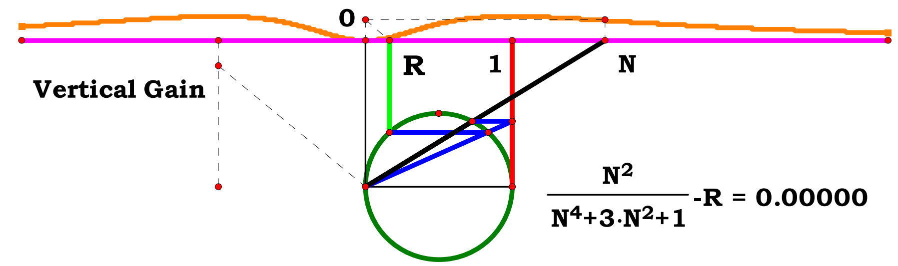
$$AR := \frac{BF \cdot AB}{DF} \qquad AR - \frac{1 - AN}{\left(AN - AN^2\right)^{\frac{1}{2}}} = 0$$





$AB := 1$

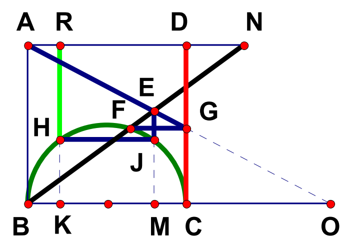
$AN := 3$



$$EK := \frac{AN}{AN^2+1} \quad CF := EK \quad AO := \frac{AB^2}{CF} \quad BJ := \frac{1}{AO^2+1} \quad AR := BJ$$

$$AR - \frac{AN^2}{AN^4+3AN^2+1} = 0$$

$$AO - \frac{AN^2+1}{AN} = 0$$



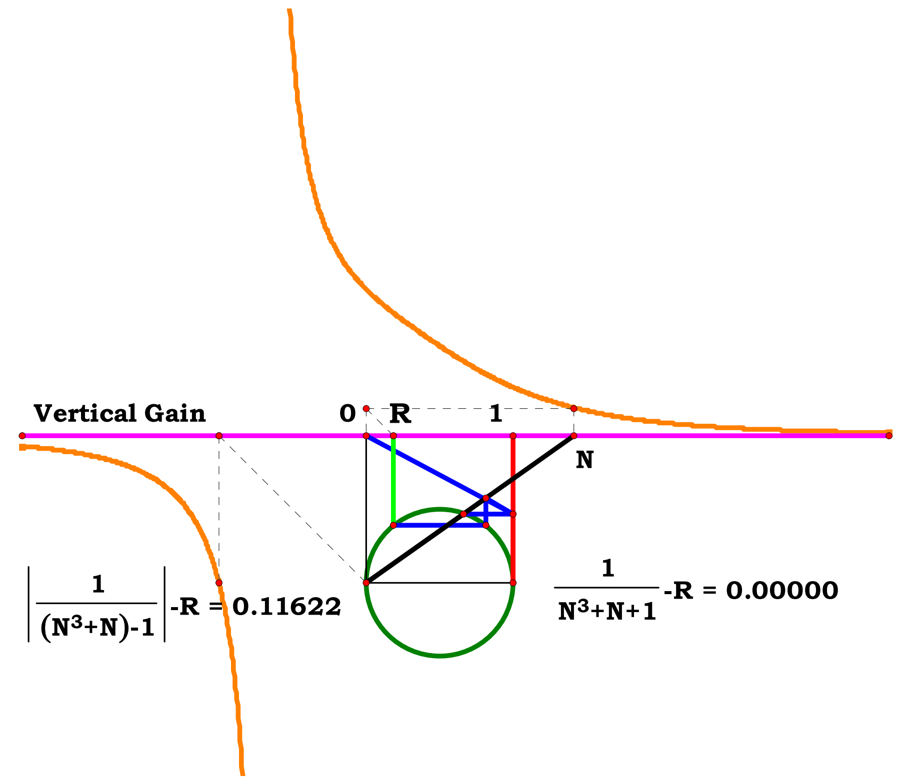
$$AB := 1$$

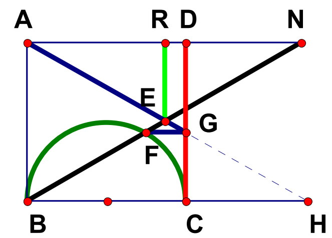
$$AN := 2$$

$$CG := \frac{AN}{AN^2 + 1} \qquad BO := \frac{AB^2}{AB - CG} \qquad BM := \frac{AN \cdot BO}{AN + BO} \qquad CM := AB - BM$$

$$BK := CM \qquad AR := BK \qquad AR - \frac{1}{AN^3 + AN + 1} = 0$$

$$BM - \frac{AN^3 + AN}{AN^3 + AN + 1} = 0$$

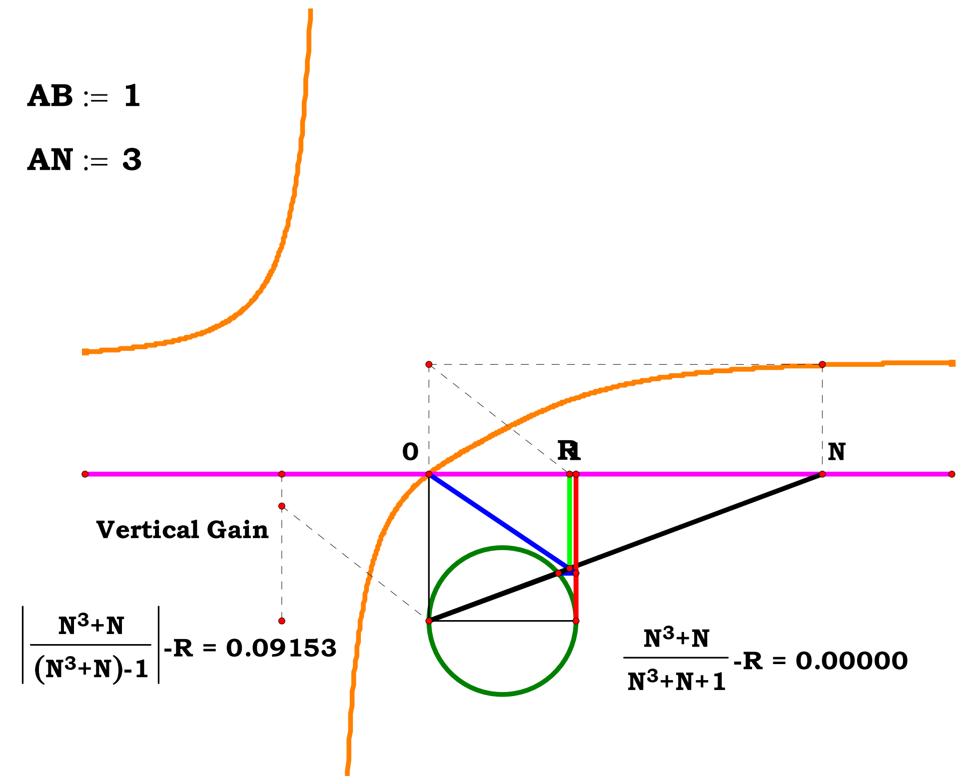


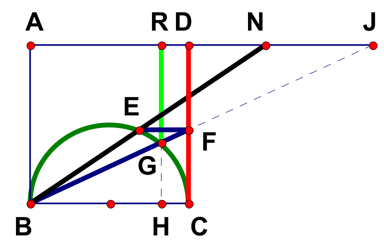


$\mathbf{AB} := \mathbf{1}$

$\mathbf{AN} := \mathbf{3}$

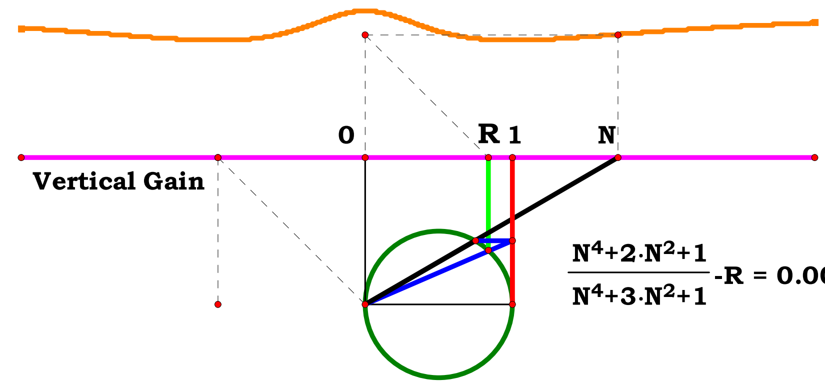
$$\mathbf{AR} := \frac{\mathbf{AN}^{3} + \mathbf{AN}}{\mathbf{AN}^{3} + \mathbf{AN} + \mathbf{1}}$$





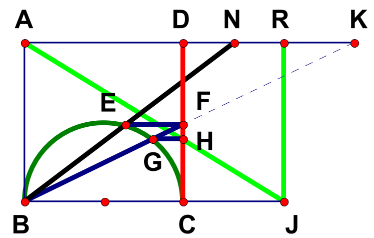
$AB := 1$

$AN := 3$



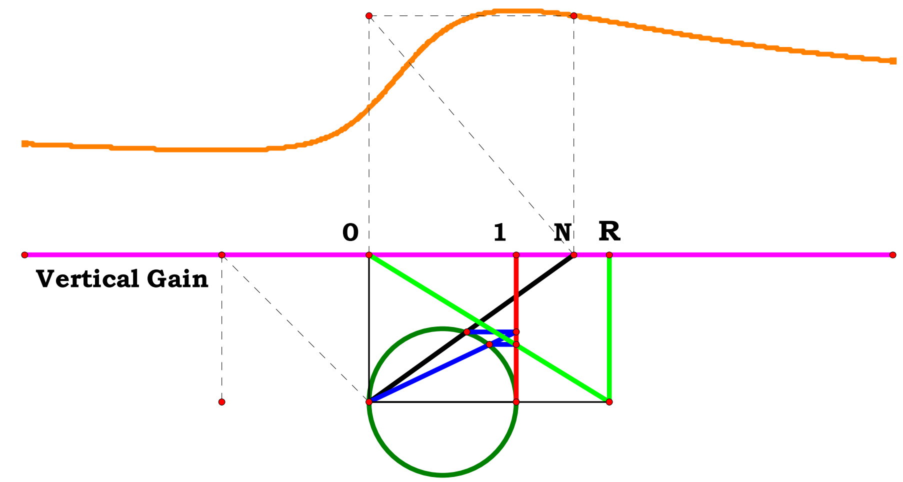
$\frac{N^4+2 \cdot N^2+1}{N^4+3 \cdot N^2+1}$-R = 0.00000

$CF := \frac{AN}{AN^2+1}$ $AJ := \frac{AB^2}{CF}$ $AR := \frac{AJ^2}{AJ^2+1}$ $AR - \frac{AN^4+2 \cdot AN^2+1}{AN^4+3 \cdot AN^2+1} = 0$



$AB := 1$

$AN := 3$

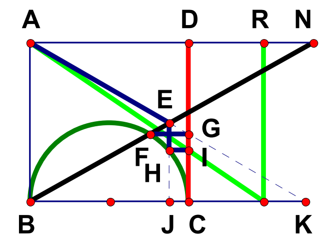


$$\frac{N^4+3\cdot N^2+1}{N^4+N^3+3\cdot N^2+N+1}-R = -0.90951 \qquad \frac{N^4+3\cdot N^2+1}{(((N^4-N^3)+3\cdot N^2)-N)+1}-R = 0.00000$$

$$AK := \frac{AN^2+1}{AN} \qquad CH := \frac{AK}{AK^2+1} \qquad BJ := \frac{AB^2}{AB-CH} \qquad AR := BJ$$

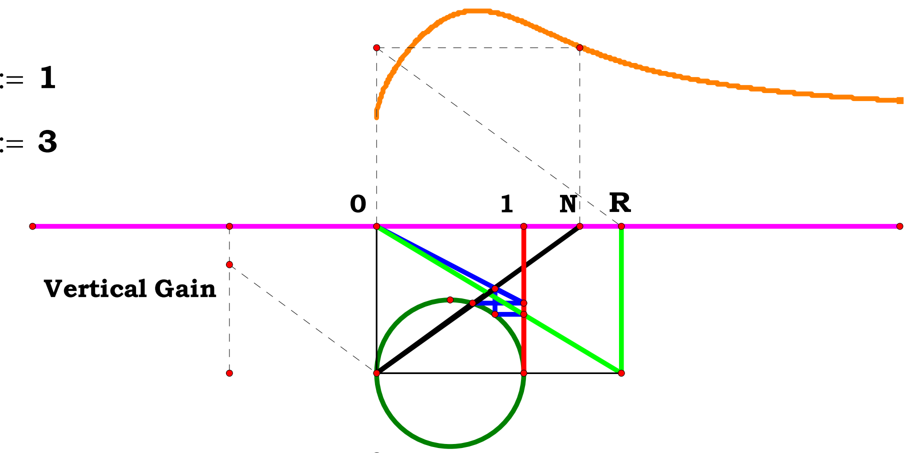
$$AR - \frac{AN^4+3\cdot AN^2+1}{AN^4-AN^3+3\cdot AN^2-AN+1} = 0$$

$$CH - \frac{AN^3+AN}{AN^4+3\cdot AN^2+1} = 0$$



$$AB := 1$$

$$AN := 3$$



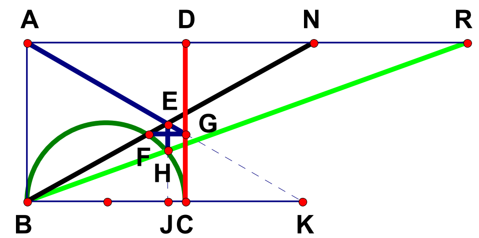
$$\frac{N^3+N+1}{(N^3+N+1)-\sqrt{N^3+N}}\text{-}R = 0.00000$$

$$BJ := \frac{AN^3 + AN}{AN^3 + AN + 1} \qquad CJ := AB - BJ \qquad HJ := \sqrt{BJ \cdot CJ}$$

$$CI := HJ \qquad BK := \frac{AB^2}{AB - CI} \qquad AR := BK$$

$$AR - \frac{AN^3 + AN + 1}{AN^3 + AN + 1 - \left(AN^3 + AN\right)^{\frac{1}{2}}} = 0$$

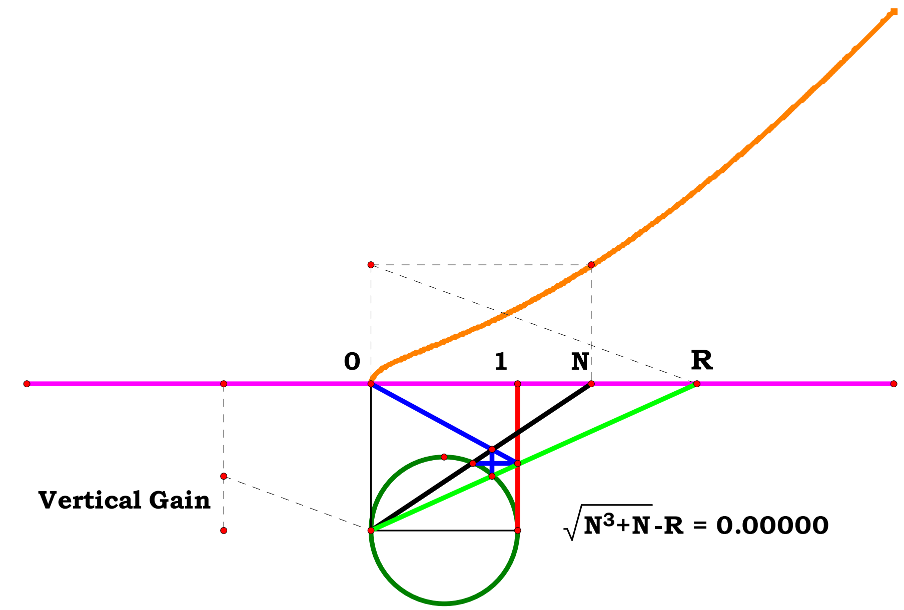
$$HJ - \frac{\left(AN^3 + AN\right)^{\frac{1}{2}}}{AN^3 + AN + 1} = 0$$

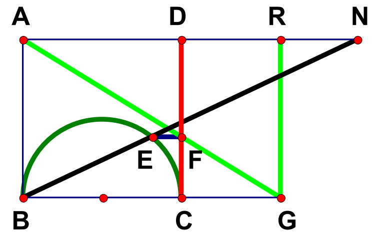


$$AB := 1$$

$$AN := 3$$

$$BJ := \frac{AN^3 + AN}{AN^3 + AN + 1} \qquad HJ := \frac{\left(AN^3 + AN\right)^{\frac{1}{2}}}{AN^3 + AN + 1} \qquad AR := \frac{BJ \cdot AB}{HJ} \qquad AR - \left(AN^3 + AN\right)^{\frac{1}{2}} = 0$$

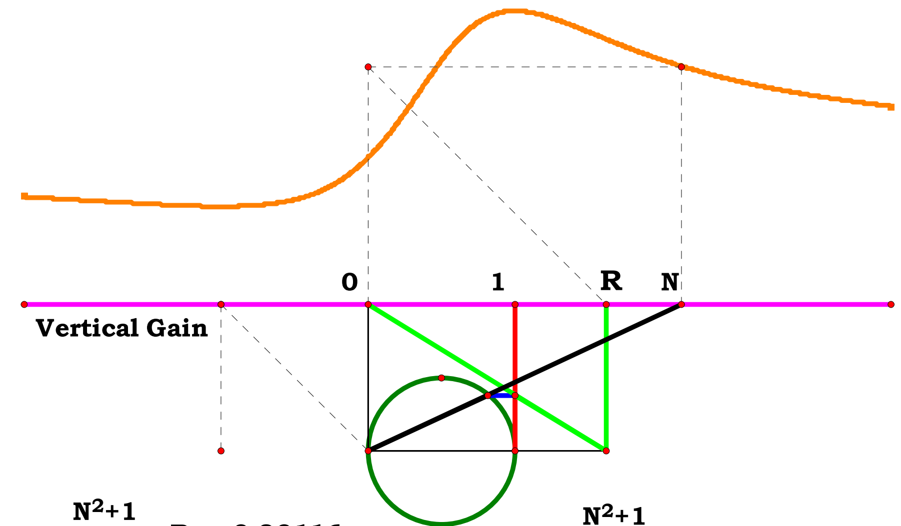




$AB := 1$

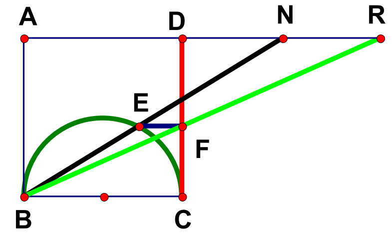
$AN := 5$

$$CF := \frac{AN}{AN^2 + 1} \qquad BG := \frac{AB^2}{AB - CF} \qquad AR := BG \qquad AR - \frac{AN^2 + 1}{AN^2 - AN + 1} = 0$$



$$\frac{N^2+1}{N^2+N+1} - R = -0.90116$$

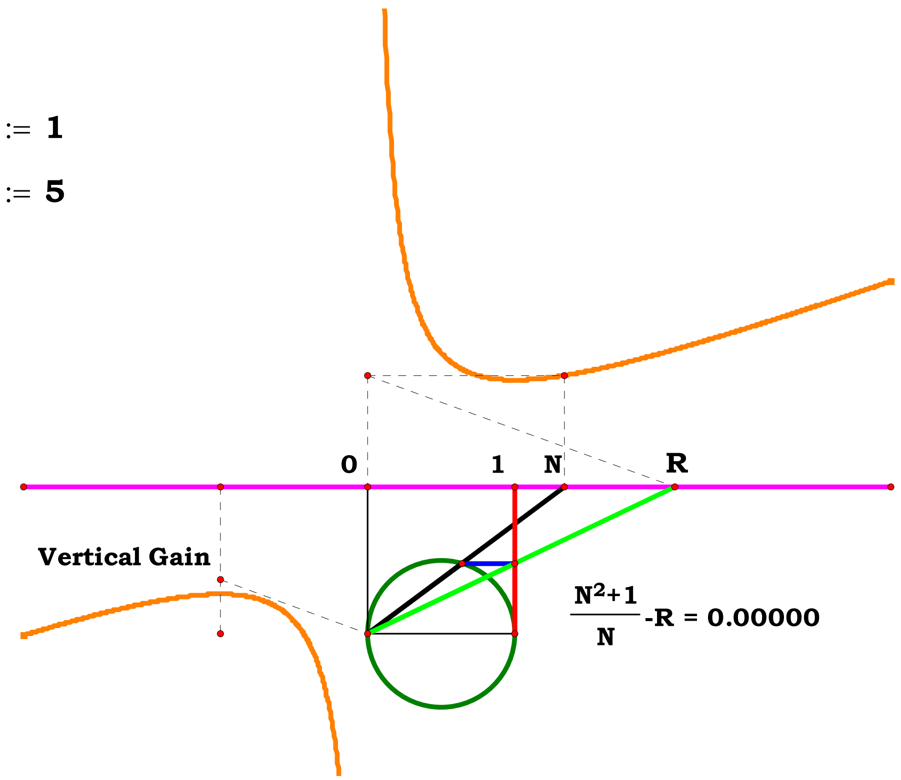
$$\frac{N^2+1}{(N^2-N)+1} - R = 0.00000$$

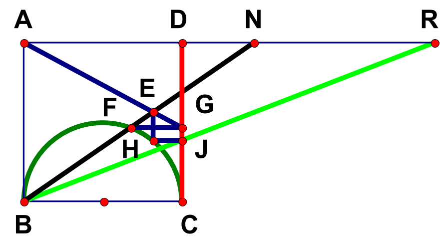


$AB := 1$

$AN := 5$

$$CF := \frac{AN}{AN^2 + 1} \qquad AR := \frac{AB^2}{CF} \qquad AR - \frac{AN^2 + 1}{AN} = 0$$

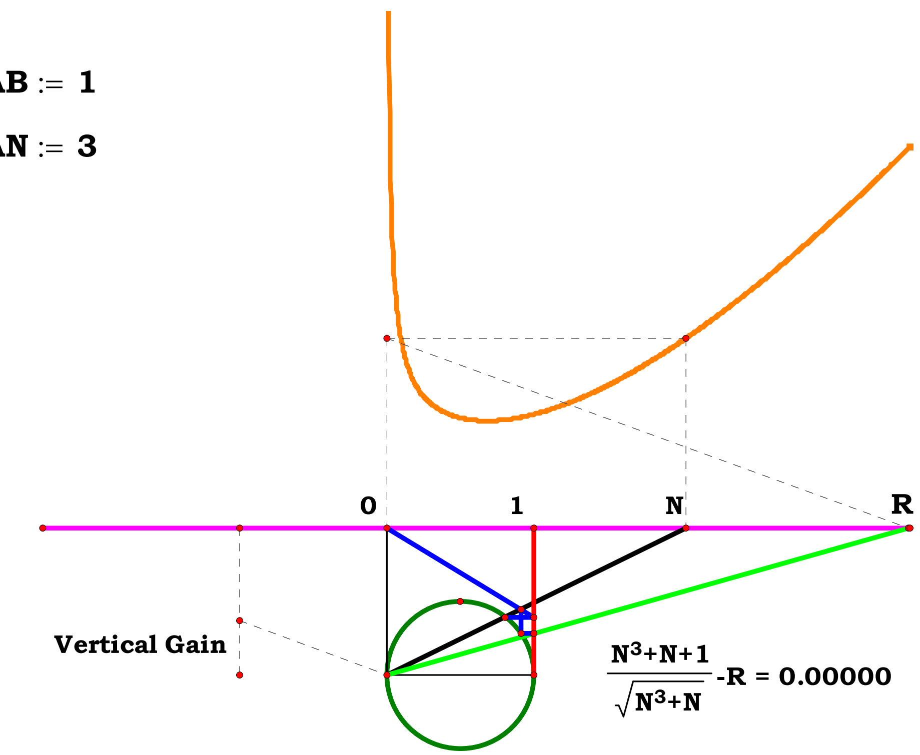


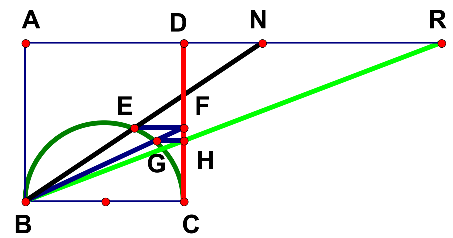


$$AB := 1$$

$$AN := 3$$

$$CJ := \frac{\left(AN^3 + AN\right)^{\frac{1}{2}}}{AN^3 + AN + 1} \qquad AR := \frac{AB^2}{CJ} \qquad AR - \frac{AN^3 + AN + 1}{\left(AN^3 + AN\right)^{\frac{1}{2}}} = 0$$

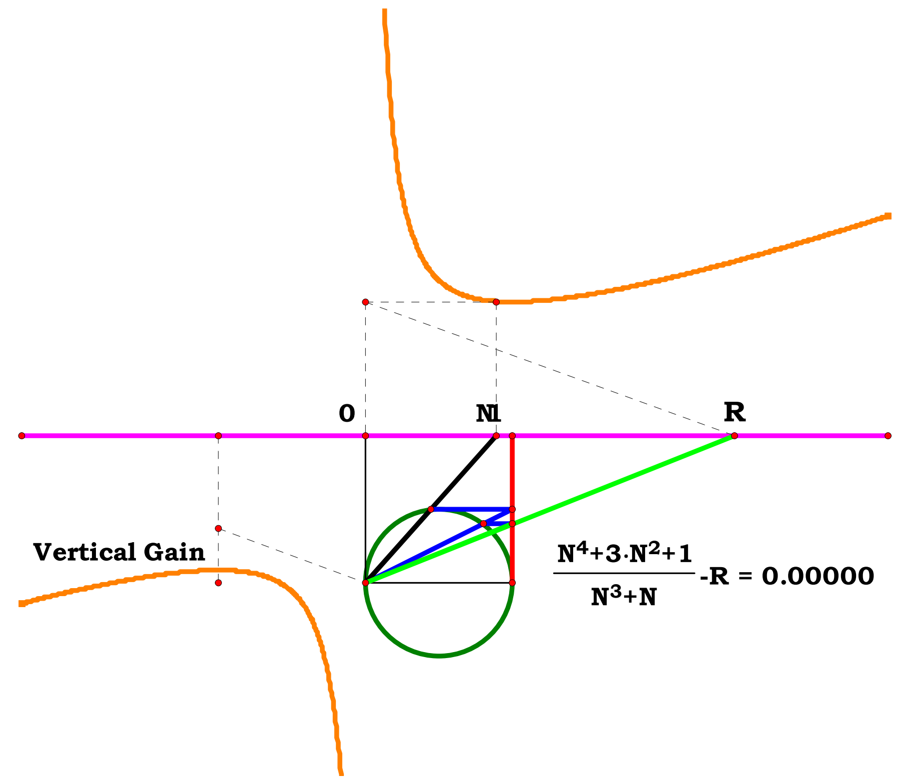


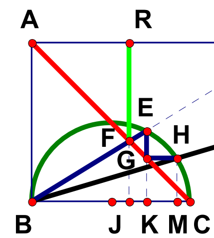


$$AB := 1$$

$$AN := 5$$

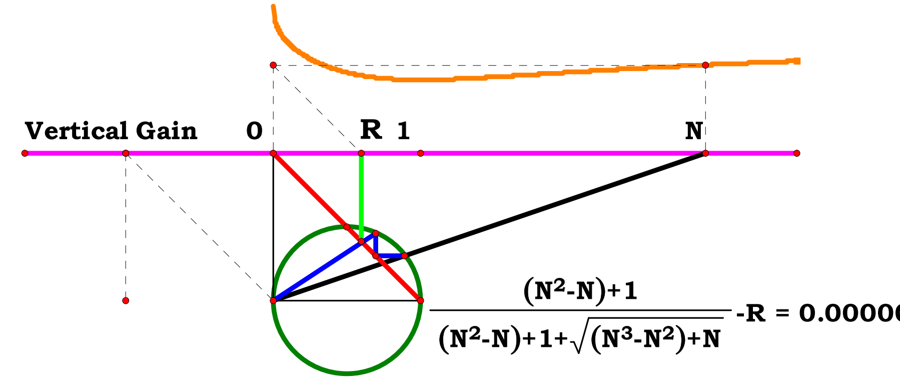
$$CH := \frac{AN^3 + AN}{AN^4 + 3 \cdot AN^2 + 1} \qquad AR := \frac{AB^2}{CH} \qquad AR - \left(\frac{AN^4 + 3 \cdot AN^2 + 1}{AN^3 + AN} \right) = 0$$





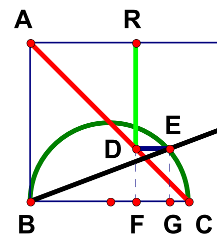
$$AB := 1$$

$$AN := 5$$



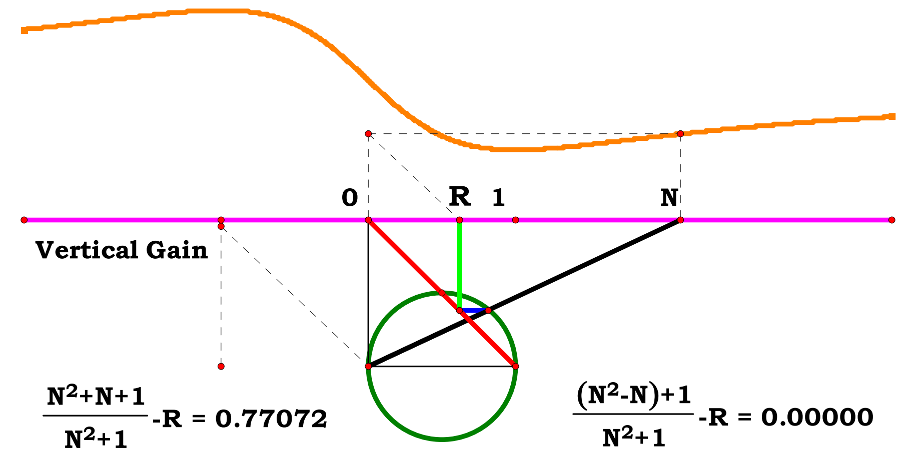
$$HM := \frac{AN}{AN^2 + 1} \qquad CK := HM \qquad BK := AB - CK \qquad EK := \sqrt{BK \cdot CK}$$

$$AD := \frac{BK \cdot AB}{EK} \qquad AR := \frac{AD \cdot AB}{AD + AB} \qquad AR - \frac{AN^2 - AN + 1}{AN^2 - AN + \left(AN^3 - AN^2 + AN\right)^{\frac{1}{2}} + 1} = 0$$

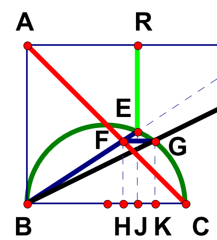


$\mathbf{AB} := \mathbf{1}$

$\mathbf{AN} := \mathbf{5}$



$$\mathbf{EG} := \frac{\mathbf{AN}}{\mathbf{AN}^2 + 1} \qquad \mathbf{CF} := \mathbf{EG} \qquad \mathbf{AR} := \mathbf{AB} - \mathbf{CF} \qquad \mathbf{AR} - \frac{\mathbf{AN}^2 - \mathbf{AN} + 1}{\mathbf{AN}^2 + 1} = \mathbf{0}$$



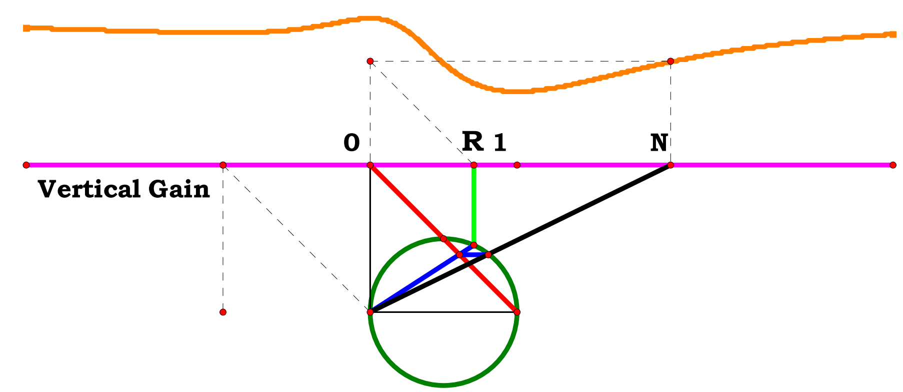
$$AB := 1$$

$$AN := 3$$

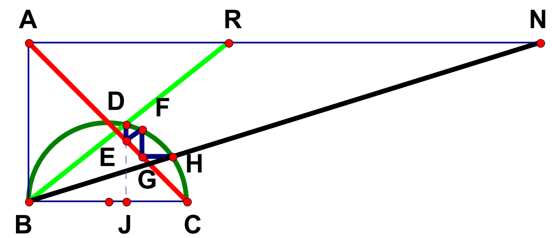
$$GK := \frac{AN}{AN^2 + 1} \quad BH := AB - GK \quad AD := \frac{BH \cdot AB}{GK} \quad BJ := \frac{AD^2}{AD^2 + 1}$$

$$AR := BJ \qquad AR - \frac{AN^4 - 2 \cdot AN^3 + 3 \cdot AN^2 - 2 \cdot AN + 1}{AN^4 - 2 \cdot AN^3 + 4 \cdot AN^2 - 2 \cdot AN + 1} = 0$$

$$\frac{(((N^4-2 \cdot N^3)+3 \cdot N^2)-2 \cdot N)+1}{(((N^4-2 \cdot N^3)+4 \cdot N^2)-2 \cdot N)+1} - R = 0.00000$$

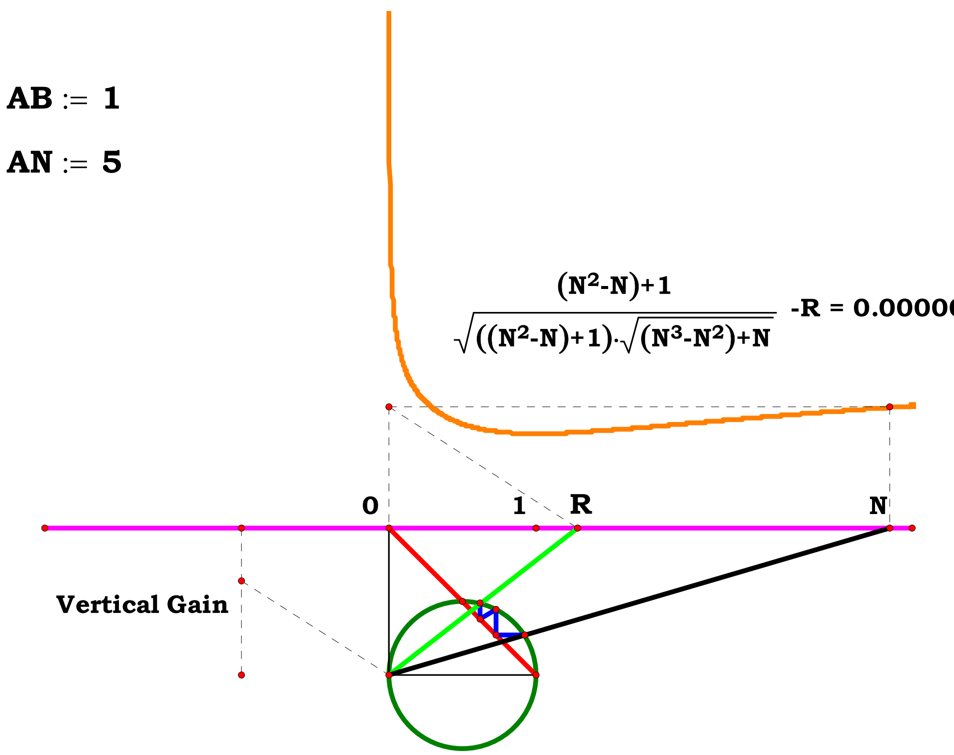


$$\frac{N^4+2 \cdot N^3+3 \cdot N^2+2 \cdot N+1}{N^4+2 \cdot N^3+4 \cdot N^2+2 \cdot N+1} - R = 0.22353$$



$AB := 1$

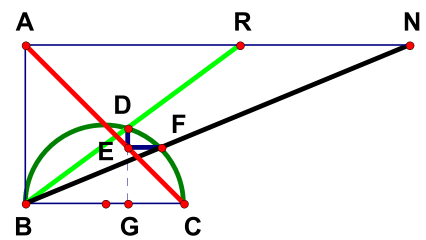
$AN := 5$



$$BJ := \frac{AN^2 - AN + 1}{AN^2 - AN + \left(AN^3 - AN^2 + AN\right)^{\frac{1}{2}} + 1} \qquad CJ := AB - BJ$$

$$DJ := \sqrt{BJ \cdot CJ} \qquad AR := \frac{BJ \cdot AB}{DJ}$$

$$AR - \frac{AN^2 - AN + 1}{\left[\left(AN^2 + 1 - AN\right)\cdot\left(AN^3 - AN^2 + AN\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}} = 0$$

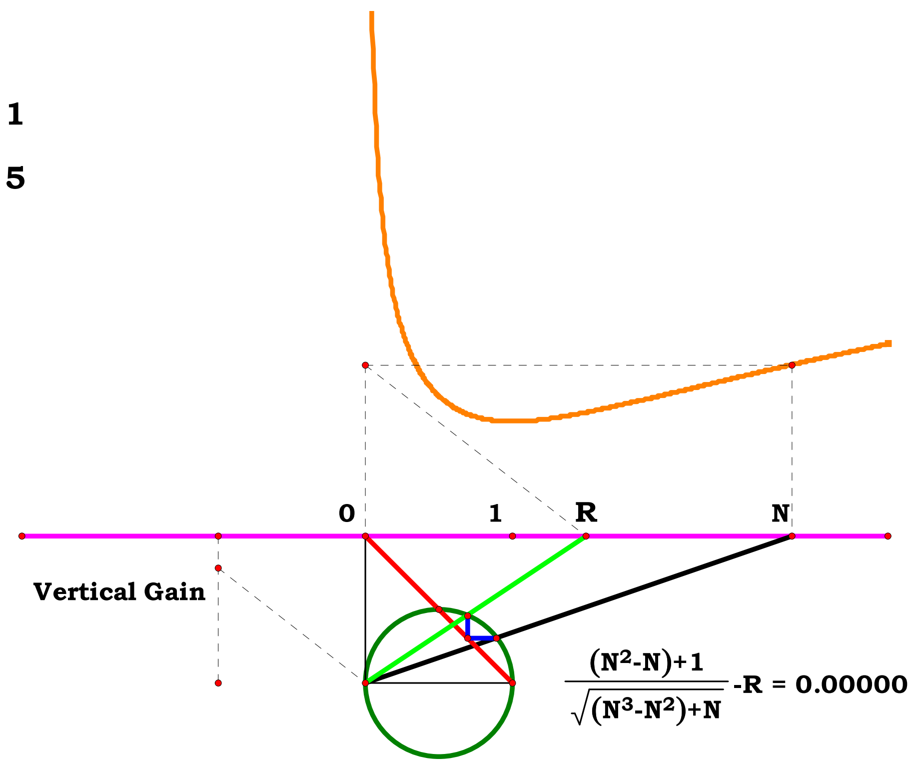


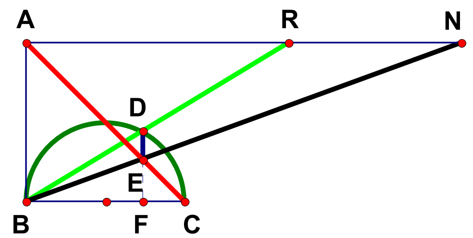
$AB := 1$

$AN := 5$

$$CG := \frac{AN}{AN^2 + 1} \qquad BG := AB - CG \qquad DG := \sqrt{BG \cdot CG} \qquad AR := \frac{BG \cdot AB}{DG}$$

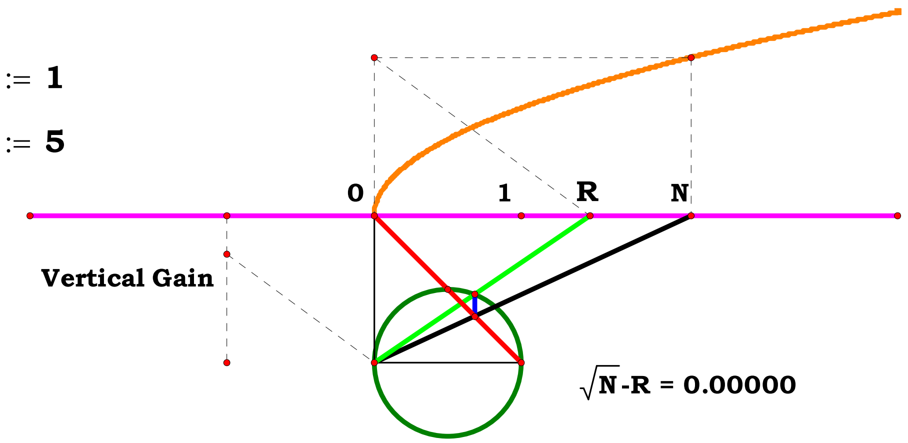
$$AR - \frac{AN^2 - AN + 1}{\left(AN^3 - AN^2 + AN\right)^{\frac{1}{2}}} = 0$$





$AB := 1$

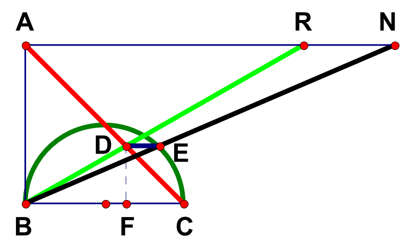
$AN := 5$



$$BF := \frac{AN}{AN+1} \quad CF := AB - BF \quad DF := \sqrt{BF \cdot CF} \quad AR := \frac{BF \cdot AB}{DF}$$

$$AR - \sqrt{AN} = 0$$

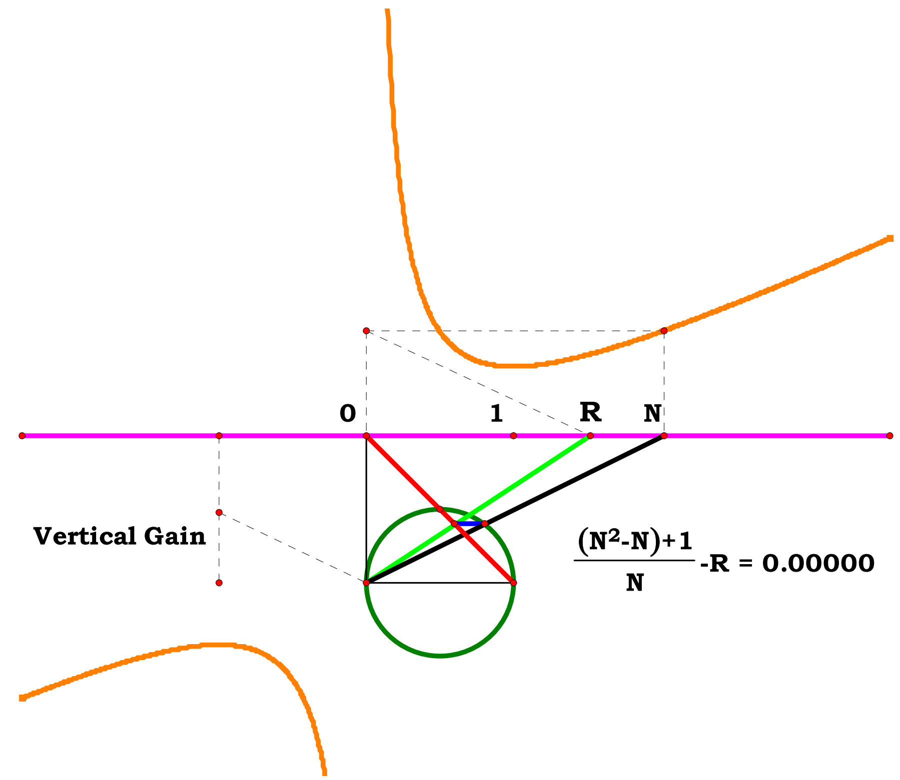
$$DF - \frac{AN^{\frac{1}{2}}}{AN+1} = 0$$

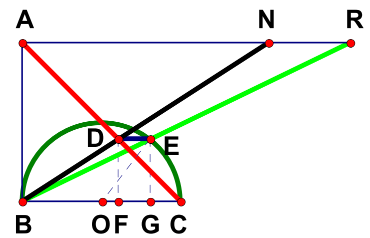


$AB := 1$

$AN := 5$

$DF := \dfrac{AN}{AN^2 + 1}$ $\quad BF := AB - DF$ $\quad AR := \dfrac{BF \cdot AB}{DF}$ $\quad AR - \dfrac{AN^2 - AN + 1}{AN} = 0$





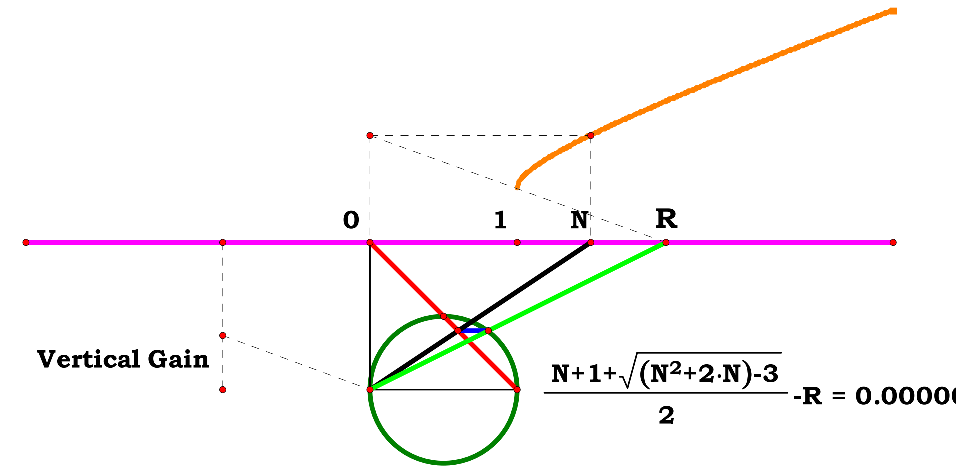
$AB := 1$

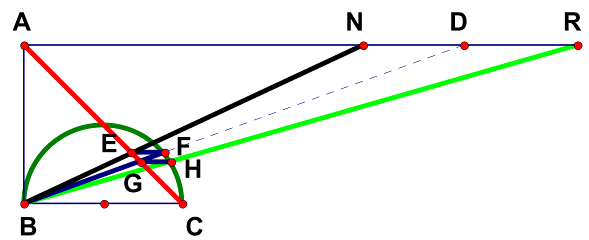
$AN := 3$

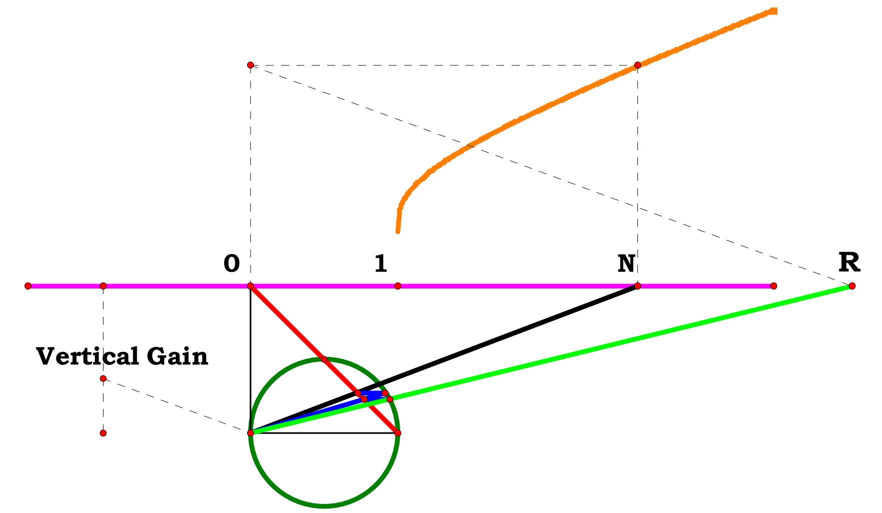
$BF := \dfrac{AN}{AN+1} \quad CF := AB - BF \quad EG := CF \quad EO := \dfrac{AB}{2} \quad GO := \sqrt{EO^2 - EG^2}$

$BG := EO + GO \quad AR := \dfrac{BG \cdot AB}{CF} \quad AR - \dfrac{AN + 1 + \sqrt{AN^2 + 2 \cdot AN - 3}}{2} = 0$

$$BG - \frac{AN + 1 + \left(AN^2 + 2 \cdot AN - 3\right)^{\frac{1}{2}}}{2AN + 2} = 0$$







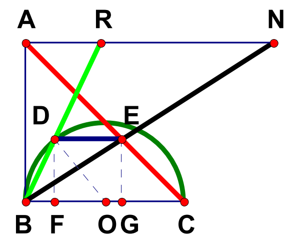
$$\frac{N+3+\sqrt{(N^2+2\cdot N)-3}+\sqrt{((2\cdot N^2+8\cdot N+2\cdot N\cdot\sqrt{(N^2+2\cdot N)-3})-10)+6\cdot\sqrt{(N^2+2\cdot N)-3}}}{4}-R=0$$

$AB := 1$

$AN := 5$

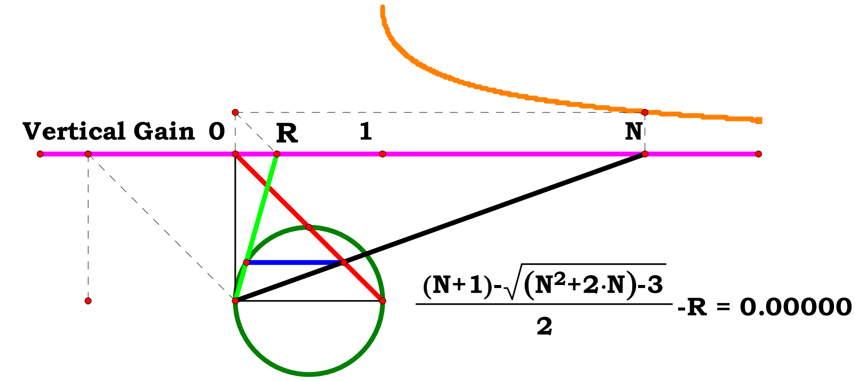
$$AD := \frac{AN + 1 + \sqrt{AN^2 + 2\cdot AN - 3}}{2} \qquad AR := \frac{AD + 1 + \sqrt{AD^2 + 2\cdot AD - 3}}{2}$$

$$AR - \frac{AN + 3 + \left(AN^2 + 2\cdot AN - 3\right)^{\frac{1}{2}} + \left[2\cdot AN^2 + 8\cdot AN + 2\cdot AN\cdot\left(AN^2 + 2\cdot AN - 3\right)^{\frac{1}{2}} - 10 + 6\cdot\left(AN^2 + 2\cdot AN - 3\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}}{4} = 0$$



$$AB := 1$$

$$AN := 3$$

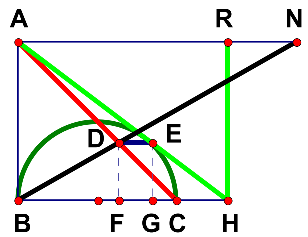


$$BG := \frac{AN}{AN+1} \quad CG := AB - BG \quad DF := CG \quad DO := \frac{AB}{2}$$

$$FO := \sqrt{DO^2 - DF^2} \quad BF := AB - \left(\frac{AB}{2} + FO\right) \quad AR := \frac{BF \cdot AB}{DF}$$

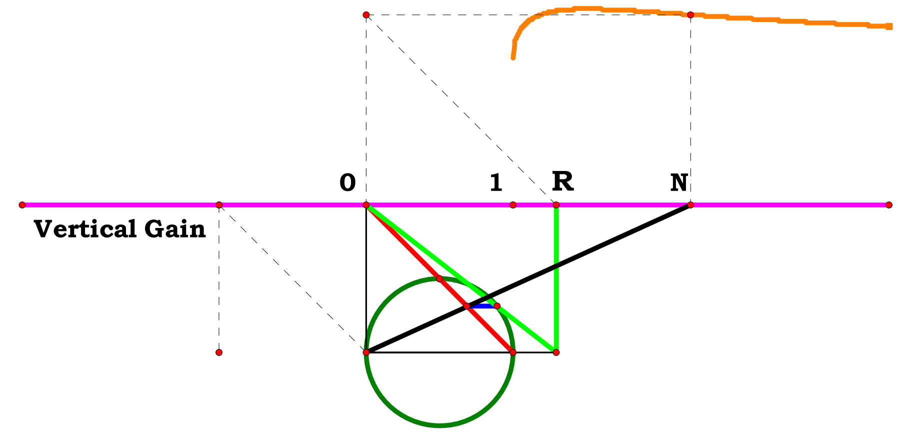
$$AR - \frac{AN + 1 - \left(AN^2 + 2\cdot AN - 3\right)^{\frac{1}{2}}}{2} = 0$$

$$BF - \frac{AN + 1 - \left(AN^2 + 2\cdot AN - 3\right)^{\frac{1}{2}}}{2AN + 2} = 0$$



$$AB := 1$$

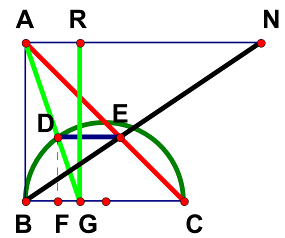
$$AN := 3$$



$$\frac{N+1+\sqrt{(N^2+2\cdot N)-3}}{2\cdot N}-R = 0.00000$$

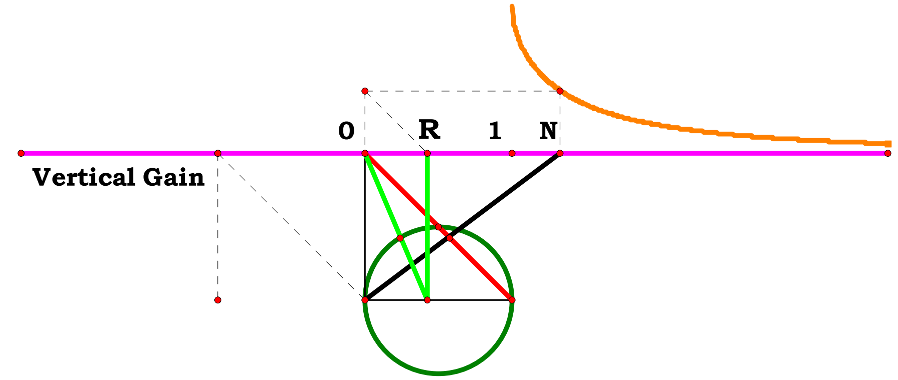
$$BG := \frac{AN+1+\left(AN^2+2\cdot AN-3\right)^{\frac{1}{2}}}{2AN+2} \qquad EG := \frac{1}{AN+1} \qquad BH := \frac{BG\cdot AB}{AB-EG}$$

$$AR := BH \qquad AR - \frac{AN+1+\left(AN^2+2\cdot AN-3\right)^{\frac{1}{2}}}{2AN} = 0$$



$\mathbf{AB} := 1$

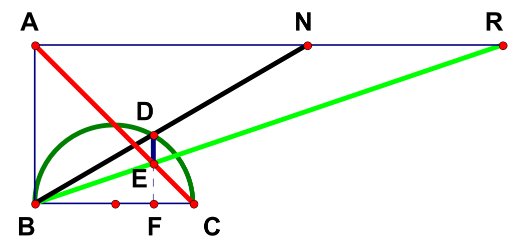
$\mathbf{AN} := 3$



$$\mathbf{DF} := \frac{1}{\mathbf{AN}+1} \qquad \mathbf{BF} := \frac{\mathbf{AN}+1-\left(\mathbf{AN}^2+2\cdot\mathbf{AN}-3\right)^{\frac{1}{2}}}{2\mathbf{AN}+2} \qquad \mathbf{BG} := \frac{\mathbf{BF}\cdot\mathbf{AB}}{\mathbf{AB}-\mathbf{DF}}$$

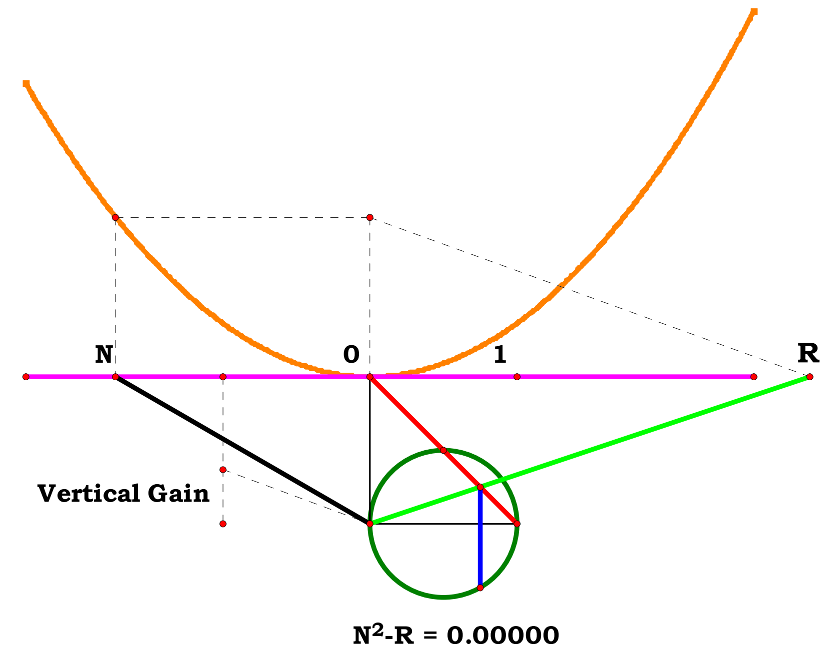
$$\frac{(N+1)-\sqrt{(N^2+2\cdot N)-3}}{2\cdot N}-R = 0.00000$$

$$\mathbf{AR} := \mathbf{BG} \qquad \mathbf{AR} - \frac{\mathbf{AN}+1-\left(\mathbf{AN}^2+2\cdot\mathbf{AN}-3\right)^{\frac{1}{2}}}{2\mathbf{AN}} = 0$$

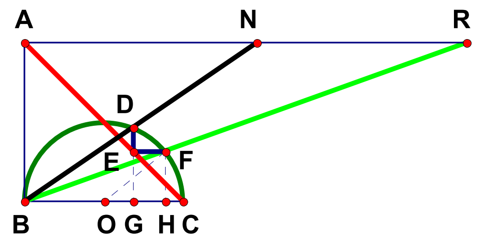


$$AB := 1$$

$$AN := 3$$

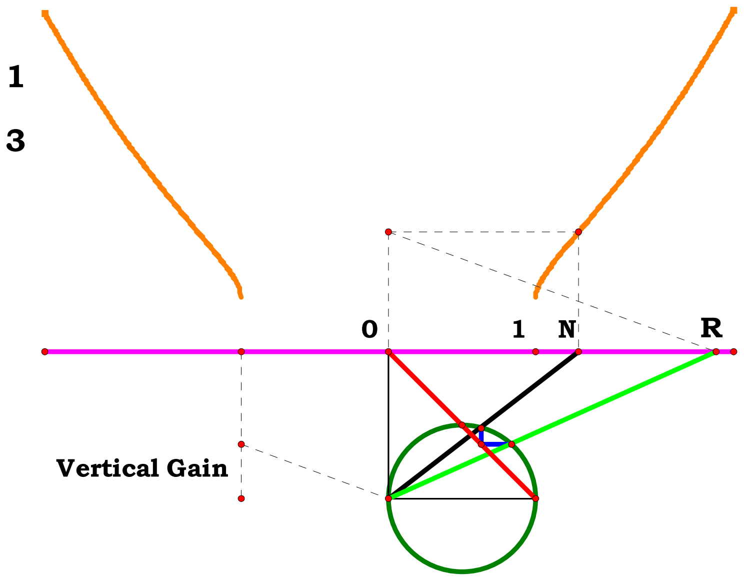


$$BF := \frac{AN^2}{AN^2 + 1} \qquad EF := \frac{1}{AN^2 + 1} \qquad AR := \frac{BF \cdot AB}{EF} \qquad AR - AN^2 = 0$$



$AB := 1$

$AN := 3$

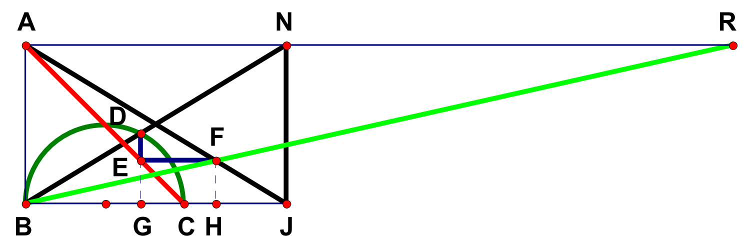


$$\frac{N^2+1+\sqrt{(N^4+2\cdot N^2)-3}}{2}-R = 0.00000$$

$$BG := \frac{AN^2}{AN^2+1} \quad CG := AB - BG \quad FH := CG \quad FO := \frac{AB}{2}$$

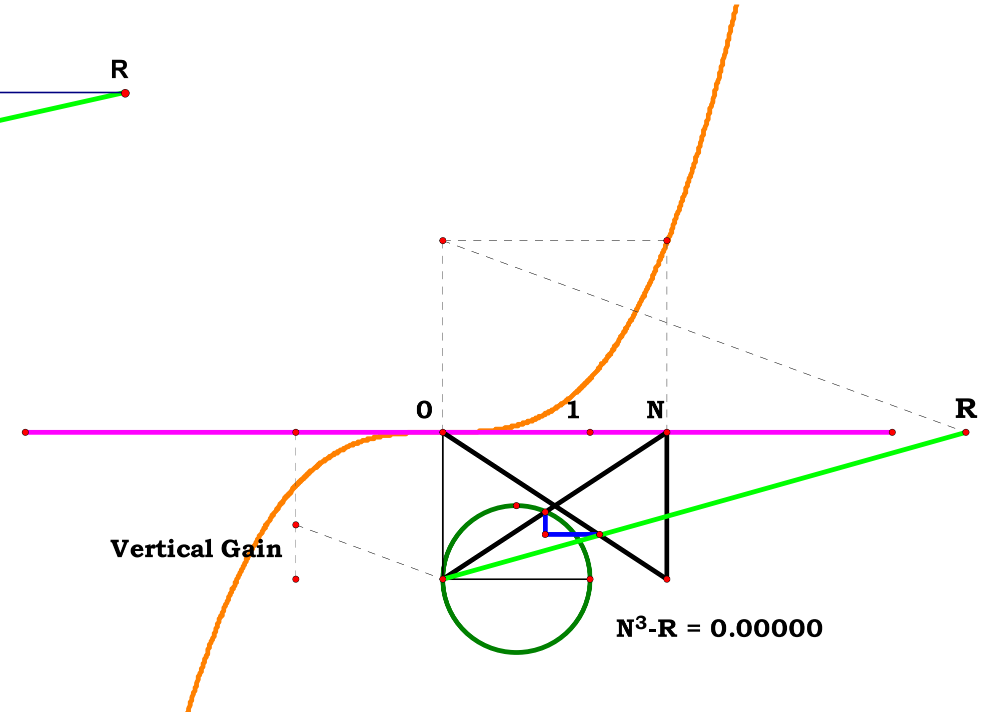
$$HO := \sqrt{FO^2 - FH^2} \quad BH := \frac{AB}{2} + HO \quad AR := \frac{BH \cdot AB}{FH}$$

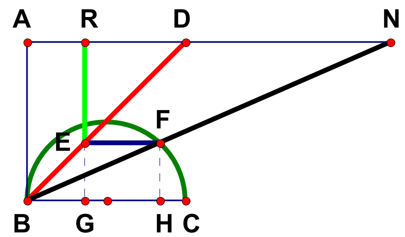
$$AR - \frac{AN^2 + 1 + \left(AN^4 + 2\cdot AN^2 - 3\right)^{\frac{1}{2}}}{2} = 0$$



$\mathbf{AB} := 1 \quad \mathbf{AN} := 3 \quad \mathbf{BG} := \frac{\mathbf{AN}^2}{\mathbf{AN}^2 + 1} \quad \mathbf{CG} := \mathbf{AB} - \mathbf{BG} \quad \mathbf{FH} := \mathbf{CG}$

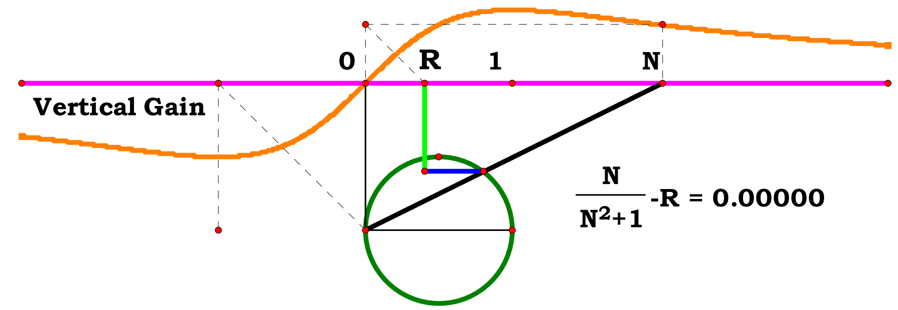
$\mathbf{HJ} := \frac{\mathbf{AN} \cdot \mathbf{FH}}{\mathbf{AB}} \quad \mathbf{BH} := \mathbf{AN} - \mathbf{HJ} \quad \mathbf{AR} := \frac{\mathbf{BH} \cdot \mathbf{AB}}{\mathbf{FH}} \quad \mathbf{AR} - \mathbf{AN}^3 = 0$



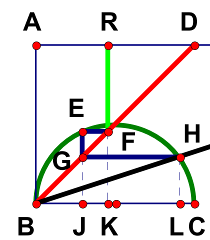


$$AB := 1$$

$$AN := 3$$

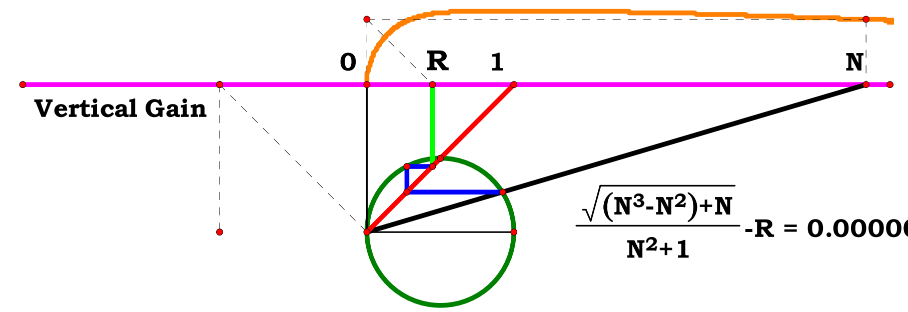


$$FH := \frac{AN}{AN^2+1} \qquad AR := FH \qquad AR - \frac{AN}{AN^2+1} = 0$$



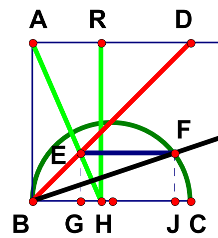
$AB := 1$

$AN := 5$



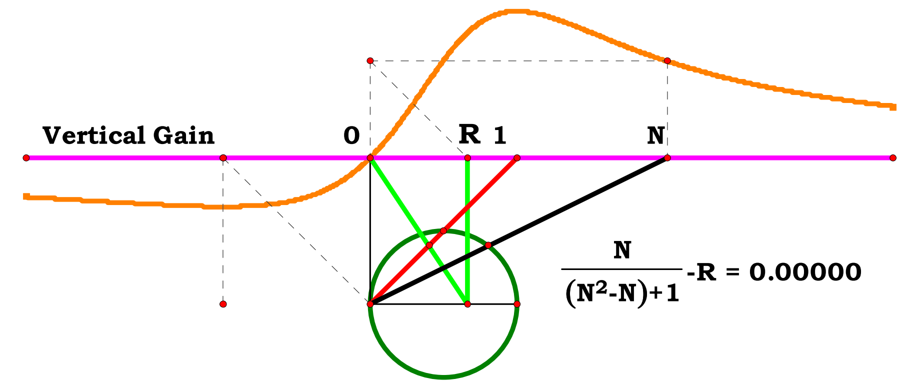
$$BJ := \frac{AN}{AN^2+1} \quad CJ := AB - BJ \quad EJ := \sqrt{BJ \cdot CJ} \quad AR := EJ$$

$$AR - \frac{\left(AN^3 - AN^2 + AN\right)^{\frac{1}{2}}}{AN^2+1} = 0$$

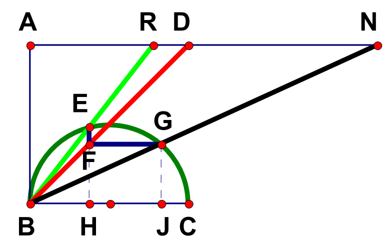


$$AB := 1$$

$$AN := 3$$

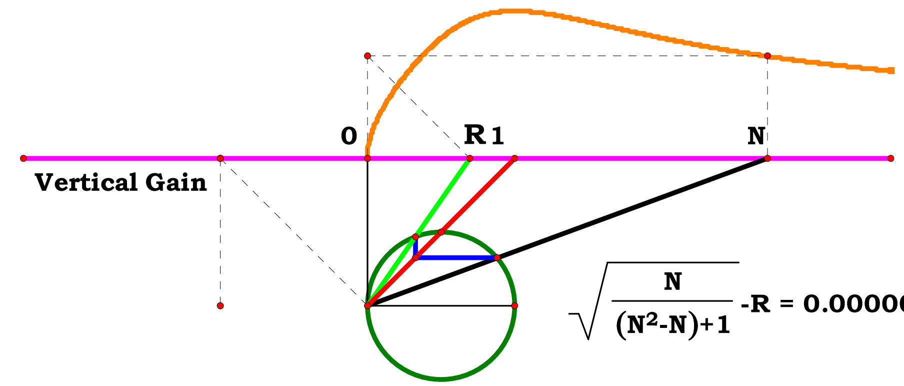


$$EG := \frac{AN}{AN^2 + 1} \qquad BH := \frac{EG \cdot AB}{AB - EG} \qquad AR := BH \qquad AR - \frac{AN}{AN^2 - AN + 1} = 0$$



$$\mathbf{AB} := 1$$

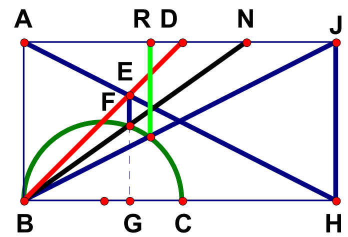
$$\mathbf{AN} := 3$$



$$\mathbf{BH} := \frac{\mathbf{AN}}{\mathbf{AN}^2 + 1} \qquad \mathbf{CH} := \mathbf{AB} - \mathbf{BH} \qquad \mathbf{EH} := \sqrt{\mathbf{BH} \cdot \mathbf{CH}}$$

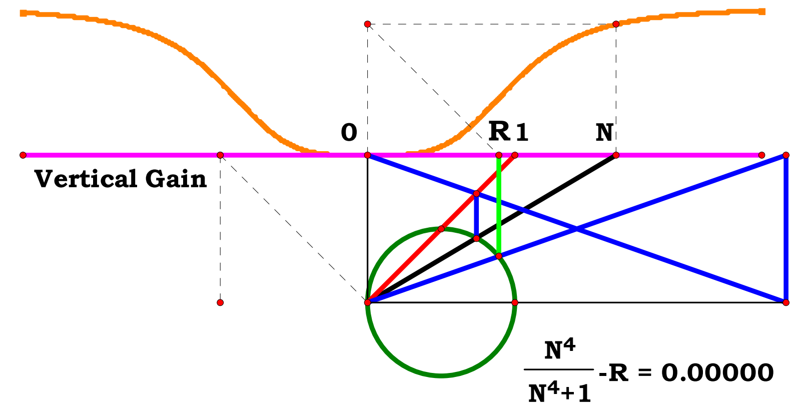
$$\mathbf{AR} := \frac{\mathbf{BH} \cdot \mathbf{AB}}{\mathbf{EH}} \qquad \mathbf{AR} - \frac{\mathbf{AN}}{\left(\mathbf{AN}^3 - \mathbf{AN}^2 + \mathbf{AN}\right)^{\frac{1}{2}}} = 0$$

$$\mathbf{AR} - \frac{\sqrt{\mathbf{AN}}}{\left(\mathbf{AN}^2 - \mathbf{AN} + 1\right)^{\frac{1}{2}}} = 0$$



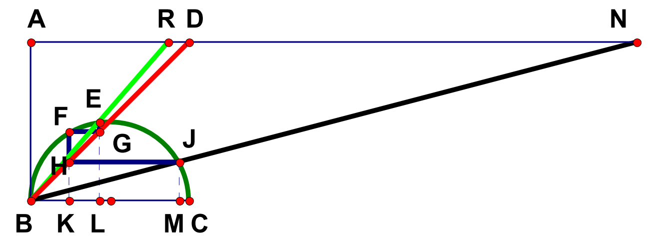
$$AB := 1$$

$$AN := 3$$

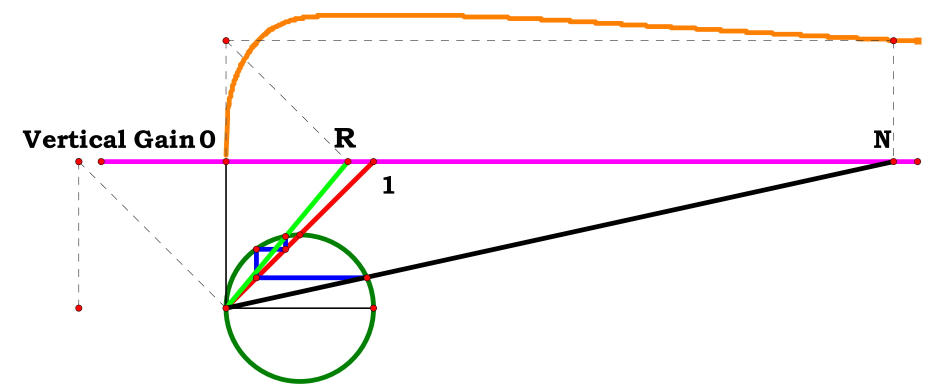


$$BG := \frac{AN^2}{AN^2+1} \qquad BH := \frac{BG \cdot AB}{AB - BG} \qquad AJ := BH \qquad AR := \frac{AJ^2}{AJ^2+1}$$

$$AR - \frac{AN^4}{AN^4+1} = 0$$



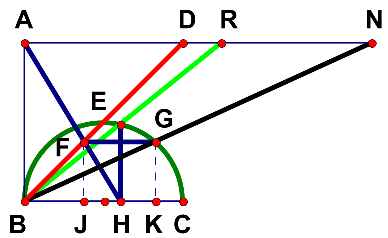
$$\frac{\sqrt{(N^3-N^2)+N}}{\sqrt{(((N^2\cdot\sqrt{(N^3-N^2)+N}+\sqrt{(N^3-N^2)+N})-N^3)+N^2)-N}}-R = 0.00000$$



$$AB := 1 \quad AN := 4 \quad BL := \frac{\left(AN^3 - AN^2 + AN\right)^{\frac{1}{2}}}{AN^2 + 1} \quad CL := AB - BL \quad EL := \sqrt{BL \cdot CL}$$

$$AR := \frac{BL \cdot AB}{EL}$$

$$AR - \frac{\left(AN^3 - AN^2 + AN\right)^{\frac{1}{2}}}{\left[\left(AN^3 - AN^2 + AN\right)^{\frac{1}{2}} \cdot AN^2 + \left(AN^3 - AN^2 + AN\right)^{\frac{1}{2}} - AN^3 + AN^2 - AN\right]^{\frac{1}{2}}} = 0$$

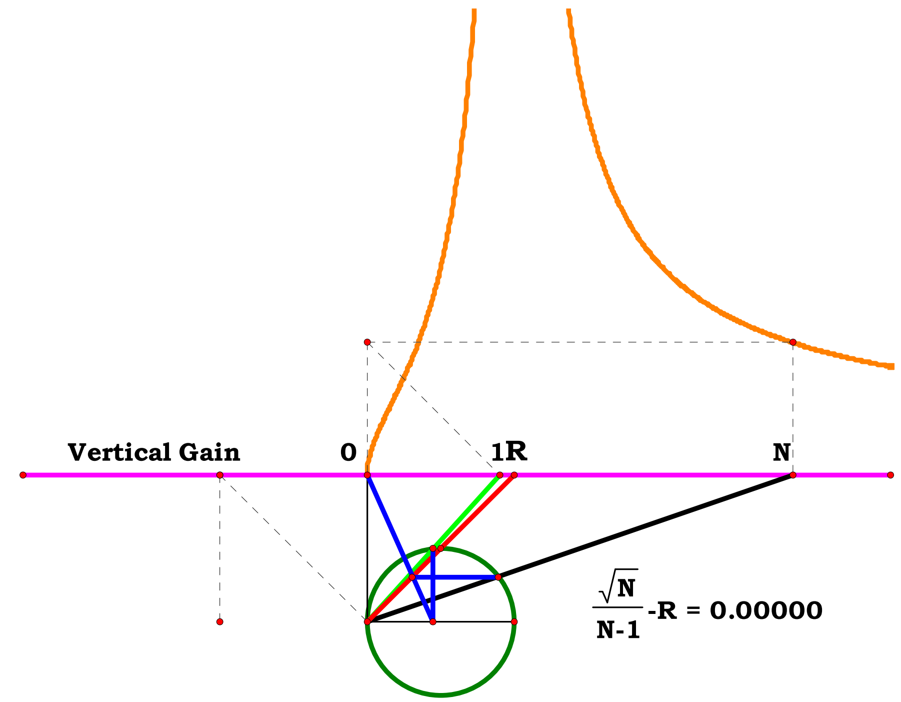


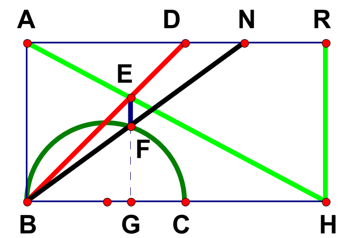
$AB := 1$

$AN := 3$

$$GK := \frac{AN}{AN^2 + 1} \quad BH := \frac{GK \cdot AB}{AB - GK} \quad CH := AB - BH \quad EH := \sqrt{BH \cdot CH}$$

$$AR := \frac{BH \cdot AB}{EH} \quad AR - \frac{AN}{\left(AN^3 - 2 \cdot AN^2 + AN\right)^{\frac{1}{2}}} = 0 \quad AR - \frac{\sqrt{AN}}{AN - 1} = 0$$

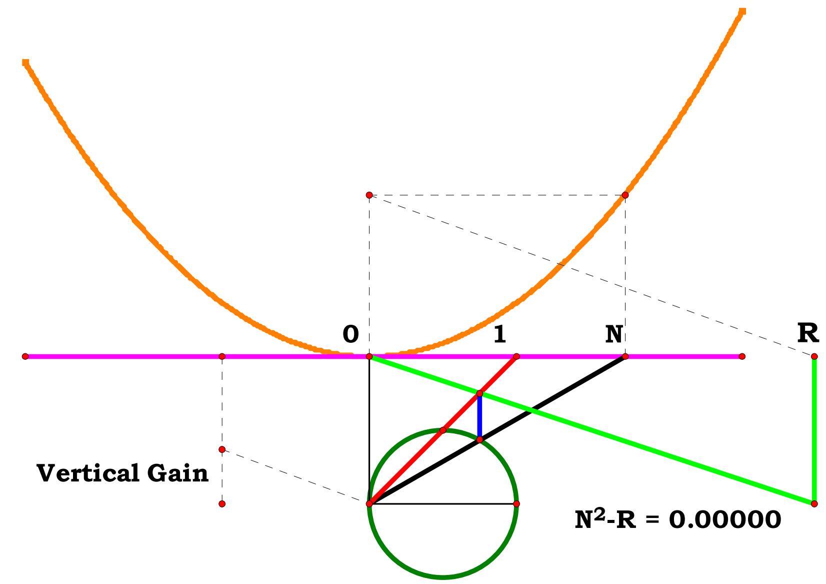


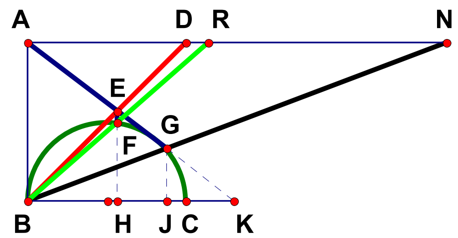


$AB := 1$

$AN := 3$

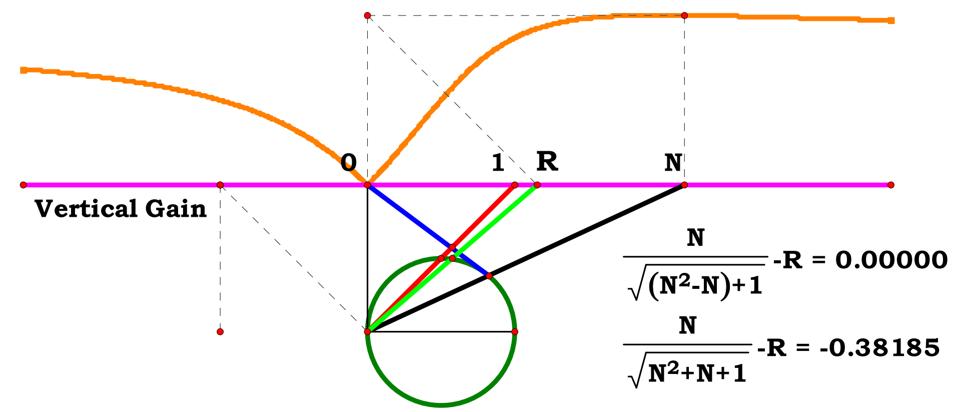
$$BG := \frac{AN^2}{AN^2 + 1} \qquad BH := \frac{BG \cdot AB}{AB - BG} \qquad AR := BH \qquad AR - AN^2 = 0$$





$AB := 1$

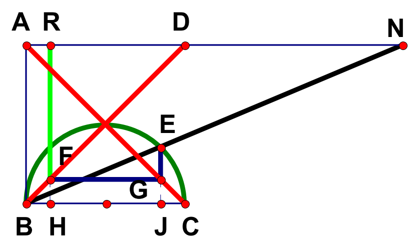
$AN := 3$



$$BJ := \frac{AN^2}{AN^2+1} \qquad GJ := \frac{AN}{AN^2+1} \qquad BK := \frac{BJ \cdot AB}{AB - GJ}$$

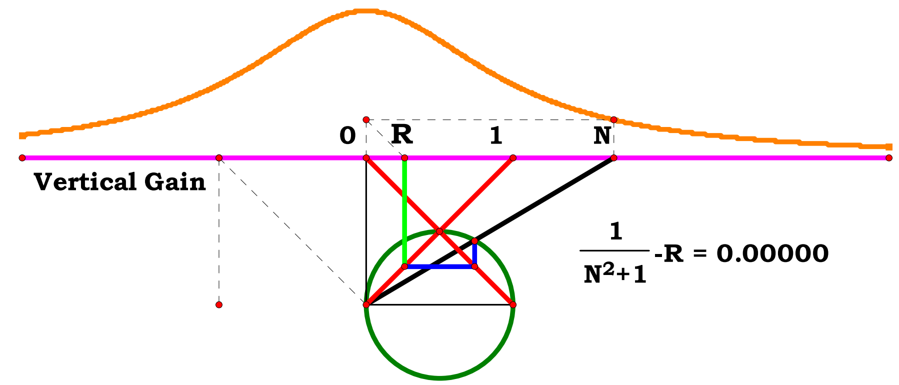
$$BH := \frac{AB \cdot BK}{AB + BK} \qquad CH := AB - BH \qquad FH := \sqrt{BH \cdot CH}$$

$$AR := \frac{BH \cdot AB}{FH} \qquad AR - \frac{AN}{\left(AN^2 - AN + 1\right)^{\frac{1}{2}}} = 0$$

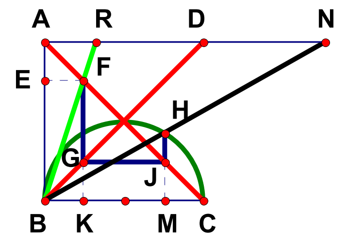


$AB := 1$

$AN := 3$



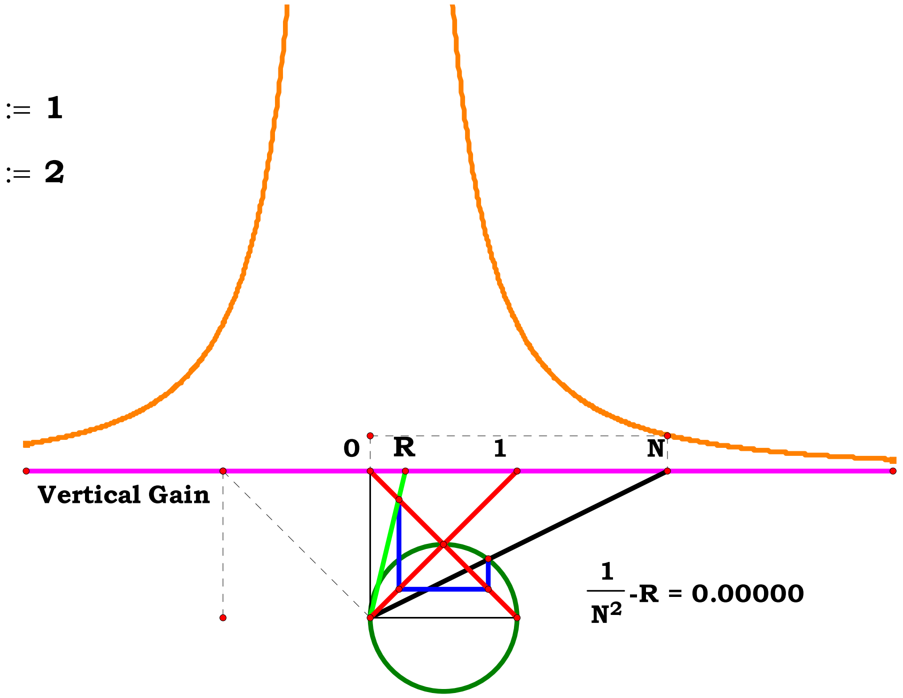
$$BH := \frac{1}{AN^2 + 1} \qquad AR := BH \qquad AR - \frac{1}{AN^2 + 1} = 0$$

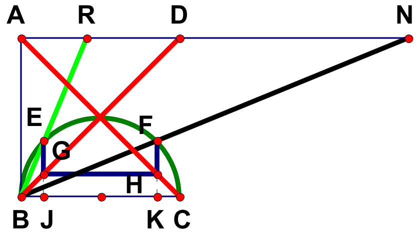


$$\mathbf{AB} := 1$$

$$\mathbf{AN} := 2$$

$$\mathbf{EF} := \frac{1}{\mathbf{AN}^2 + 1} \qquad \mathbf{AR} := \frac{\mathbf{EF} \cdot \mathbf{AB}}{\mathbf{AB} - \mathbf{EF}} \qquad \mathbf{AR} - \frac{1}{\mathbf{AN}^2} = 0$$

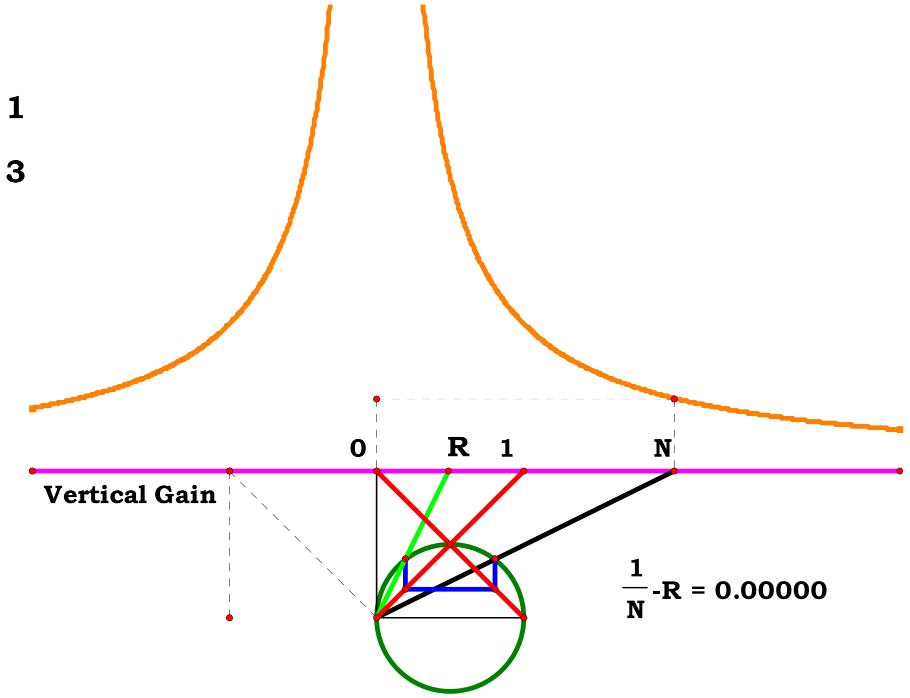


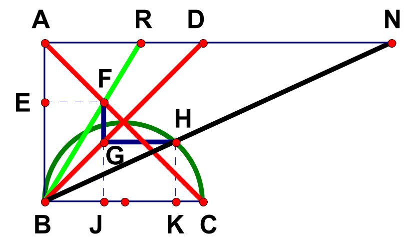


$\mathbf{AB} := \mathbf{1}$

$\mathbf{AN} := \mathbf{3}$

$$\mathbf{BJ} := \frac{\mathbf{1}}{\mathbf{AN}^{\mathbf{2}} + \mathbf{1}} \qquad \mathbf{EJ} := \frac{\mathbf{AN}}{\mathbf{AN}^{\mathbf{2}} + \mathbf{1}} \qquad \mathbf{AR} := \frac{\mathbf{BJ} \cdot \mathbf{AB}}{\mathbf{EJ}} \qquad \mathbf{AR} - \frac{\mathbf{1}}{\mathbf{AN}} = \mathbf{0}$$

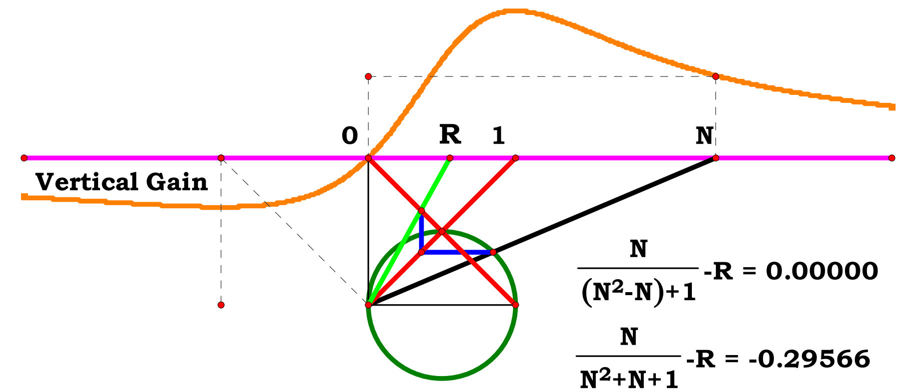




$AB := 1$

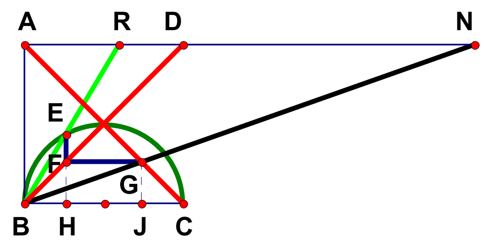
$AN := 3$

$$HK := \frac{AN}{AN^2 + 1} \qquad AR := \frac{HK \cdot AB}{AB - HK} \qquad AR - \frac{AN}{AN^2 - AN + 1} = 0$$



$$\frac{N}{(N^2 - N) + 1} - R = 0.00000$$

$$\frac{N}{N^2 + N + 1} - R = -0.29566$$

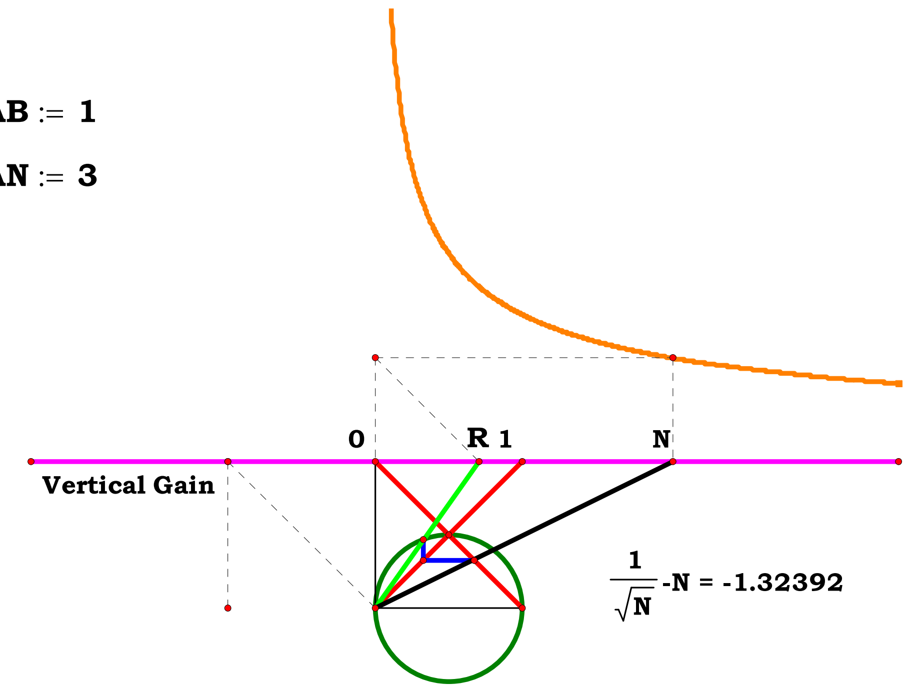


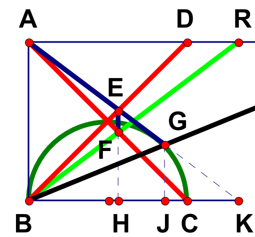
$AB := 1$

$AN := 3$

$CJ := \dfrac{1}{AN + 1}$ $\quad BH := CJ$ $\quad CH := AB - BH$ $\quad EH := \sqrt{BH \cdot CH}$

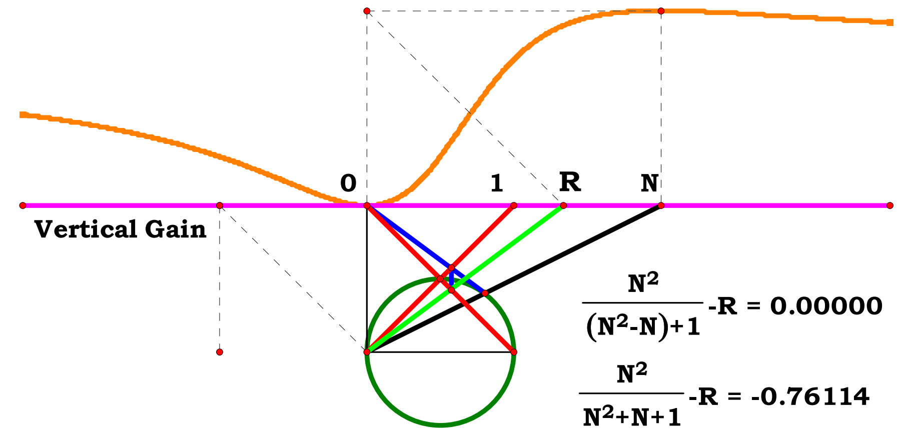
$AR := \dfrac{BH \cdot AB}{EH}$ $\quad AR - \dfrac{1}{AN^{.5}} = 0$





$$AB := 1$$

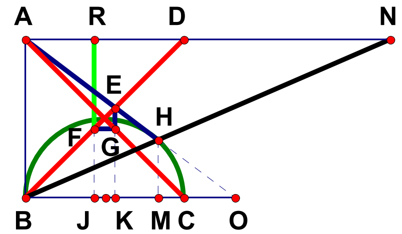
$$AN := 3$$



$$BJ := \frac{AN^2}{AN^2+1} \qquad GJ := \frac{AN}{AN^2+1} \qquad BK := \frac{BJ \cdot AB}{AB - GJ} \qquad BH := \frac{AB \cdot BK}{AB + BK}$$

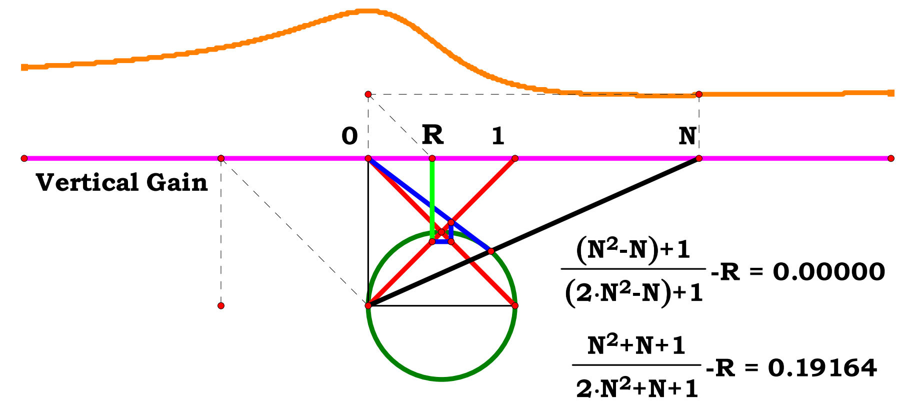
$$CH := AB - BH \qquad AR := \frac{BH \cdot AB}{CH} \qquad AR - \frac{AN^2}{AN^2 - AN + 1} = 0$$

$$BK - \frac{AN^2}{AN^2 - AN + 1} = 0 \qquad BH - \frac{AN^2}{2 \cdot AN^2 - AN + 1} = 0$$



$AB := 1$

$AN := 3$

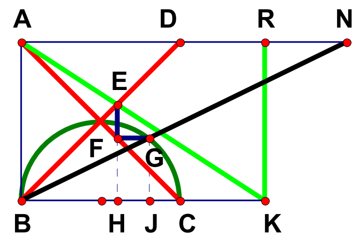


$$\frac{(N^2 - N)+1}{(2 \cdot N^2 - N)+1} - R = 0.00000$$

$$\frac{N^2 + N + 1}{2 \cdot N^2 + N + 1} - R = 0.19164$$

$$BK := \frac{AN^2}{2 \cdot AN^2 - AN + 1} \qquad CK := AB - BK \qquad BJ := CK \qquad AR := BJ$$

$$AR - \frac{AN^2 - AN + 1}{2AN^2 - AN + 1} = 0$$

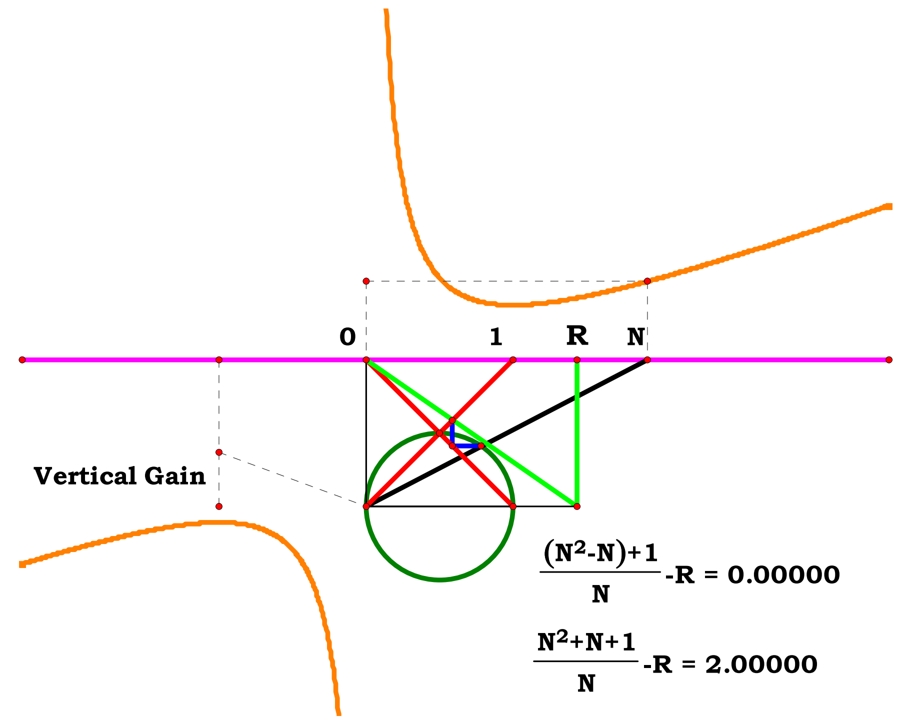


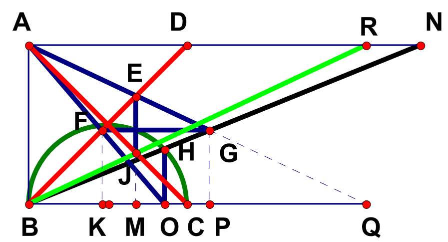
$AB := 1$

$AN := 3$

$GJ := \dfrac{AN}{AN^2 + 1} \quad CH := GJ \quad BH := AB - CH \quad BK := \dfrac{BH \cdot AB}{AB - BH}$

$AR := BK \quad AR - \dfrac{AN^2 - AN + 1}{AN} = 0$



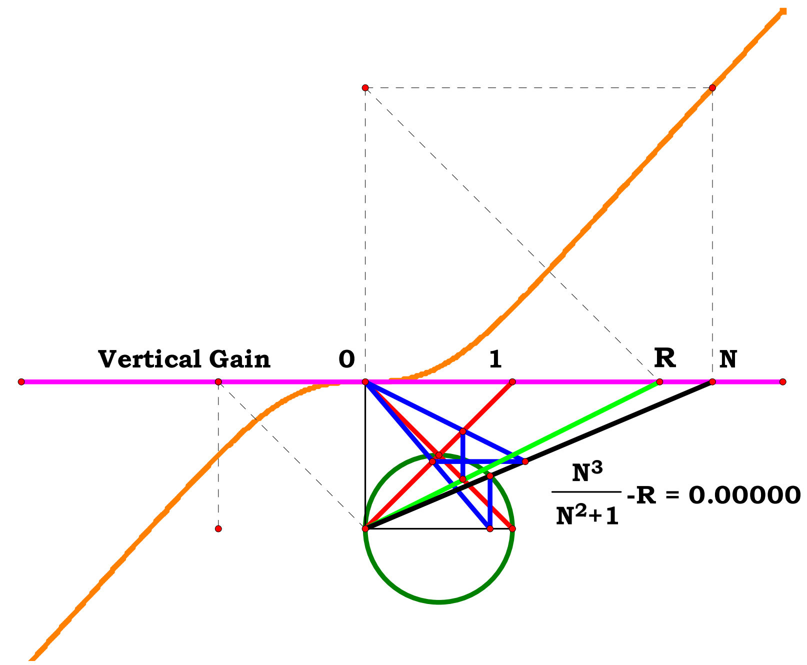


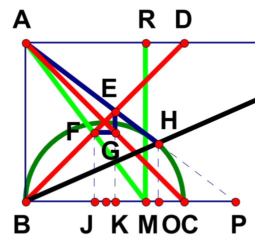
$AB := 1$

$AN := 3$

$$BO := \frac{AN^2}{AN^2 + 1} \qquad BK := \frac{AB \cdot BO}{AB + BO} \qquad BQ := \frac{AN \cdot BK}{AB - BK}$$

$$BM := \frac{AB \cdot BQ}{AB + BQ} \qquad CM := AB - BM \qquad AR := \frac{BM \cdot AB}{CM} \qquad AR - \frac{AN^3}{AN^2 + 1} = 0$$



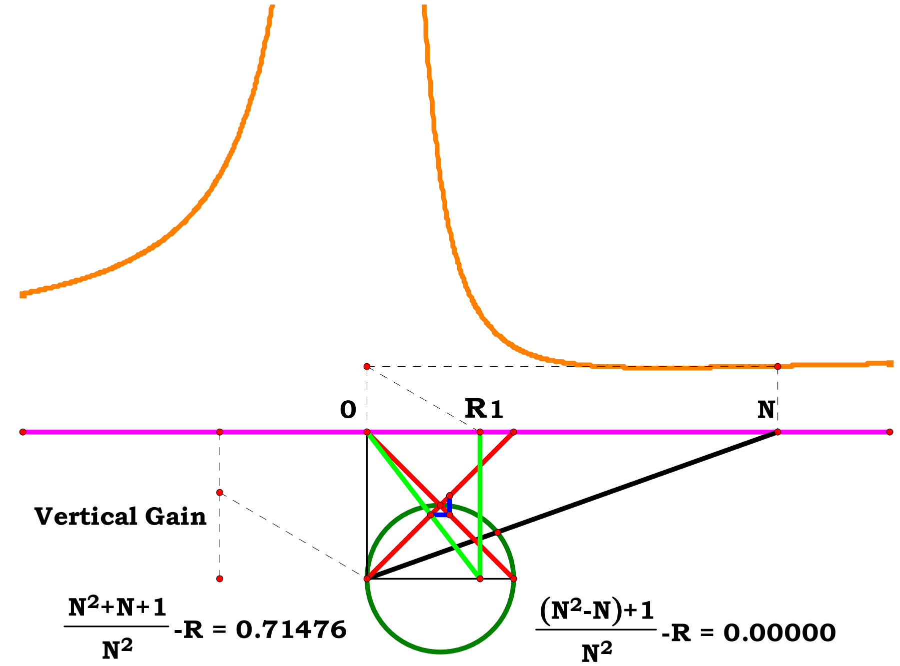


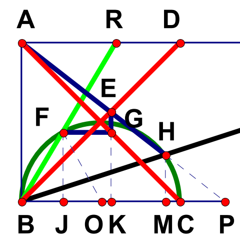
$$AB := 1$$

$$AN := 3$$

$$BP := \frac{AN^2}{AN^2 - AN + 1} \qquad BK := \frac{AB \cdot BP}{AB + BP} \qquad CK := AB - BK$$

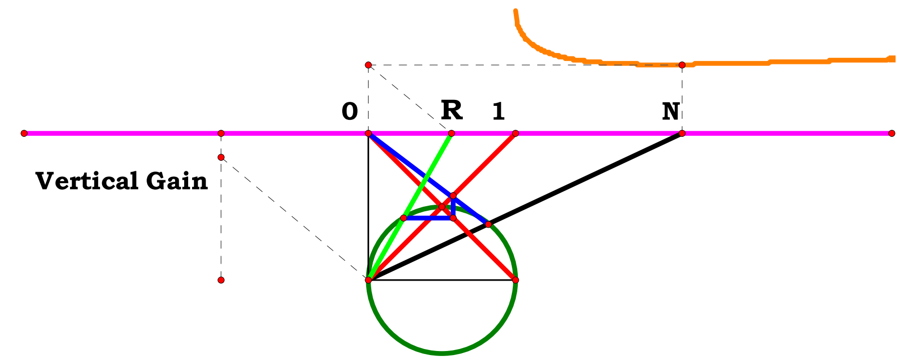
$$BJ := CK \qquad BM := \frac{BJ \cdot AB}{AB - BJ} \qquad AR := BM \qquad AR - \frac{AN^2 - AN + 1}{AN^2} = 0$$





$$AB := 1$$

$$AN := 3$$

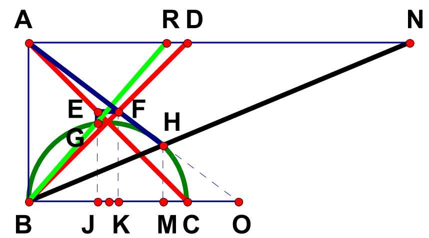


$$\frac{(2\cdot N^2+1)-N-\sqrt{((4\cdot N^3-7\cdot N^2)+6\cdot N)-3}}{(2\cdot N^2-2\cdot N)+2}-R = 0.00000$$

$$BK := \frac{AN^2}{2\cdot AN^2 - AN + 1} \qquad CK := AB - BK \qquad FO := \frac{AB}{2} \qquad FJ := CK$$

$$JO := \sqrt{FO^2 - FJ^2} \qquad BJ := FO - JO \qquad AR := \frac{BJ\cdot AB}{FJ}$$

$$AR - \frac{2\cdot AN^2 - AN + 1 - \left(4\cdot AN^3 - 7\cdot AN^2 + 6\cdot AN - 3\right)^{\frac{1}{2}}}{2\left(AN^2 - AN + 1\right)} = 0$$

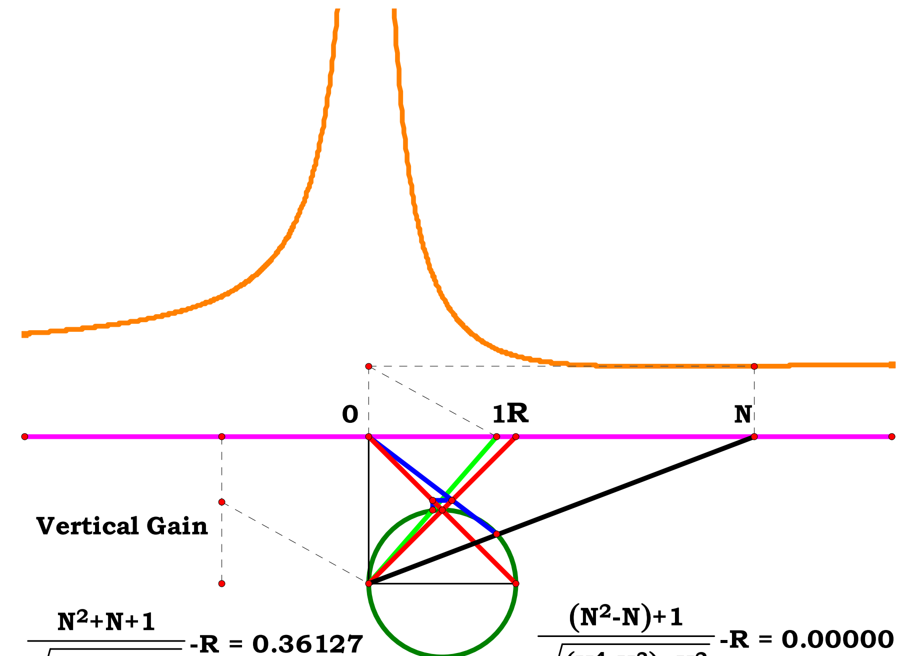


$$AB := 1$$

$$AN := 3$$

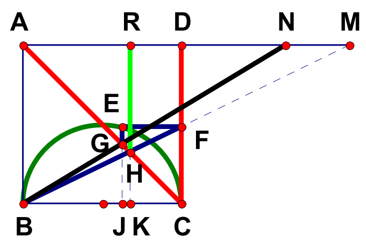
$$BK := \frac{AN^2}{2 \cdot AN^2 - AN + 1} \qquad CK := AB - BK \qquad BJ := CK \qquad CJ := AB - BJ$$

$$EJ := \sqrt{BJ \cdot CJ} \qquad AR := \frac{BJ \cdot AB}{EJ} \qquad AR - \frac{AN^2 - AN + 1}{\left(AN^4 - AN^3 + AN^2\right)^{\frac{1}{2}}} = 0$$



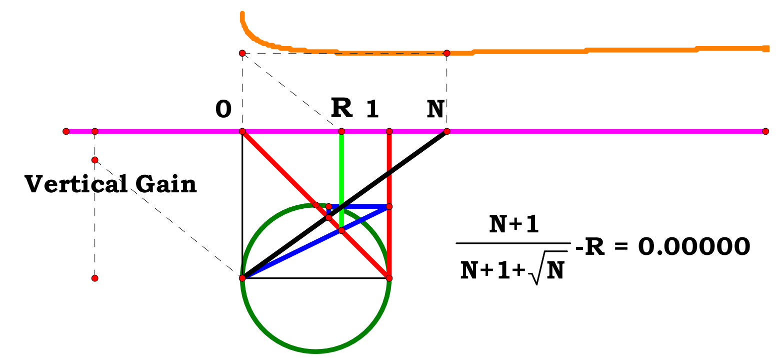
$$\frac{N^2+N+1}{\sqrt{N^4+N^3+N^2}}\text{-R} = 0.36127$$

$$\frac{(N^2-N)+1}{\sqrt{(N^4-N^3)+N^2}}\text{-R} = 0.00000$$



$$AB := 1$$

$$AN := 2$$

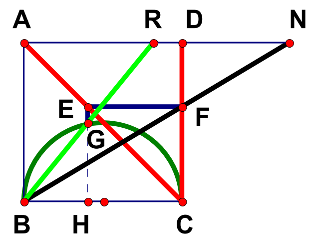


$$\frac{N+1}{N+1+\sqrt{N}}-R = 0.00000$$

$$CJ := \frac{1}{AN+1} \quad BJ := AB - CJ \quad EJ := \sqrt{BJ \cdot CJ} \quad CF := EJ$$

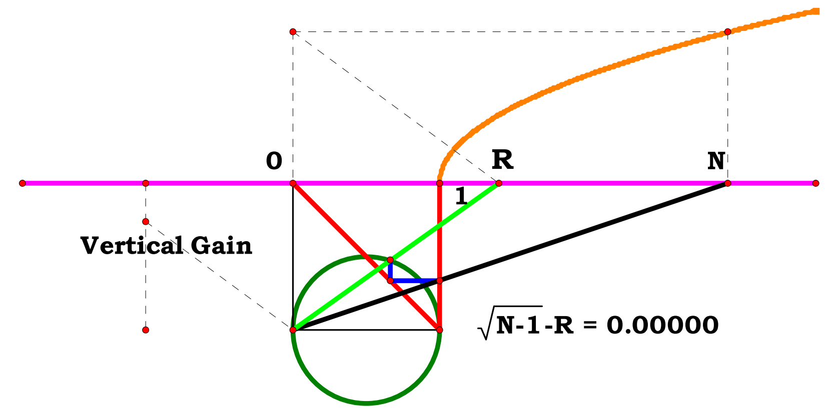
$$AM := \frac{AB^2}{CF} \quad BK := \frac{AB \cdot AM}{AB + AM} \quad AR := BK \quad AR - \frac{AN+1}{AN+1+AN^{.5}} = 0$$

$$EJ - \frac{AN^{\frac{1}{2}}}{AN+1} = 0 \quad AM - \frac{AN+1}{AN^{\frac{1}{2}}} = 0$$



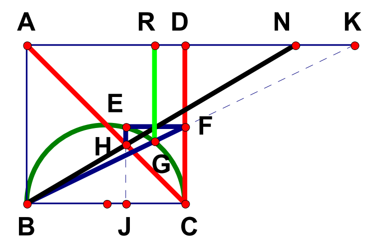
$\mathbf{AB} := \mathbf{1}$

$\mathbf{AN} := \mathbf{2}$



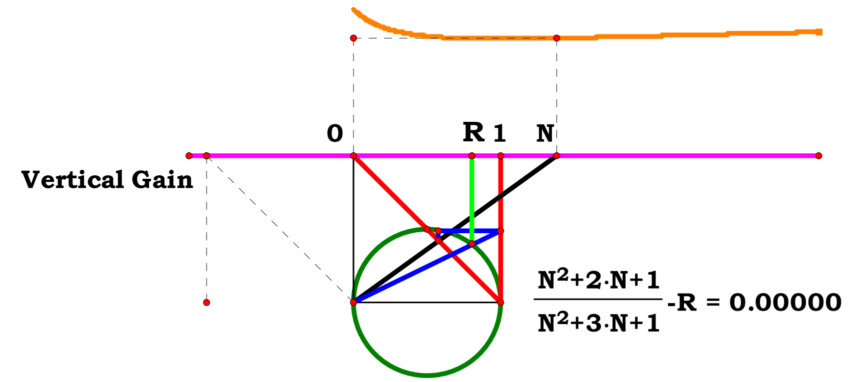
$$\mathbf{CF} := \frac{\mathbf{1}}{\mathbf{AN}} \qquad \mathbf{CH} := \mathbf{CF} \qquad \mathbf{BH} := \mathbf{AB} - \mathbf{CH} \qquad \mathbf{GH} := \sqrt{\mathbf{BH} \cdot \mathbf{CH}}$$

$$\mathbf{AR} := \frac{\mathbf{BH} \cdot \mathbf{AB}}{\mathbf{GH}} \qquad \mathbf{AR} - (\mathbf{AN} - \mathbf{1})^{.5} = \mathbf{0}$$

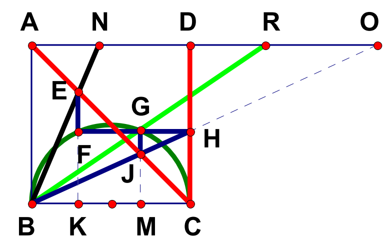


$$AB := 1$$

$$AN := 2$$

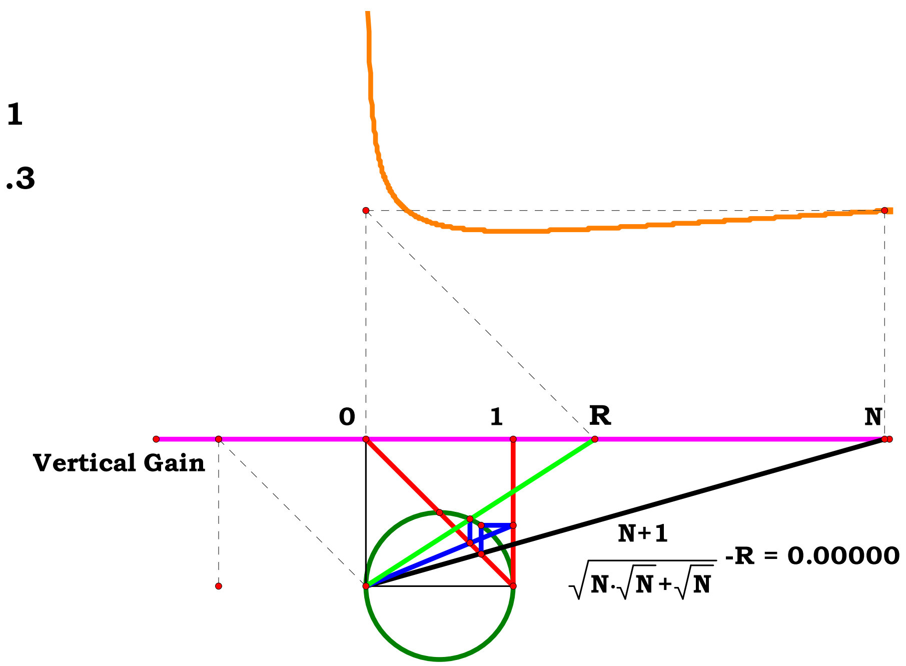


$$AK := \frac{AN+1}{AN^{\frac{1}{2}}} \qquad AR := \frac{AK^2}{AK^2+1} \qquad AR - \frac{AN^2+2\cdot AN+1}{AN^2+3\cdot AN+1} = 0$$



$AB := 1$

$AN := .3$

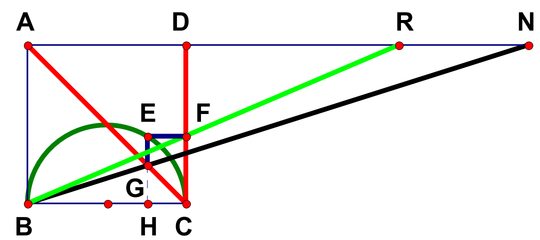


$$FK := \frac{AN^{.5}}{AN+1} \qquad CH := FK \qquad AO := \frac{AB^2}{CH} \qquad BM := \frac{AO}{AO+1}$$

$$GM := \frac{AO^{.5}}{AO+1} \qquad AR := \frac{BM\cdot AB}{GM} \qquad AR - \frac{(AN+1.)^{\frac{1}{2}}}{AN^{\frac{1}{4}}} = 0$$

$$AR - \frac{AN+1}{AN^{\frac{1}{4}}\cdot(AN+1)^{\frac{1}{2}}} = 0$$

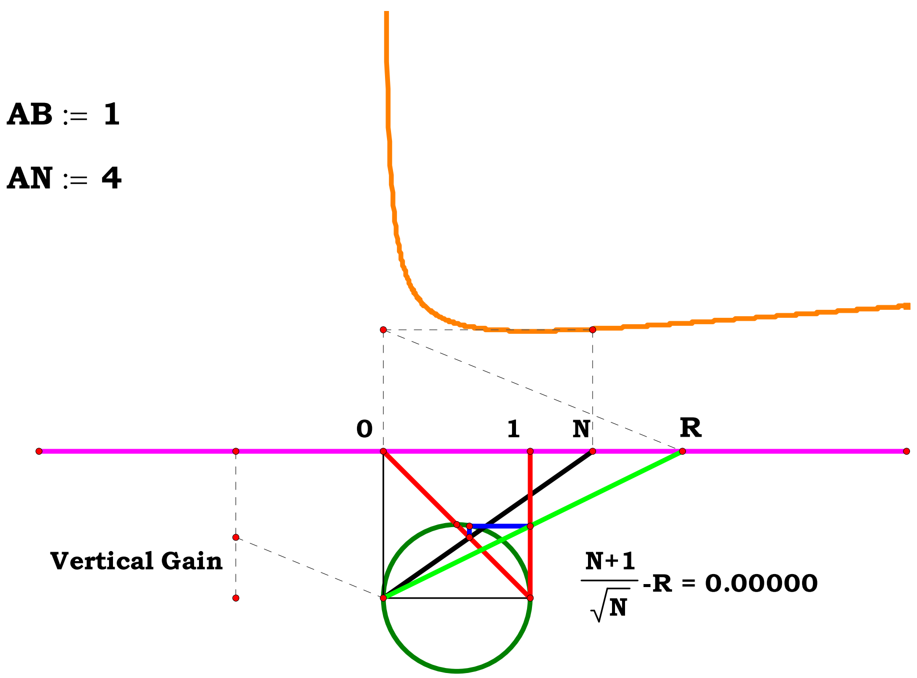
$$AO - \frac{AN+1}{AN^{.5}} = 0$$

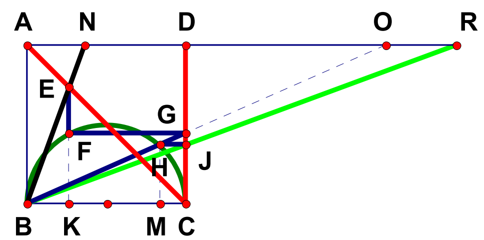


$AB := 1$

$AN := 4$

$EH := \dfrac{AN^{.5}}{AN + 1}$ $\quad CF := EH$ $\quad AR := \dfrac{AB^2}{CF}$ $\quad AR - \dfrac{AN + 1}{AN^{.5}} = 0$

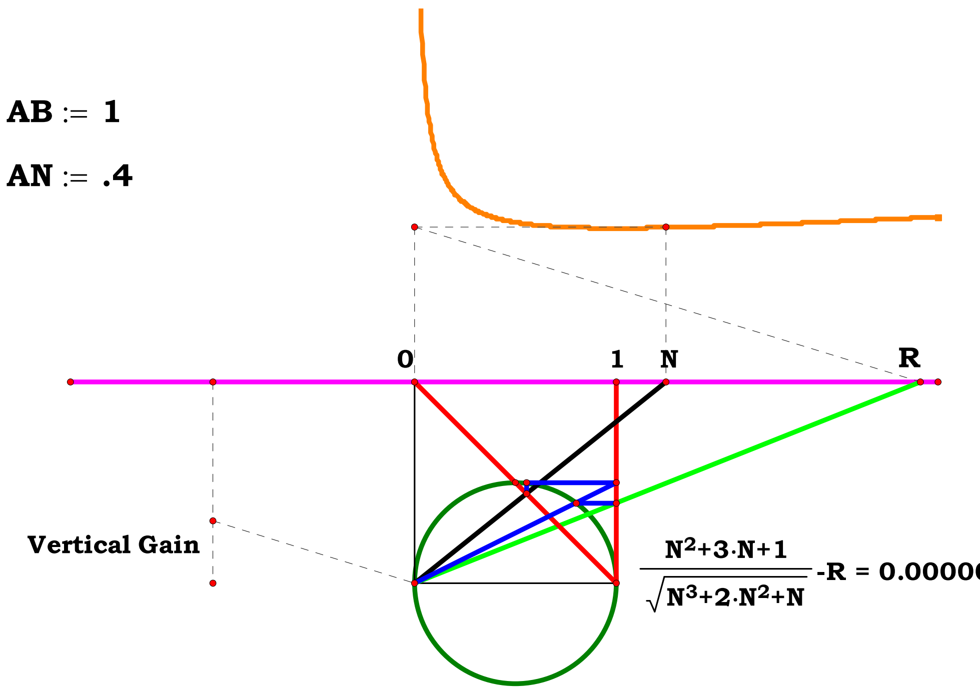


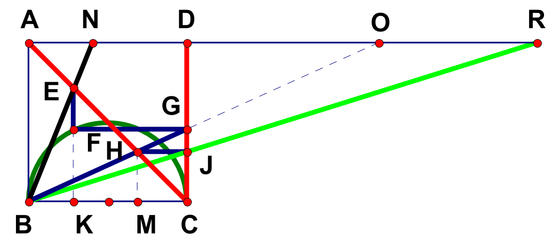


$AB := 1$

$AN := .4$

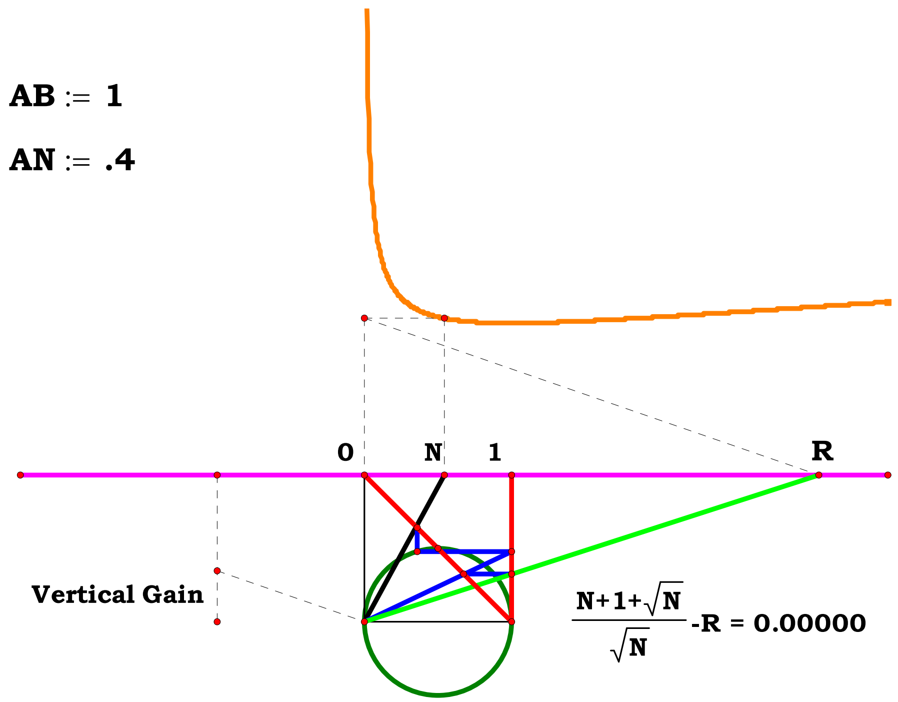
$$AO := \frac{AN + 1}{AN^{.5}} \qquad AR := \frac{AO^2 + 1}{AO} \qquad AR - \frac{AN^2 + 3 \cdot AN + 1}{AN^{\frac{3}{2}} + AN^{\frac{1}{2}}} = 0$$



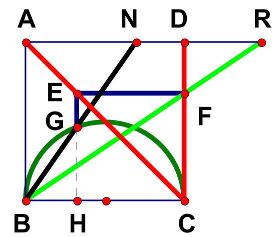


AB := 1

AN := .4



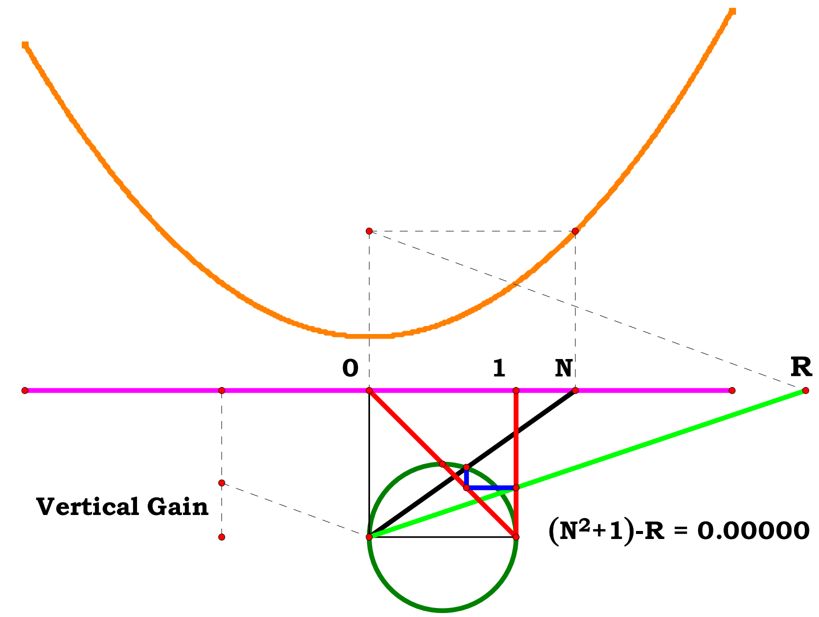
$$AO := \frac{AN + 1}{AN^{.5}} \qquad AR := AO + 1 \qquad AR - \frac{AN + 1. + AN^{\frac{1}{2}}}{AN^{\frac{1}{2}}} = 0$$

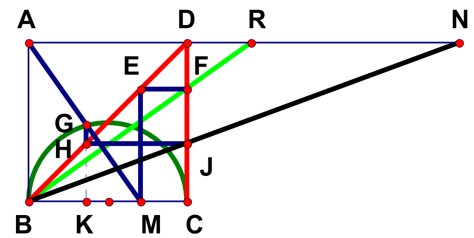


$\mathbf{AB} := 1$

$\mathbf{AN} := .7$

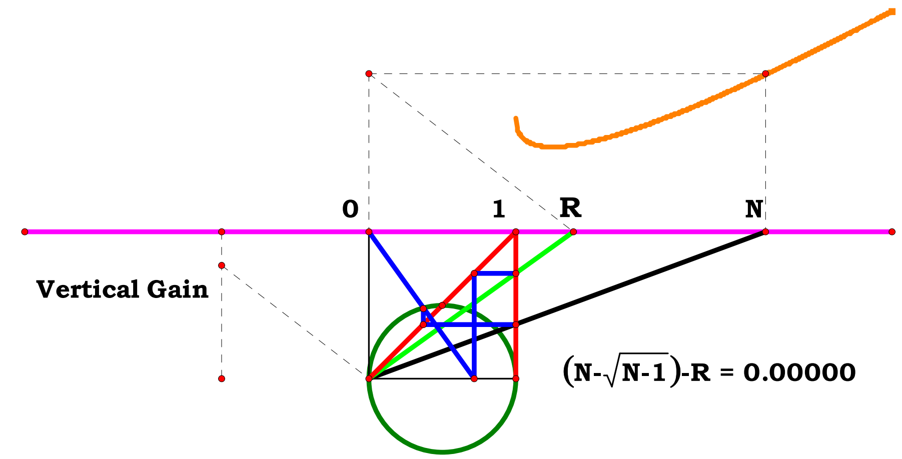
$$\mathbf{BH} := \frac{\mathbf{AN}^2}{\mathbf{AN}^2 + 1} \qquad \mathbf{DF} := \mathbf{BH} \qquad \mathbf{AR} := \frac{\mathbf{AB}^2}{\mathbf{AB} - \mathbf{DF}} \qquad \mathbf{AR} - \left(\mathbf{AN}^2 + 1\right) = 0$$





$AB := 1$

$AN := 3$

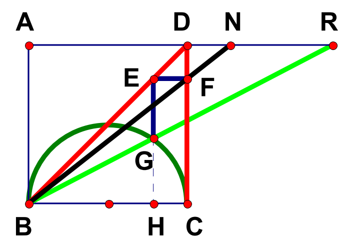


$CJ := \frac{1}{AN}$ $\quad BK := CJ$ $\quad CK := AB - BK$ $\quad GK := \sqrt{BK \cdot CK}$

$BM := \frac{BK \cdot AB}{AB - GK}$ $\quad CM := AB - BM$ $\quad AR := \frac{AB^2}{AB - CM}$

$$AR - \left[AN - (AN - 1)^{\frac{1}{2}}\right] = 0$$

$$GK - \frac{(AN - 1)^{\frac{1}{2}}}{AN} = 0$$

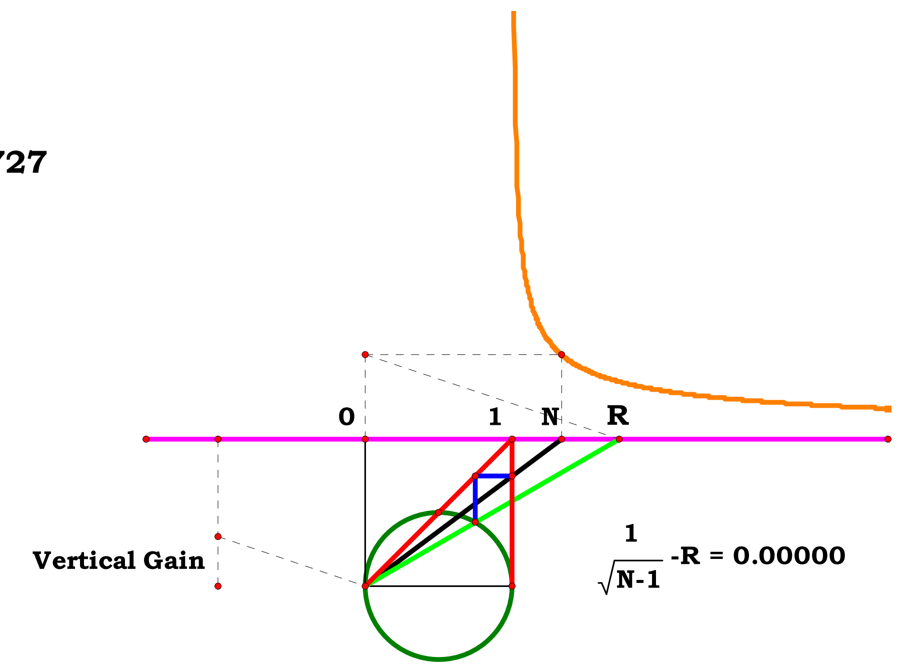


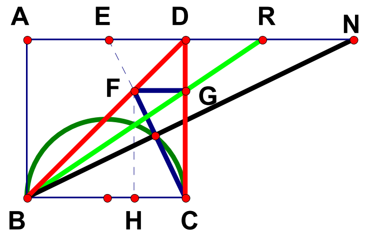
$\mathbf{AB} := \mathbf{1}$

$\mathbf{AN} := \mathbf{1.27272727}$

$\mathbf{CH} := \dfrac{\mathbf{AN} - \mathbf{1}}{\mathbf{AN}} \qquad \mathbf{BH} := \mathbf{AB} - \mathbf{CH} \qquad \mathbf{GH} := \sqrt{\mathbf{CH} \cdot \mathbf{BH}} \qquad \mathbf{AR} := \dfrac{\mathbf{BH} \cdot \mathbf{AB}}{\mathbf{GH}}$

$$\mathbf{AR} - \frac{\mathbf{1}}{(\mathbf{AN} - \mathbf{1})^{\frac{1}{2}}} = \mathbf{0}$$



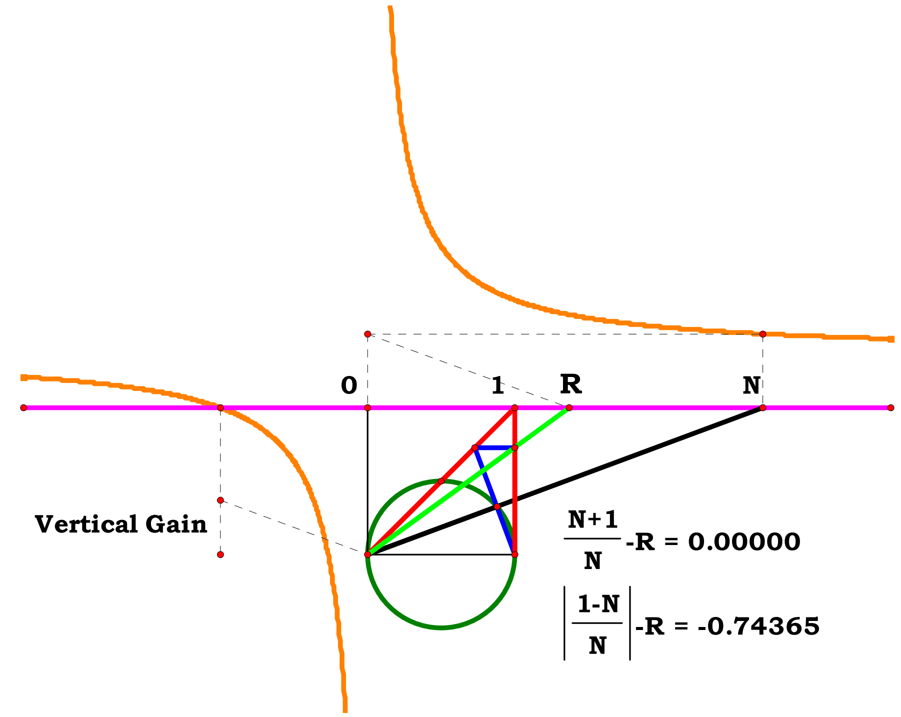
$AB := 1$

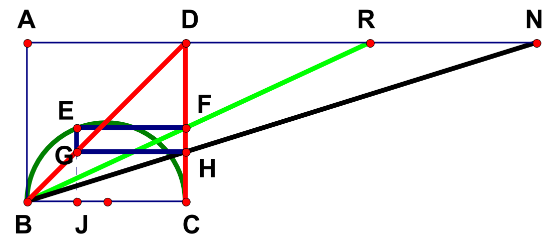
$AN := 3$

$DE := \dfrac{AB^2}{AN} \qquad CH := \dfrac{AB \cdot DE}{DE + AB} \qquad DG := CH \qquad AR := \dfrac{AB^2}{AB - DG}$

$AR - \dfrac{AN + 1}{AN} = 0$

$BH := AB - CH \quad BH - \dfrac{AN}{AN + 1} = 0 \qquad CH - \dfrac{1}{1 + AN} = 0$

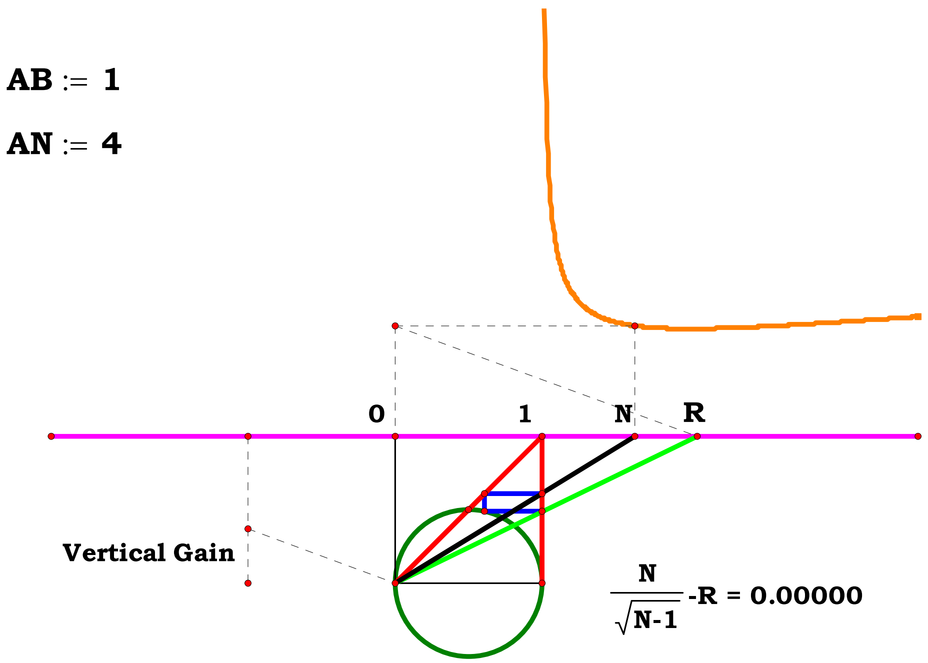


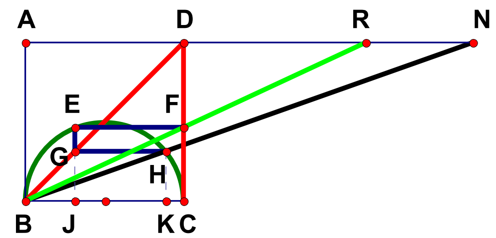


$$AB := 1$$

$$AN := 4$$

$$EJ := \frac{(AN-1)^{\frac{1}{2}}}{AN} \qquad CF := EJ \qquad AR := \frac{AB^2}{CF} \qquad AR - \frac{AN}{(AN-1)^{.5}} = 0$$



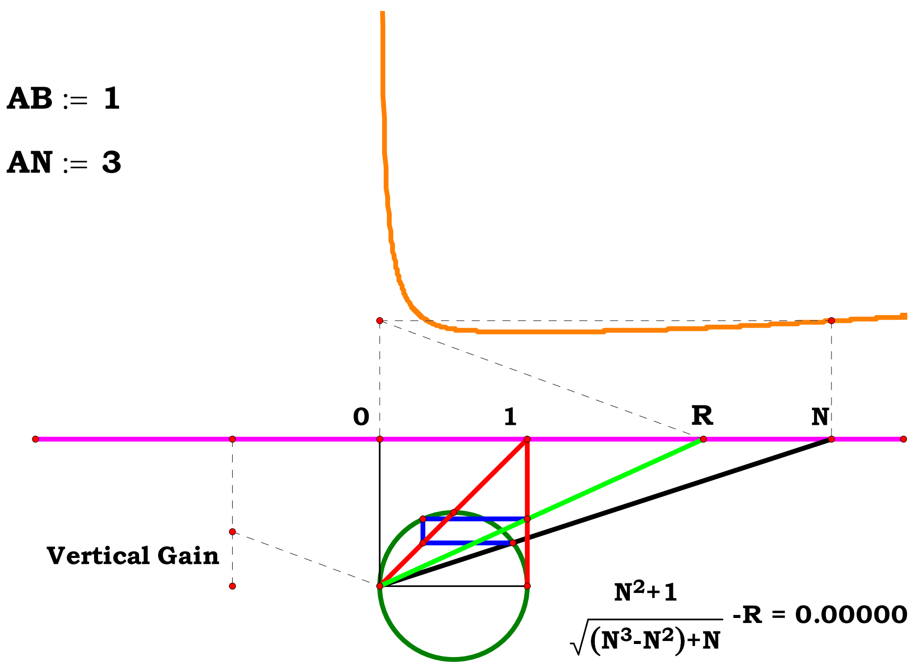


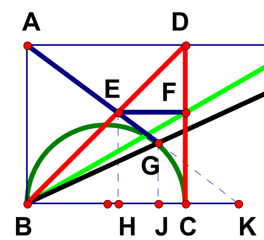
$$HK := \frac{AN}{AN^2 + 1} \qquad BJ := HK \qquad CJ := AB - BJ \qquad EJ := \sqrt{BJ \cdot CJ}$$

$$CF := EJ \qquad AR := \frac{AB^2}{CF} \qquad AR - \frac{AN^2 + 1}{\left(AN^3 - AN^2 + AN\right)^{\frac{1}{2}}} = 0$$

$AB := 1$

$AN := 3$



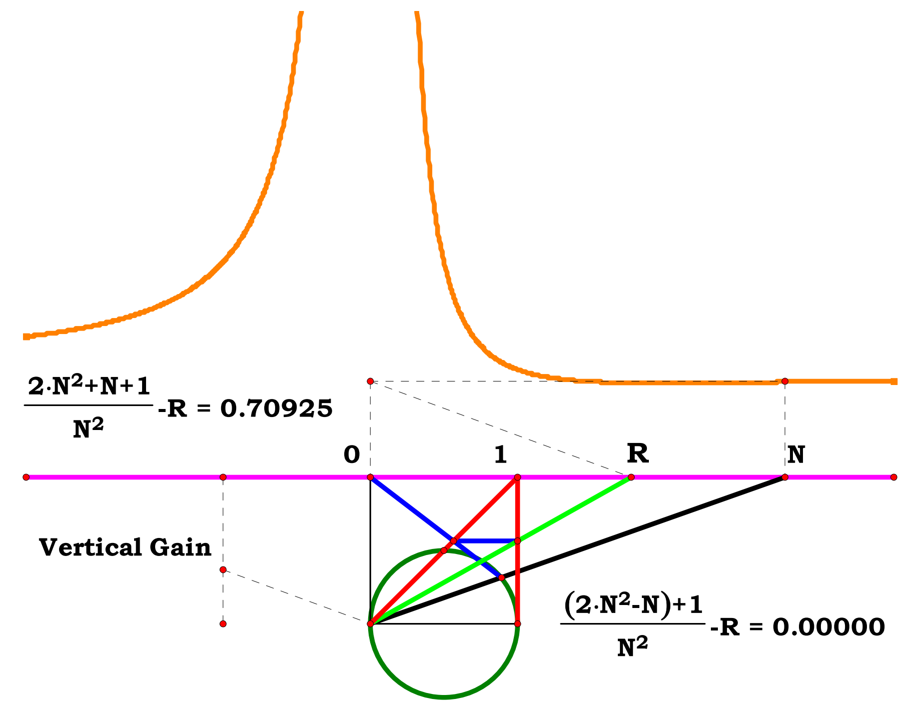


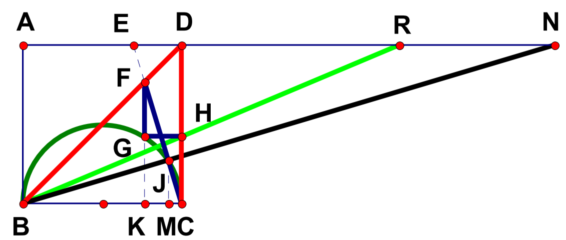
$$AB := 1$$

$$AN := 3$$

$$GJ := \frac{AN}{AN^2 + 1} \qquad BJ := \frac{AN^2}{AN^2 + 1} \qquad BK := \frac{BJ \cdot AB}{AB - GJ} \qquad BH := \frac{AB \cdot BK}{AB + BK}$$

$$CF := BH \qquad AR := \frac{AB^2}{CF} \qquad AR - \frac{2 \cdot AN^2 + 1 - AN}{AN^2} = 0$$



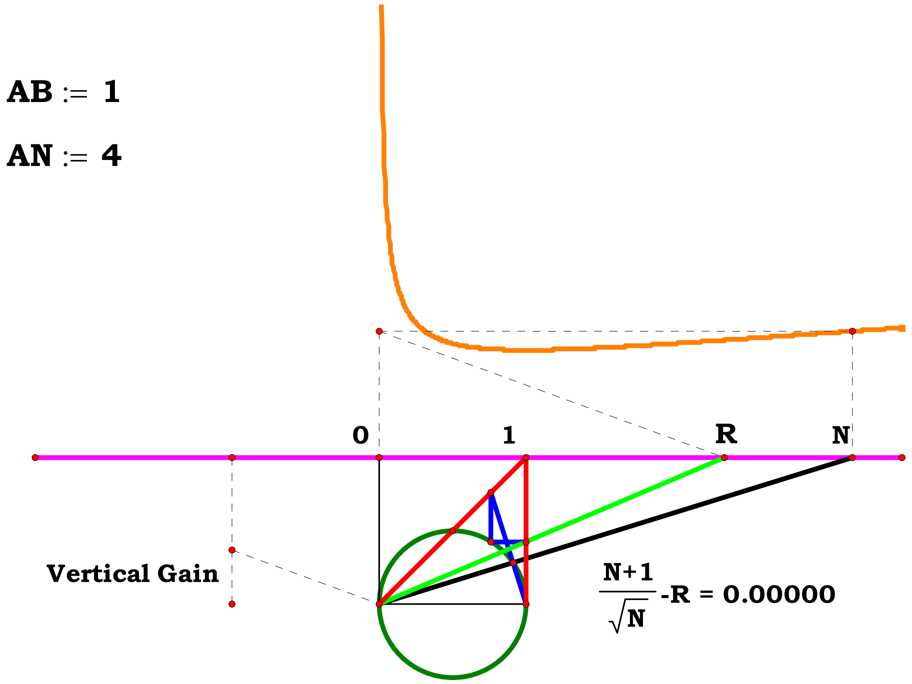


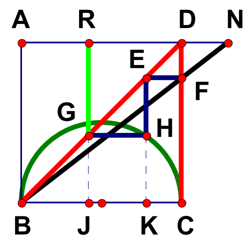
$AB := 1$

$AN := 4$

$$BK := \frac{AN}{AN+1} \qquad CK := \frac{1}{1+AN} \qquad GK := \sqrt{BK \cdot CK} \qquad CH := GK$$

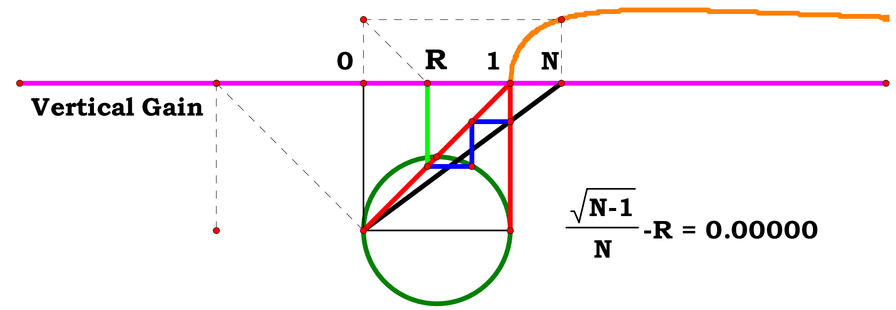
$$AR := \frac{AB^2}{CH} \qquad AR - \frac{AN+1}{AN^{\frac{1}{2}}} = 0$$





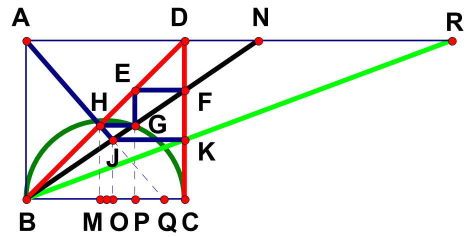
$$AB := 1$$

$$AN := 3$$



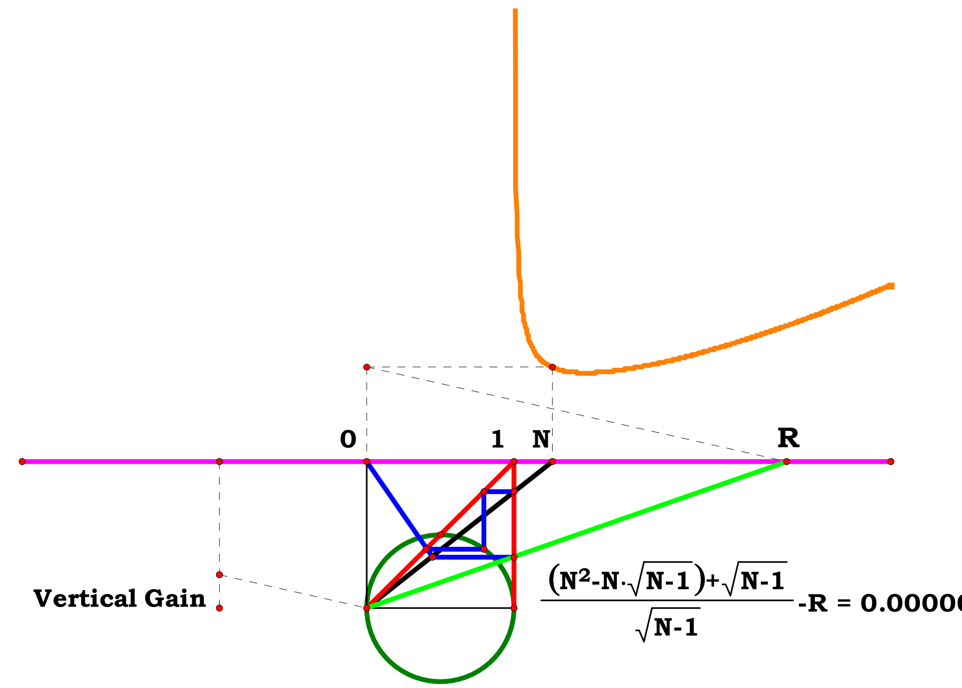
$$CK := \frac{AN - 1}{AN} \qquad BK := AB - CK \qquad HK := \sqrt{BK \cdot CK}$$

$$AR := HK \qquad AR - \frac{(AN - 1)^{\frac{1}{2}}}{AN} = 0$$



$$AB := 1$$

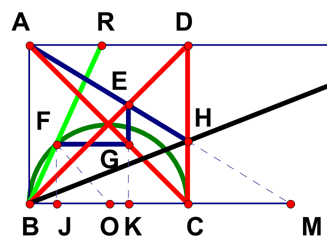
$$AN := 2$$



$$DF := \frac{AN - 1}{AN} \quad CP := DF \quad BP := AB - CP \quad GP := \sqrt{BP \cdot CP} \quad BM := GP$$

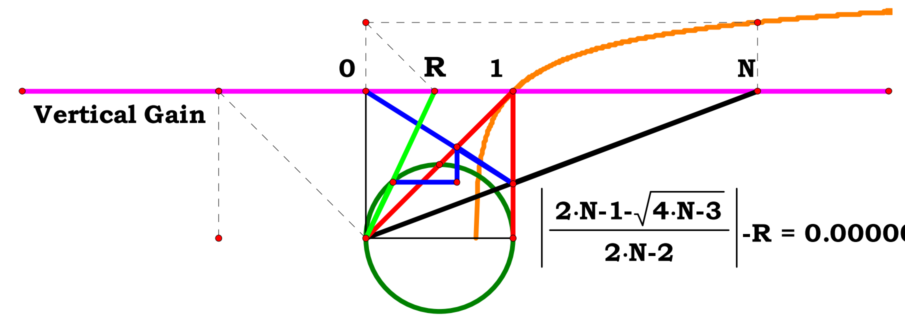
$$BQ := \frac{BM \cdot AB}{AB - GP} \quad BO := \frac{BQ \cdot AN}{BQ + AN} \quad JO := \frac{AB \cdot BO}{AN} \quad CK := JO \quad AR := \frac{AB^2}{CK}$$

$$AR - \frac{AN^2 - AN \cdot \sqrt{AN - 1} + \sqrt{AN - 1}}{\sqrt{AN - 1}} = 0$$



$$AB := 1$$

$$AN := 3$$

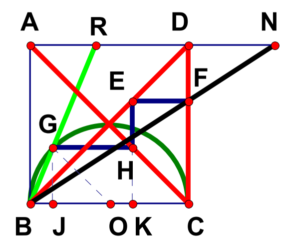


$$DH := \frac{AN - 1}{AN} \qquad BM := \frac{AB^2}{DH} \qquad BK := \frac{AB \cdot BM}{AB + BM} \qquad CK := AB - BK$$

$$FJ := CK \qquad JO := \sqrt{\left(\frac{AB}{2}\right)^2 - FJ^2} \qquad BJ := \frac{AB}{2} - JO \qquad AR := \frac{BJ \cdot AB}{FJ}$$

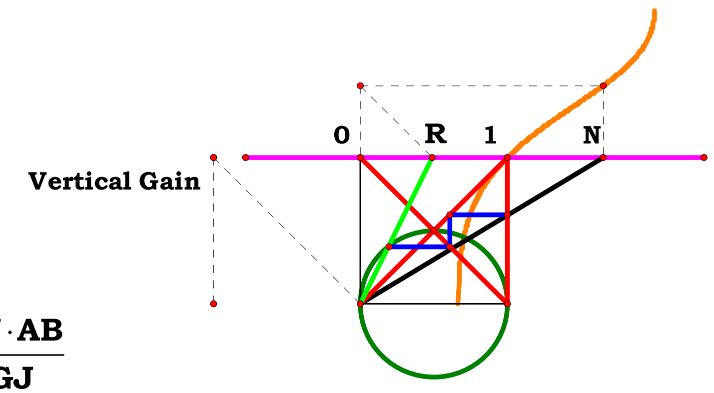
$$AR - \frac{2 \cdot AN - 1 - (4 \cdot AN - 3)^{\frac{1}{2}}}{2AN - 2} = 0$$

$$CK - \frac{AN - 1}{2 \cdot AN - 1} = 0$$



$$AB := 1$$

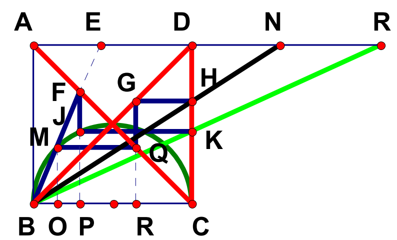
$$AN := 3$$



$$\frac{N-\sqrt{8\cdot N-3\cdot N^2-4}}{2\cdot N-2}-R = 0.00000$$

$$GJ := \frac{AN-1}{AN} \qquad JO := \sqrt{\left(\frac{AB}{2}\right)^2 - GJ^2} \qquad BJ := \frac{AB}{2} - JO \qquad AR := \frac{BJ\cdot AB}{GJ}$$

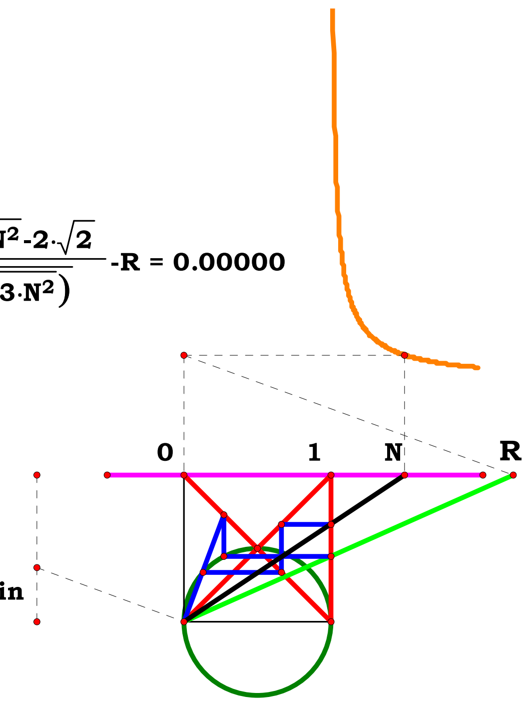
$$AR - \frac{AN - \left(8\cdot AN - 3\cdot AN^2 - 4\right)^{\frac{1}{2}}}{2AN-2} = 0$$



$$AB := 1$$

$$AN := 3$$

$$\frac{3\cdot N\cdot\sqrt{2}-\sqrt{2}\cdot\sqrt{8\cdot N-4-3\cdot N^2}-2\cdot\sqrt{2}}{2\cdot\sqrt{(N-1)\cdot\left(N-\sqrt{8\cdot N-4-3\cdot N^2}\right)}}-R = 0.00000$$

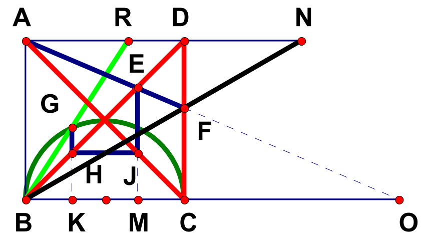


$$AE := \frac{AN-\left(8\cdot AN-3\cdot AN^2-4\right)^{\frac{1}{2}}}{2AN-2} \qquad BP := \frac{AB\cdot AE}{AB+AE} \qquad CP := AB-BP$$

$$JP := \sqrt{BP\cdot CP} \qquad CK := JP \qquad AR := \frac{AB^2}{CK}$$

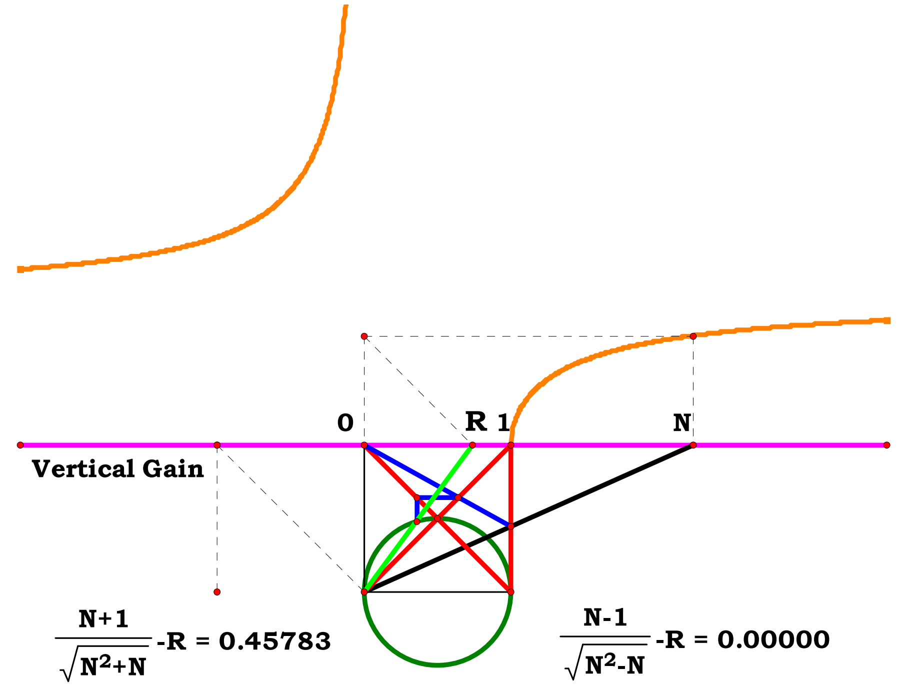
$$AR-\frac{2^{\frac{1}{2}}\cdot\left[3\cdot AN-2-\left(8\cdot AN-3\cdot AN^2-4\right)^{\frac{1}{2}}\right]}{2\cdot\left[AN^2-AN-\left(8\cdot AN-3\cdot AN^2-4\right)^{\frac{1}{2}}\cdot AN+\left(8\cdot AN-3\cdot AN^2-4\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}}=0$$

$$BP-\frac{AN-\left(8\cdot AN-3\cdot AN^2-4\right)^{\frac{1}{2}}}{3\cdot AN-2-\left(8\cdot AN-3\cdot AN^2-4\right)^{\frac{1}{2}}}=0$$



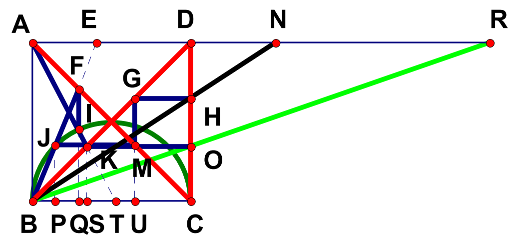
$$AB := 1$$

$$AN := 3$$



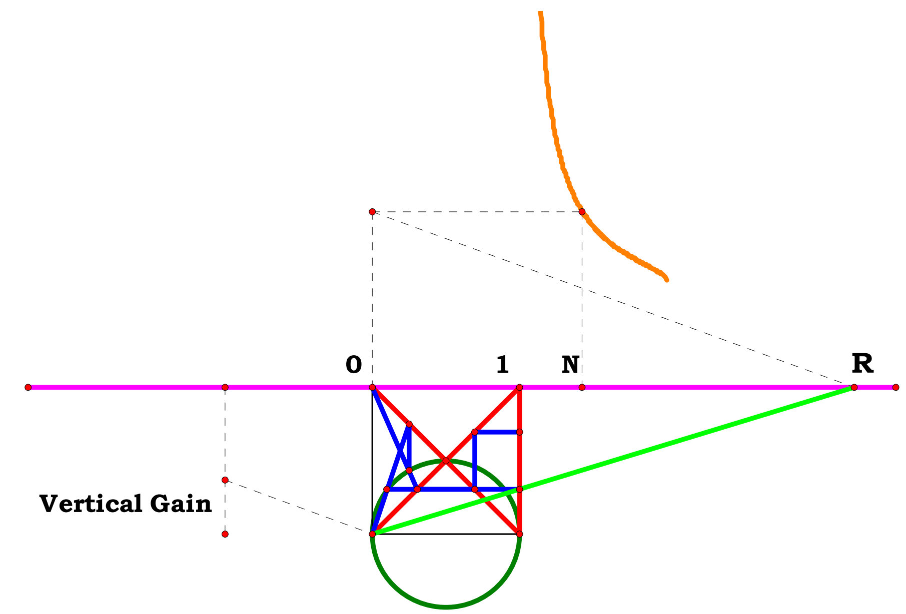
$$CM := \frac{AN - 1}{2 \cdot AN - 1} \quad BK := CM \quad CK := AB - BK \quad GK := \sqrt{BK \cdot CK}$$

$$AR := \frac{BK \cdot AB}{GK} \quad AR - \frac{AN - 1}{\left(AN^2 - AN\right)^{\frac{1}{2}}} = 0$$



$AB := 1$

$AN := 3$



$$\frac{4\cdot N-2-2\cdot\sqrt{8\cdot N-3\cdot N^2-4}-\sqrt{2}\cdot\sqrt{\left(N^2-N-\sqrt{8\cdot N-3\cdot N^2-4}\cdot N\right)+\sqrt{8\cdot N-3\cdot N^2-4}}}{N-\sqrt{8\cdot N-3\cdot N^2-4}}-R = 0.00000$$

$$BQ := \frac{AN-\left(8\cdot AN-3\cdot AN^2-4\right)^{\frac{1}{2}}}{3\cdot AN-2-\left(8\cdot AN-3\cdot AN^2-4\right)^{\frac{1}{2}}} \qquad CQ := AB-BQ \qquad IQ := \sqrt{BQ\cdot CQ}$$

$$BT := \frac{BQ\cdot AB}{AB-IQ} \qquad BS := \frac{AB\cdot BT}{AB+BT} \qquad CO := BS \qquad AR := \frac{AB^2}{CO}$$

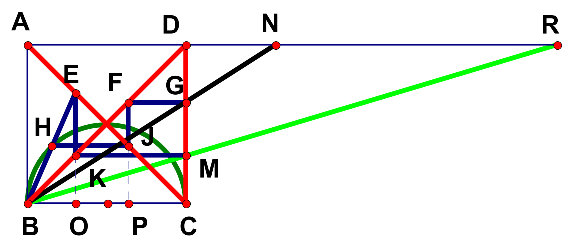
$$AR - \frac{4\cdot AN-2-2\cdot\left(8\cdot AN-3\cdot AN^2-4\right)^{\frac{1}{2}}-2^{\frac{1}{2}}\cdot\left[(AN-1)\cdot\left[AN-\left(8\cdot AN-3\cdot AN^2-4\right)^{\frac{1}{2}}\right]\right]^{\frac{1}{2}}}{AN-\left(8\cdot AN-3\cdot AN^2-4\right)^{\frac{1}{2}}} = 0$$

$$CQ - \frac{2AN-2}{3\cdot AN-2-\left(8\cdot AN-3\cdot AN^2-4\right)^{\frac{1}{2}}} = 0$$

$$IQ - \frac{\sqrt{2\cdot AN^2-2\cdot AN-2\cdot\left(8\cdot AN-3\cdot AN^2-4\right)^{\frac{1}{2}}\cdot AN+2\cdot\left(8\cdot AN-3\cdot AN^2-4\right)^{\frac{1}{2}}}}{3\cdot AN-2-\left(8\cdot AN-3\cdot AN^2-4\right)^{\frac{1}{2}}} = 0$$

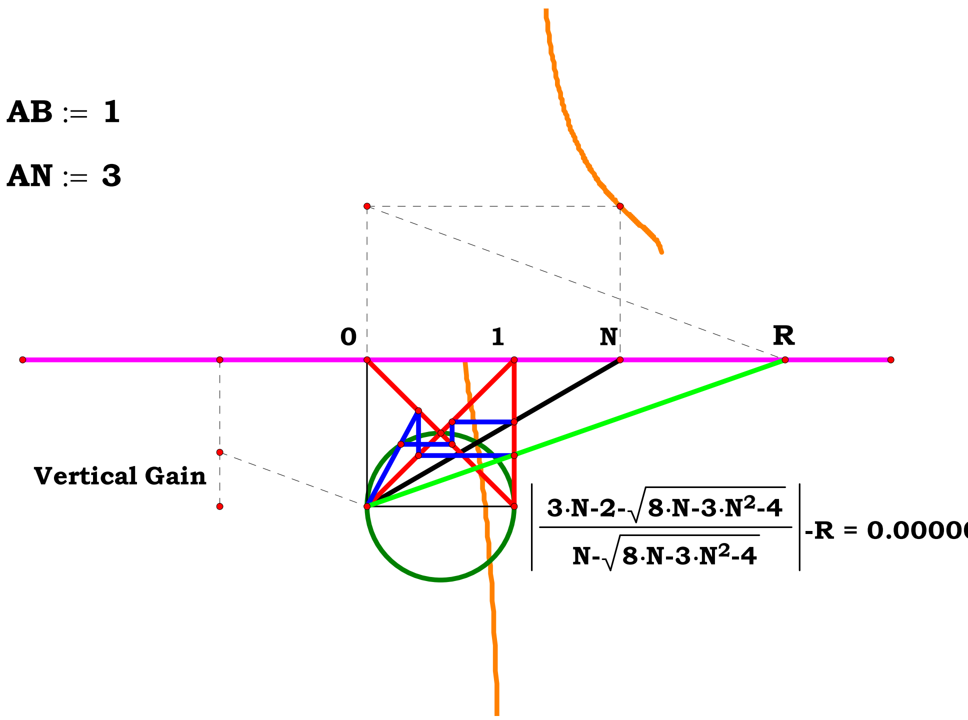
$$BS - \frac{AN-\left(8\cdot AN-3\cdot AN^2-4\right)^{\frac{1}{2}}}{4\cdot AN-2-2\cdot\left(8\cdot AN-3\cdot AN^2-4\right)^{\frac{1}{2}}-2^{\frac{1}{2}}\cdot\left[(AN-1)\cdot\left[AN-\left(8\cdot AN-3\cdot AN^2-4\right)^{\frac{1}{2}}\right]\right]^{\frac{1}{2}}} = 0$$

$$BT - \frac{(-AN)+\left(8\cdot AN-3\cdot AN^2-4\right)^{\frac{1}{2}}}{(-3)\cdot AN+2+\left(8\cdot AN-3\cdot AN^2-4\right)^{\frac{1}{2}}+2^{\frac{1}{2}}\cdot\left[[-(AN-1)]\cdot\left[(-AN)+\left(8\cdot AN-3\cdot AN^2-4\right)^{\frac{1}{2}}\right]\right]^{\frac{1}{2}}} = 0$$



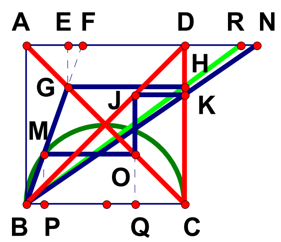
$$AB := 1$$

$$AN := 3$$



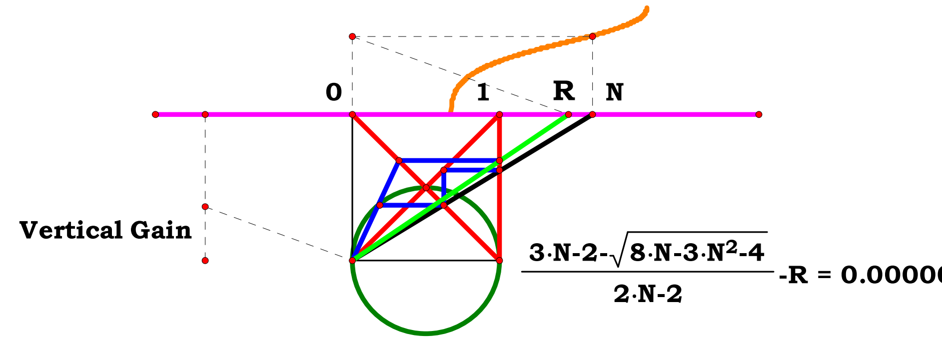
$$BO := \frac{AN - \left(8\cdot AN - 3\cdot AN^2 - 4\right)^{\frac{1}{2}}}{3\cdot AN - 2 - \left(8\cdot AN - 3\cdot AN^2 - 4\right)^{\frac{1}{2}}} \qquad CM := BO$$

$$AR := \frac{AB^2}{CM} \qquad AR - \frac{3\cdot AN - 2 - \left(8\cdot AN - 3\cdot AN^2 - 4\right)^{\frac{1}{2}}}{AN - \left(8\cdot AN - 3\cdot AN^2 - 4\right)^{\frac{1}{2}}} = 0$$



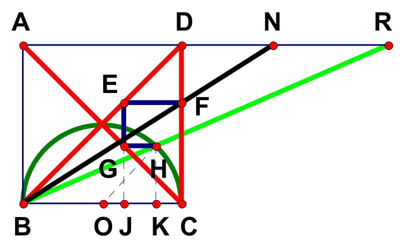
$\mathbf{AB} := 1$

$\mathbf{AN} := 3$



$$\mathbf{AE} := \frac{AN - \left(8\cdot AN - 3\cdot AN^2 - 4\right)^{\frac{1}{2}}}{3\cdot AN - 2 - \left(8\cdot AN - 3\cdot AN^2 - 4\right)^{\frac{1}{2}}} \qquad \mathbf{AR} := \frac{AB^2}{AB - AE}$$

$$AR - \frac{3\cdot AN - 2 - \left(8\cdot AN - 3\cdot AN^2 - 4\right)^{\frac{1}{2}}}{2AN - 2}$$

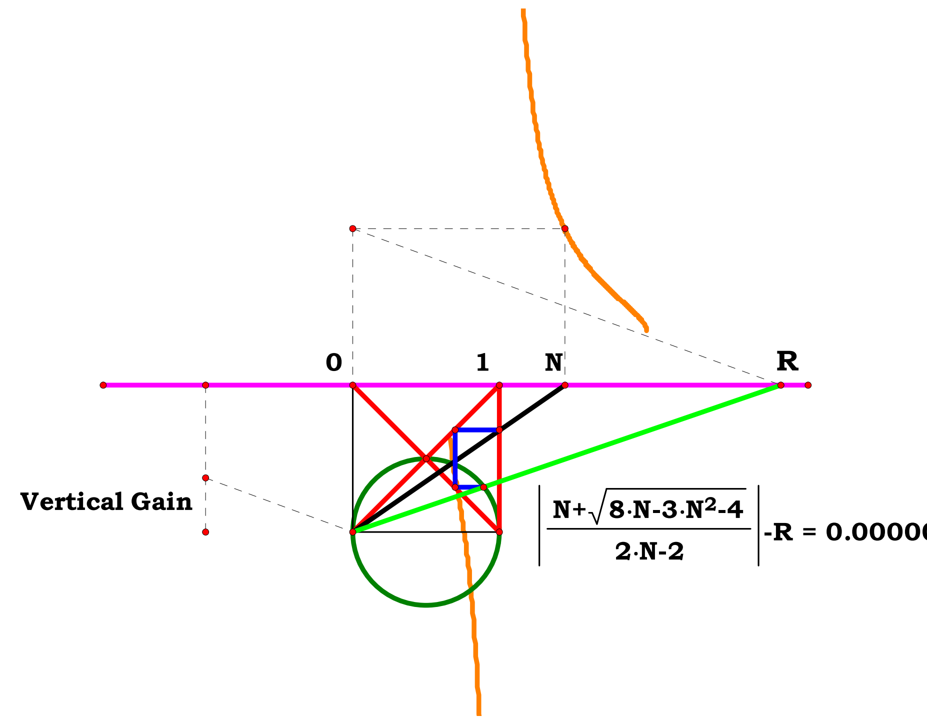


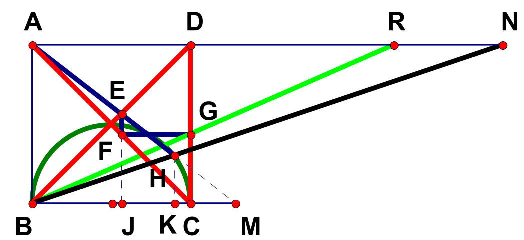
$$AB := 1$$

$$AN := 3$$

$$DF := \frac{AN - 1}{AN} \quad HK := DF \quad KO := \sqrt{\left(\frac{AB}{2}\right)^2 - HK^2} \quad BK := \frac{AB}{2} + KO$$

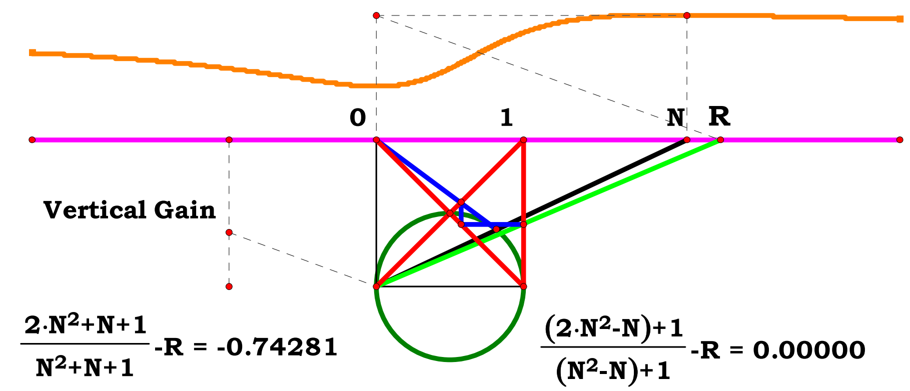
$$AR := \frac{BK \cdot AB}{HK} \quad AR - \frac{AN + \left(8 \cdot AN - 3 \cdot AN^2 - 4\right)^{\frac{1}{2}}}{2AN - 2} = 0$$





$AB := 1$

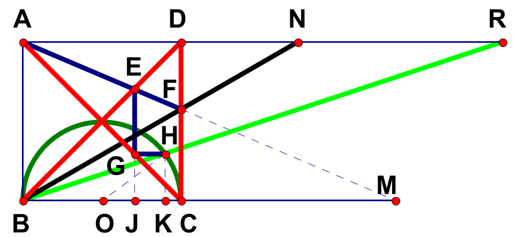
$AN := 3$



$$HK := \frac{AN}{AN^2+1} \qquad BK := \frac{AN^2}{AN^2+1} \qquad BM := \frac{BK \cdot AB}{AB - HK}$$

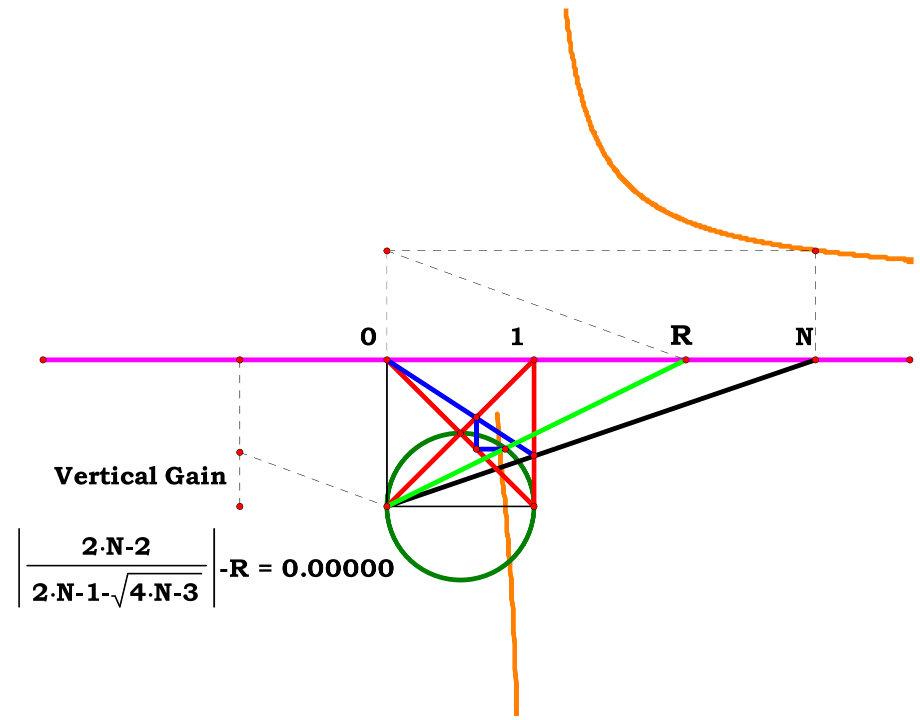
$$BJ := \frac{AB \cdot BM}{AB + BM} \qquad CJ := AB - BJ \qquad CG := CJ \qquad AR := \frac{AB^2}{CG}$$

$$AR - \frac{2 \cdot AN^2 - AN + 1}{AN^2 - AN + 1} = 0$$



$$AB := 1$$

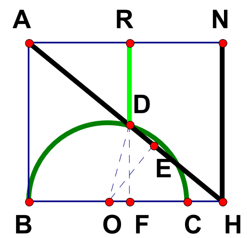
$$AN := 3$$



$$DF := \frac{AN - 1}{AN} \qquad BM := \frac{AB^2}{DF} \qquad BJ := \frac{AB \cdot BM}{AB + BM} \qquad CJ := AB - BJ$$

$$KO := \sqrt{\left(\frac{AB}{2}\right)^2 - CJ^2} \qquad BK := \frac{AB}{2} + KO \qquad AR := \frac{BK \cdot AB}{CJ}$$

$$AR - \frac{2 \cdot AN - 1 + (4 \cdot AN - 3)^{\frac{1}{2}}}{2AN - 2} = 0 \qquad AR - \frac{2AN - 2}{2 \cdot AN - 1 - (4 \cdot AN - 3)^{\frac{1}{2}}} = 0$$



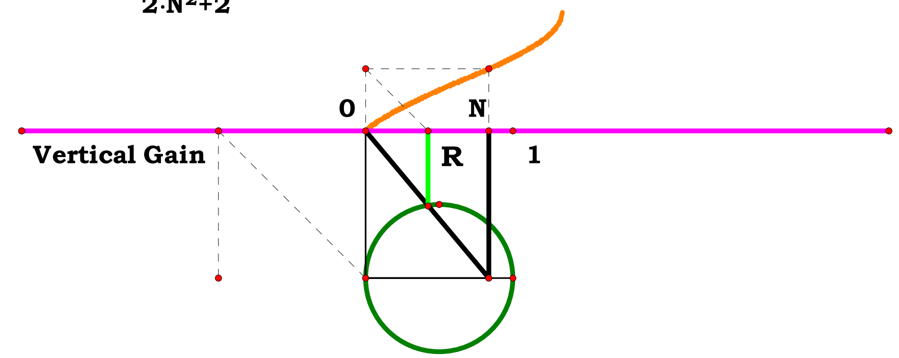
$AB := 1$

$AN := 1.2$

$BH := AN$

$BO := \frac{AB}{2}$

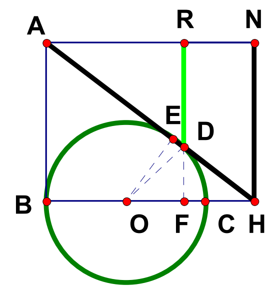
$$\frac{(N^2+2\cdot N)-N\cdot\sqrt{4\cdot N-3\cdot N^2}}{2\cdot N^2+2}-R = 0.00000$$



$$DO := BO \quad AH := \sqrt{AB^2 + AN^2} \quad HO := BH - BO \quad EH := \frac{BH \cdot HO}{AH}$$

$$EO := \frac{AB \cdot HO}{AH} \quad DE := \sqrt{DO^2 - EO^2} \quad DH := EH + DE \quad FH := \frac{BH \cdot DH}{AH}$$

$$AR := BH - FH \quad AR - \frac{AN^2 + 2AN - AN\left(4 \cdot AN - 3 \cdot AN^2\right)^{\frac{1}{2}}}{2AN^2 + 2} = 0$$

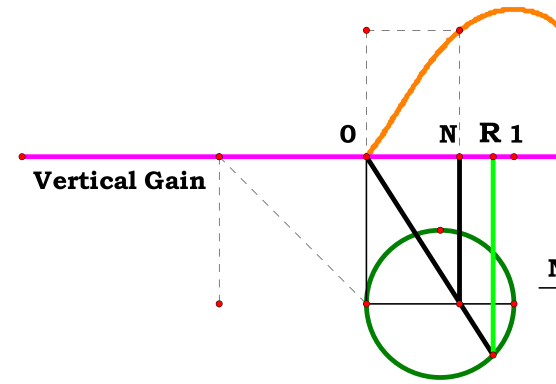


$AB := 1$

$AN := 1.3$

$BH := AN$

$BO := \dfrac{AB}{2}$

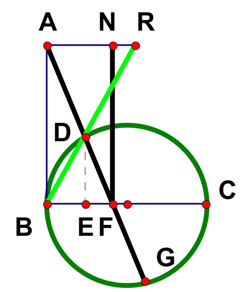


$$\frac{N^2+2\cdot N+N\cdot\sqrt{4\cdot N-3\cdot N^2}}{2\cdot N^2+2}-R = 0.00000$$

$$DO := BO \quad AH := \sqrt{AB^2+AN^2} \quad HO := BH - BO \quad EH := \frac{BH\cdot HO}{AH}$$

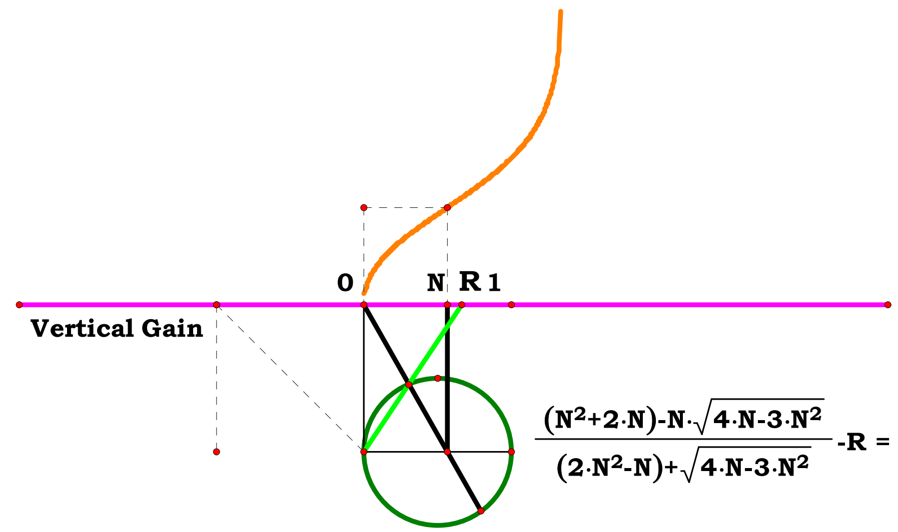
$$EO := \frac{AB\cdot HO}{AH} \quad DE := \sqrt{DO^2 - EO^2} \quad DH := EH - DE \quad FH := \frac{BH\cdot DH}{AH}$$

$$AR := BH - FH \quad AR - \frac{AN^2 + 2AN + AN\left(4\cdot AN - 3\cdot AN^2\right)^{\frac{1}{2}}}{2AN^2+2} = 0$$



$AB := 1$

$AN := .41$

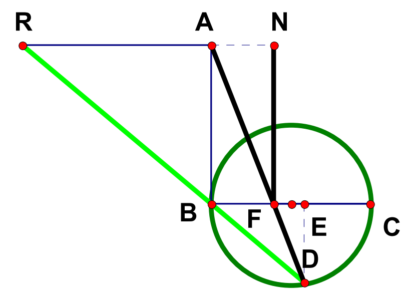


$$\frac{(N^2+2\cdot N)-N\cdot\sqrt{4\cdot N-3\cdot N^2}}{(2\cdot N^2-N)+\sqrt{4\cdot N-3\cdot N^2}}-R = 0.00000$$

$$BE := \frac{AN^2 + 2AN - AN\left(4\cdot AN - 3\cdot AN^2\right)^{\frac{1}{2}}}{2AN^2 + 2} \qquad CE := AB - BE$$

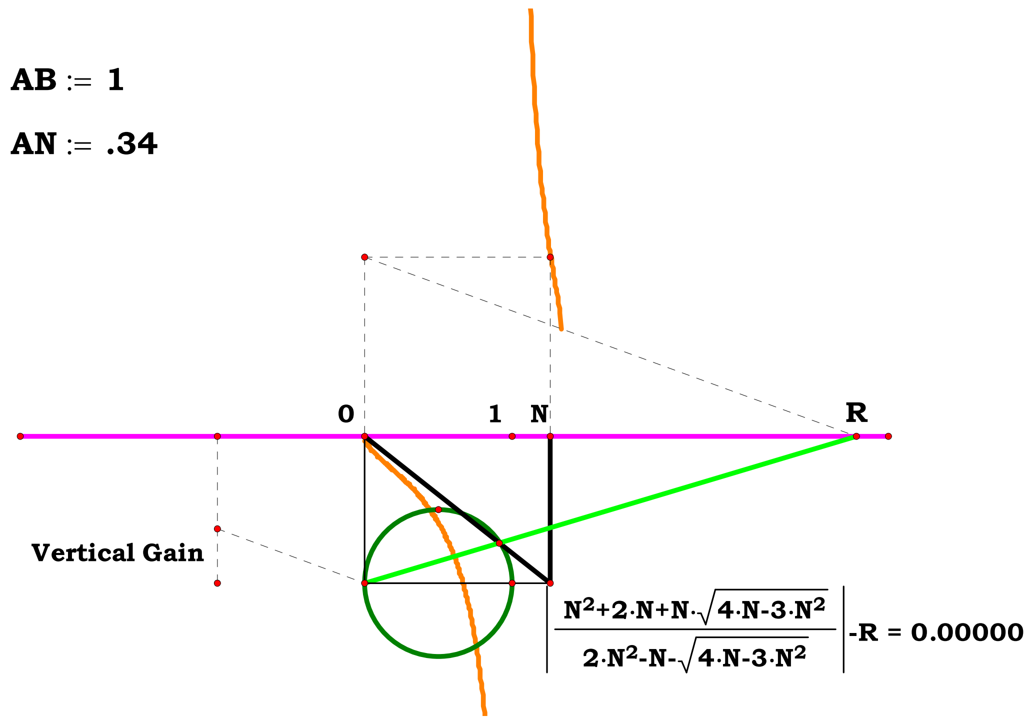
$$DE := \sqrt{BE\cdot CE} \qquad AR := \frac{BE\cdot AB}{DE}$$

$$AR - \frac{AN^2 + 2AN - AN\cdot\sqrt{4AN - 3AN^2}}{2AN^2 - AN + \sqrt{4AN - 3AN^2}} = 0$$



$$AB := 1$$

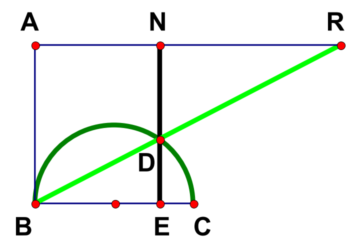
$$AN := .34$$



$$BE := \frac{AN^2 + 2AN + AN\left(4\cdot AN - 3\cdot AN^2\right)^{\frac{1}{2}}}{2AN^2 + 2} \qquad CE := AB - BE$$

$$DE := \sqrt{BE\cdot CE} \qquad AR := \frac{BE\cdot AB}{DE} \qquad AR = 1.020135$$

$$AR - \frac{AN^2 + 2AN + AN\cdot\sqrt{4AN - 3AN^2}}{\left|2AN^2 - AN - \sqrt{4AN - 3AN^2}\right|} = 0$$

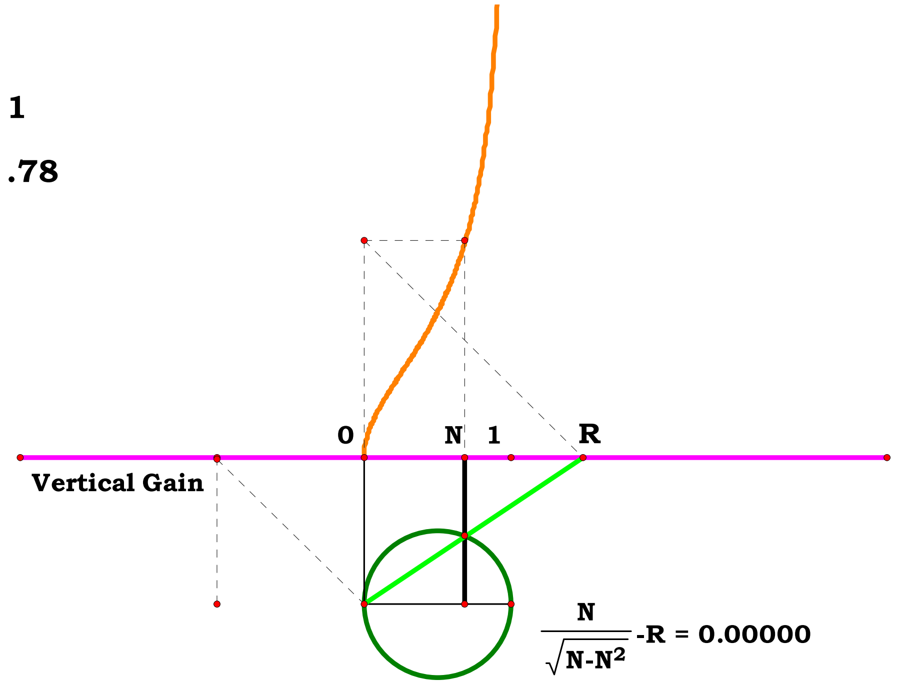


$AB := 1$

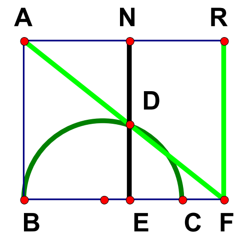
$AN := .78$

$BE := AN \quad CE := AB - BE \quad DE := \sqrt{BE \cdot CE} \quad AR := \dfrac{BE}{DE}$

$$AR - \frac{AN}{\left(AN - AN^2\right)^{\frac{1}{2}}} = 0$$

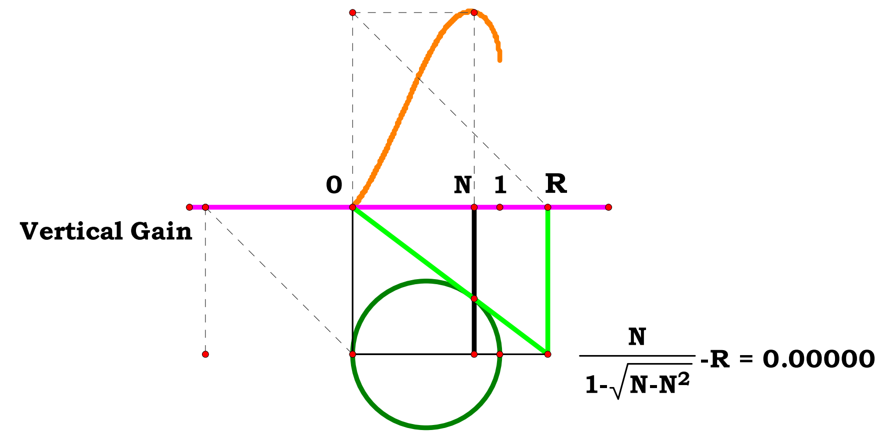


$\dfrac{N}{\sqrt{N\text{-}N^2}}\text{-}R = 0.00000$



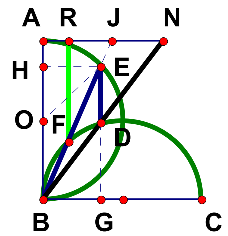
$$AB := 1$$

$$AN := .66$$



$$BE := AN \quad CE := AB - BE \quad DE := \sqrt{BE \cdot CE} \quad DN := AB - DE$$

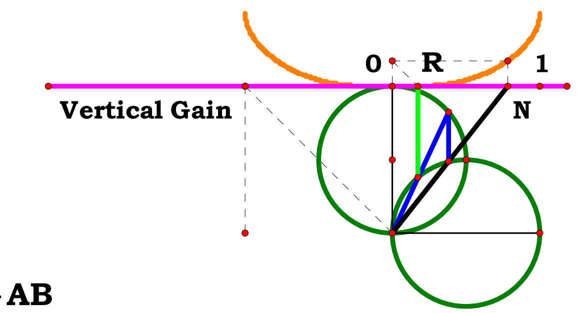
$$AR := \frac{AN}{DN} \qquad AR - \frac{AN}{1 - \left(AN - AN^2\right)^{\frac{1}{2}}} = 0$$



$AB := 1$

$AN := .75$

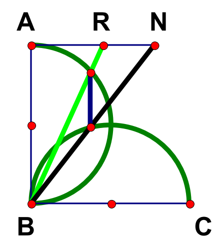
$$\frac{(N^2+1)-\sqrt{(2\cdot N^2+1)-3\cdot N^4}}{2\cdot N^2+2} - R = 0.00000$$



$$BG := \frac{AN^2}{AN^2+1} \qquad HO := \sqrt{\left(\frac{AB}{2}\right)^2 - BG^2} \qquad BH := \frac{AB}{2} + HO \qquad AJ := \frac{BG \cdot AB}{BH}$$

$$AR := \frac{AJ^2}{AJ^2+1} \qquad AR - \frac{2AN^4}{\left(AN^2+1\right)\cdot\left[AN^2+\left(2\cdot AN^2-3\cdot AN^4+1\right)^{\frac{1}{2}}+1\right]} = 0 \qquad AR - \frac{AN^2-\sqrt{2\cdot AN^2-3\cdot AN^4+1}+1}{2AN^2+2} = 0$$

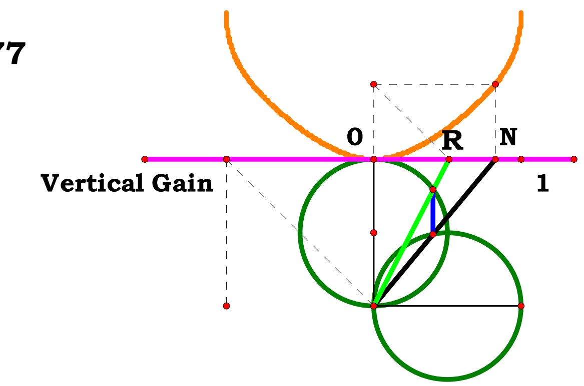
$$AJ - \frac{2AN^2}{AN^2+\left(2\cdot AN^2-3\cdot AN^4+1\right)^{\frac{1}{2}}+1} = 0$$



$$AB := 1$$

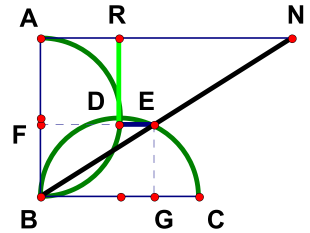
$$AN := .77$$

$$\frac{(N^2+1)-\sqrt{(2\cdot N^2+1)-3\cdot N^4}}{2\cdot N^2}-R = 0.00000$$



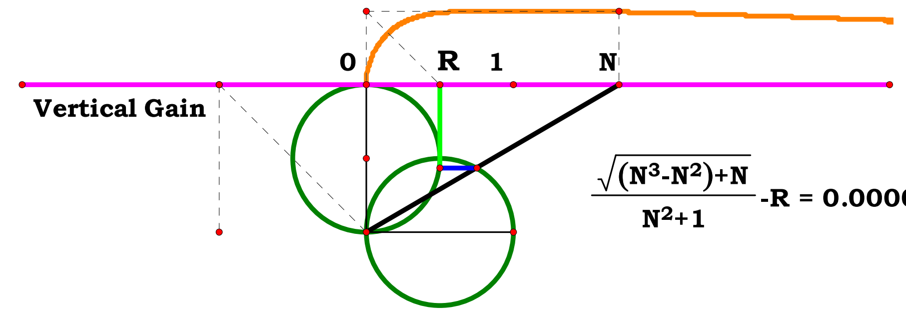
$$AR := \frac{2AN^2}{AN^2 + \left(2\cdot AN^2 - 3\cdot AN^4 + 1\right)^{\frac{1}{2}} + 1}$$

$$AR - \frac{AN^2 - \sqrt{2\cdot AN^2 - 3\cdot AN^4 + 1} + 1}{2AN^2} = 0$$



$\mathbf{AB} := 1$

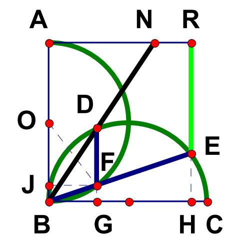
$\mathbf{AN} := 1.6$



$$\frac{\sqrt{(N^3-N^2)+N}}{N^2+1}-R = 0.00000$$

$$\mathbf{EG} := \frac{\mathbf{AN}}{\mathbf{AN}^2+1} \qquad \mathbf{BF} := \mathbf{EG} \quad \mathbf{AF} := \mathbf{AB}-\mathbf{BF} \quad \mathbf{DF} := \sqrt{\mathbf{BF}\cdot\mathbf{AF}}$$

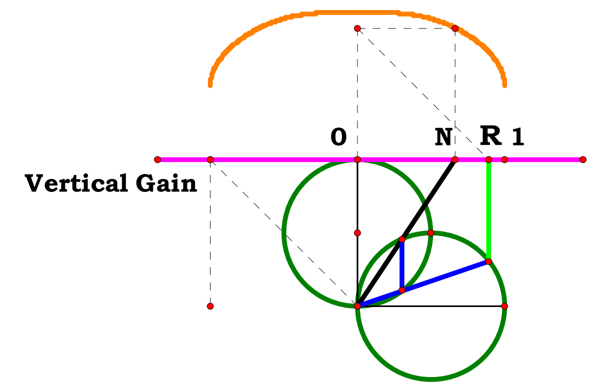
$$\mathbf{AR} := \mathbf{DF} \qquad \mathbf{AR} - \frac{\left(\mathbf{AN}^3-\mathbf{AN}^2+\mathbf{AN}\right)^{\frac{1}{2}}}{\mathbf{AN}^2+1} = 0$$



$$AB := 1$$

$$AN := .66$$

$$\frac{N^2+\sqrt{(2\cdot N^2-3\cdot N^4)+1}+1}{2\cdot N^2+2}-R = 0.00000$$



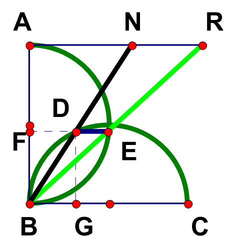
$$BG := \frac{AN^2}{AN^2+1} \quad FJ := BG \quad OJ := \sqrt{\left(\frac{AB}{2}\right)^2 - FJ^2} \quad AJ := \frac{AB}{2} + OJ$$

$$BH := AJ \quad AR := BH$$

$$AR - \frac{AN^2 + \left(2\cdot AN^2 - 3\cdot AN^4 + 1\right)^{\frac{1}{2}} + 1}{2AN^2 + 2} = 0$$

$$AR - \frac{2AN^4}{AN^4 + 2AN^2 - AN^2\cdot\left(2\cdot AN^2 - 3\cdot AN^4 + 1\right)^{\frac{1}{2}} - \left(2\cdot AN^2 - 3\cdot AN^4 + 1\right)^{\frac{1}{2}} + 1} = 0$$

$$BJ := AB - \frac{AN^2 + \left(2\cdot AN^2 - 3\cdot AN^4 + 1\right)^{\frac{1}{2}} + 1}{2AN^2 + 2} \qquad BJ - \frac{AN^2 - \left(2\cdot AN^2 - 3\cdot AN^4 + 1\right)^{\frac{1}{2}} + 1}{2AN^2 + 2} = 0$$



$AB := 1$

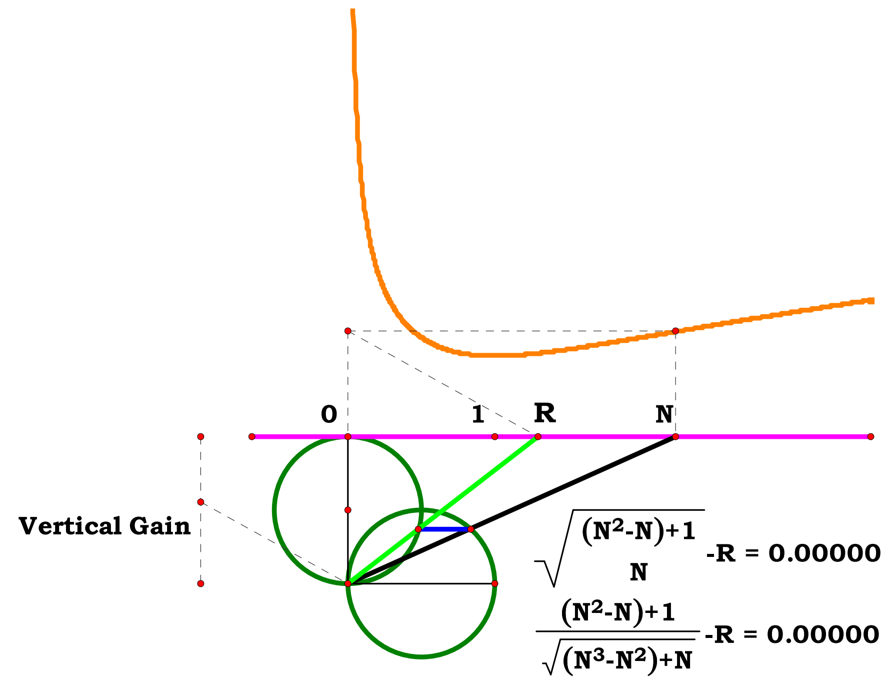
$AN := .64$

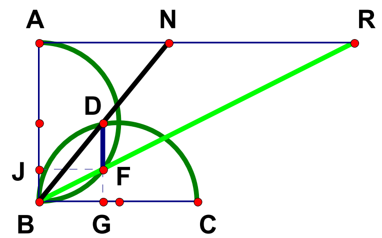
$$BG := \frac{AN^2}{AN^2 + 1} \qquad CG := AB - BG \qquad DG := \sqrt{BG \cdot CG} \qquad BF := DG$$

$$AF := AB - BF \qquad EF := \sqrt{BF \cdot AF} \qquad AR := \frac{EF \cdot AB}{BF}$$

$$AR - \frac{\left(AN^3 - AN^2 + AN\right)^{\frac{1}{2}}}{AN} = 0 \qquad AR - \frac{\sqrt{AN^2 - AN + 1}}{\sqrt{AN}} = 0$$

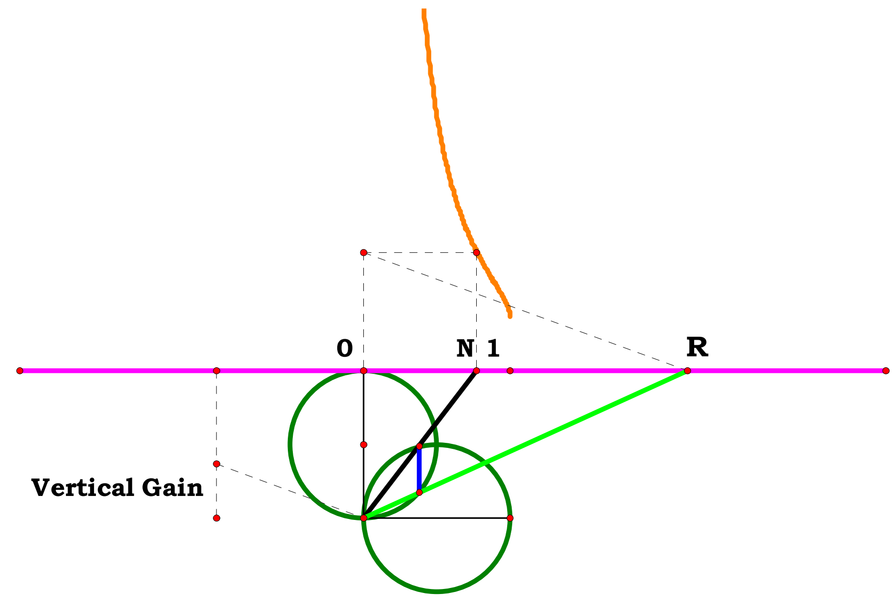
$$AR - \frac{AN^2 - AN + 1}{\sqrt{AN^3 - AN^2 + AN}} = 0$$





$$\mathbf{AB} := 1$$

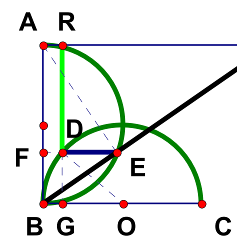
$$\mathbf{AN} := .81$$



$$\mathbf{BG} := \frac{\mathbf{AN}^2}{\mathbf{AN}^2 + 1} \qquad \mathbf{FG} := \frac{\mathbf{AN}^2 - \left(2\cdot\mathbf{AN}^2 - 3\cdot\mathbf{AN}^4 + 1\right)^{\frac{1}{2}} + 1}{2\mathbf{AN}^2 + 2}$$

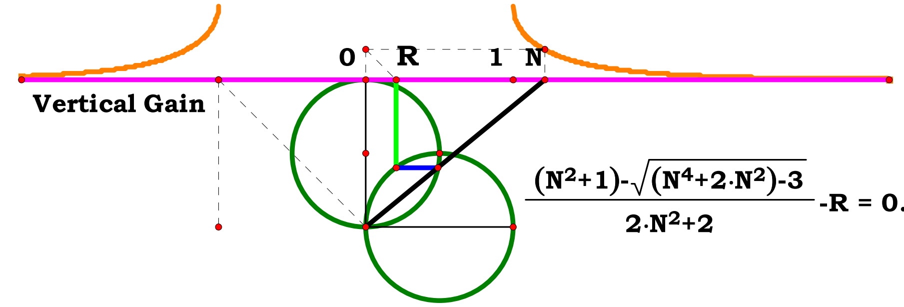
$$\mathbf{AR} := \frac{\mathbf{BG}\cdot\mathbf{AB}}{\mathbf{FG}} \qquad \mathbf{AR} - \frac{2\mathbf{AN}^2}{\mathbf{AN}^2 + 1 - \left(2\cdot\mathbf{AN}^2 - 3\cdot\mathbf{AN}^4 + 1\right)^{\frac{1}{2}}} = 0$$

$$\frac{2\cdot N^2}{(N^2+1)-\sqrt{(2\cdot N^2+1)-3\cdot N^4}}-R = 0.00000$$



$$AB := 1$$

$$AN := 1.4$$



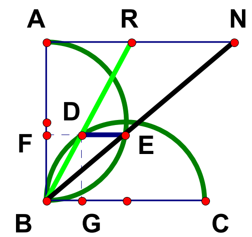
$$\frac{(N^2+1)-\sqrt{(N^4+2\cdot N^2)-3}}{2\cdot N^2+2}-R = 0.00000$$

$$BN := \sqrt{AN^2 + AB^2} \quad BE := \frac{AB^2}{BN} \quad BF := \frac{AB\cdot BE}{BN} \quad DG := BF$$

$$GO := \sqrt{\left(\frac{AB}{2}\right)^2 - DG^2} \quad BG := \frac{AB}{2} - GO \quad AR := BG$$

$$AR - \frac{AN^2 + 1 - \left(AN^4 + 2\cdot AN^2 - 3\right)^{\frac{1}{2}}}{2AN^2 + 2} = 0$$

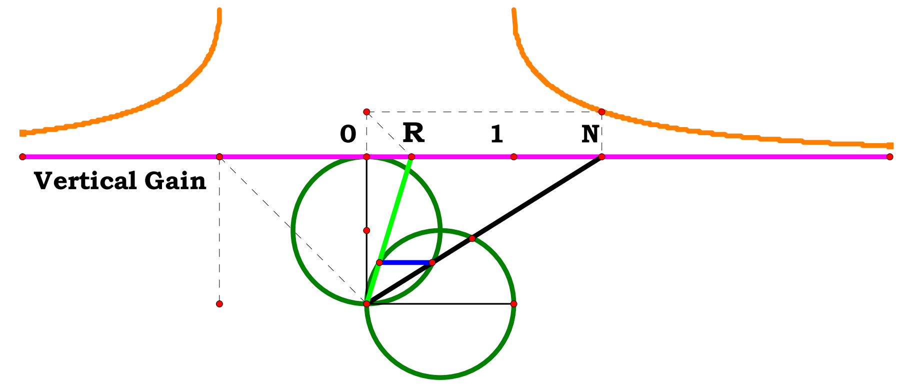
$$DG - \frac{1}{AN^2 + 1} = 0$$



$AB := 1$

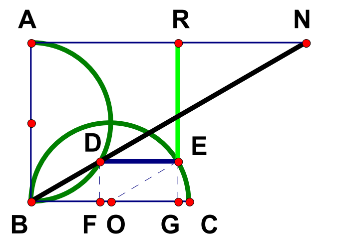
$AN := 1.1$

$$\frac{(N^2+1)-\sqrt{(N^4+2\cdot N^2)-3}}{2}-R = 0.00000$$



$$DG := \frac{1}{AN^2+1} \quad BG := \frac{AN^2+1-\left(AN^4+2\cdot AN^2-3\right)^{\frac{1}{2}}}{2AN^2+2}$$

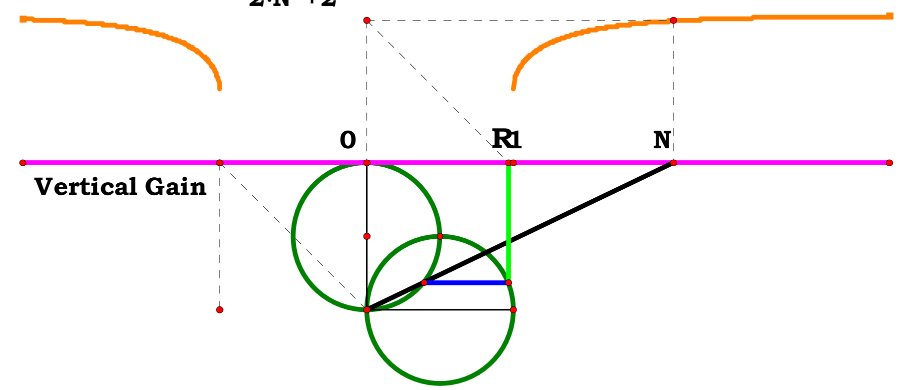
$$AR := \frac{BG\cdot AB}{DG} \quad AR - \frac{AN^2+1-\left(AN^4+2\cdot AN^2-3\right)^{\frac{1}{2}}}{2} = 0$$



$AB := 1$

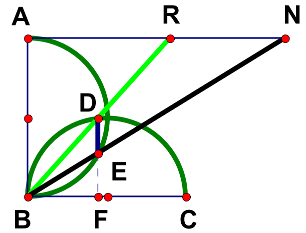
$AN := 1.7$

$$\frac{N^2+1+\sqrt{(N^4+2\cdot N^2)-3}}{2\cdot N^2+2} - R = 0.00000$$



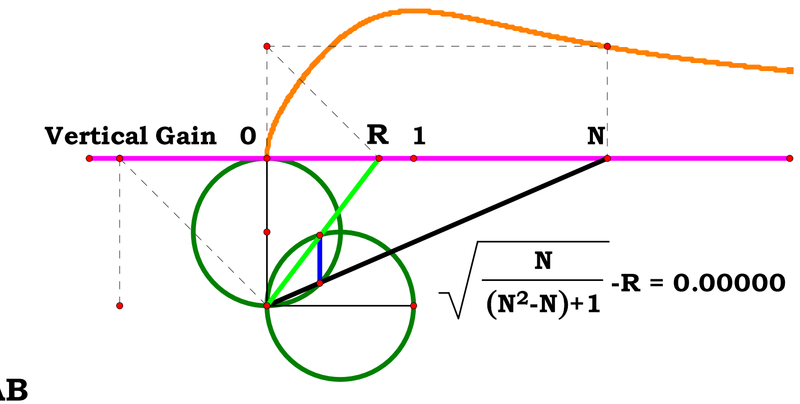
$$DF := \frac{1}{AN^2+1} \qquad GO := \sqrt{\left(\frac{AB}{2}\right)^2 - DF^2} \qquad BG := \frac{AB}{2} + GO$$

$$AR := BG \qquad AR - \frac{AN^2 + 1 + \left(AN^4 + 2\cdot AN^2 - 3\right)^{\frac{1}{2}}}{2AN^2 + 2} = 0$$



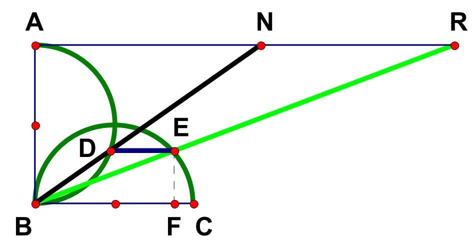
$AB := 1$

$AN := 1.6$



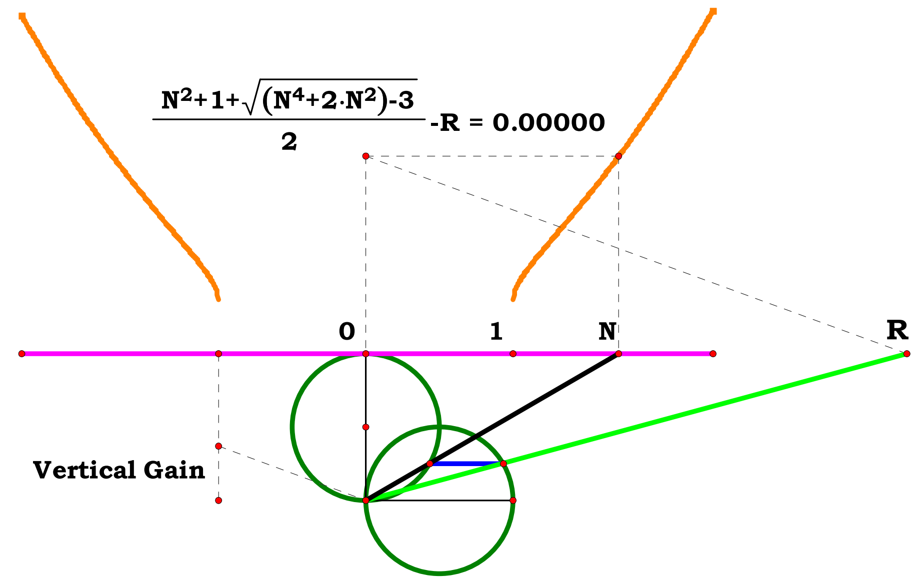
$$BF := \frac{AN}{AN^2+1} \qquad CF := AB - BF \qquad DF := \sqrt{BF \cdot CF} \qquad AR := \frac{BF \cdot AB}{DF}$$

$$AR - \frac{AN}{\left(AN^3 + AN - AN^2\right)^{\frac{1}{2}}} = 0 \qquad AR - \frac{AN^{.5}}{\left(AN^2 - AN + 1\right)^{.5}} = 0$$



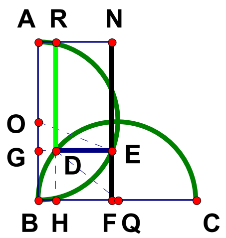
$AB := 1$

$AN := 1.4$



$$BF := \frac{AN^2+1+\left(AN^4+2\cdot AN^2-3\right)^{\frac{1}{2}}}{2AN^2+2} \qquad EF := \frac{1}{AN^2+1}$$

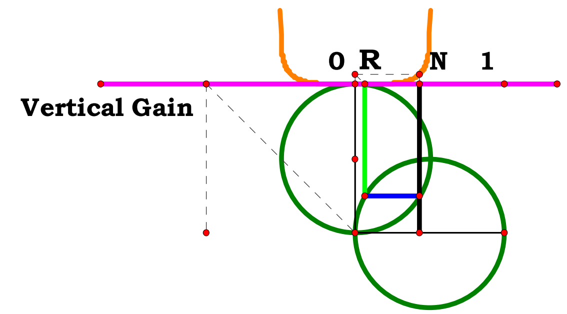
$$AR := \frac{BF\cdot AB}{EF} \qquad AR - \frac{AN^2+1+\left(AN^4+2\cdot AN^2-3\right)^{\frac{1}{2}}}{2} = 0$$



$AB := 1$

$AN := .46$

$$\frac{1-\sqrt{(4\cdot N^2-1)+2\cdot\sqrt{1-4\cdot N^2}}}{2}-R = 0.00000$$

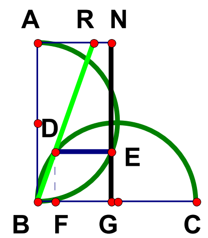


$EG := AN \quad GO := \sqrt{\left(\frac{AB}{2}\right)^2 - EG^2} \quad BG := \frac{AB}{2} - GO$

$DH := BG \quad HQ := \sqrt{\left(\frac{AB}{2}\right)^2 - DH^2} \quad BH := \frac{AB}{2} - HQ$

$$AR := BH \qquad AR - \frac{1-\left[4\cdot AN^2 + 2\cdot\left(1-4\cdot AN^2\right)^{\frac{1}{2}} - 1\right]^{\frac{1}{2}}}{2} = 0$$

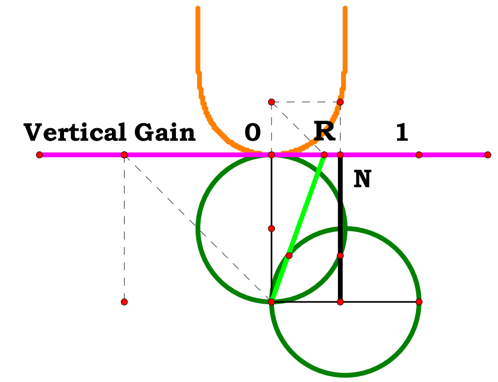
$$BG - \frac{1-\left(1-4\cdot AN^2\right)^{\frac{1}{2}}}{2} = 0 \qquad HQ - \frac{\left[4\cdot AN^2 + 2\cdot\left(1-4\cdot AN^2\right)^{\frac{1}{2}} - 1\right]^{\frac{1}{2}}}{2} = 0$$



$AB := 1$

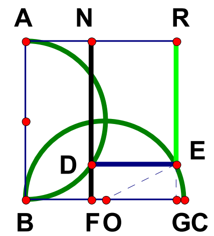
$AN := .46$

$$\frac{1-\sqrt{(4\cdot N^2-1)+2\cdot\sqrt{1-4\cdot N^2}}}{1-\sqrt{1-4\cdot N^2}}-R = 0.00000$$



$$DF := \frac{1-\left(1-4\cdot AN^2\right)^{\frac{1}{2}}}{2} \qquad BF := \frac{1-\left[4\cdot AN^2+2\cdot\left(1-4\cdot AN^2\right)^{\frac{1}{2}}-1\right]^{\frac{1}{2}}}{2}$$

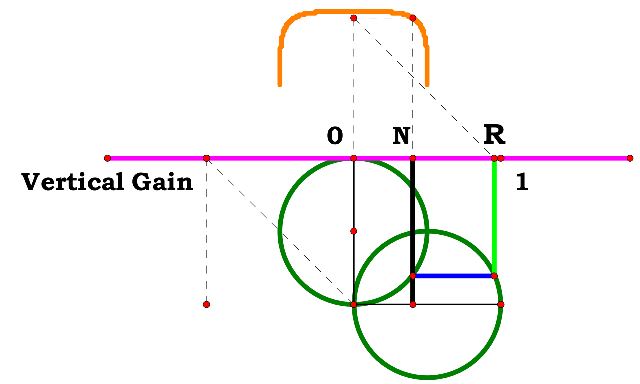
$$AR := \frac{BF\cdot AB}{DF} \qquad AR-\frac{\left[4\cdot AN^2+2\cdot\left(1-4\cdot AN^2\right)^{\frac{1}{2}}-1\right]^{\frac{1}{2}}-1}{\left(1-4\cdot AN^2\right)^{\frac{1}{2}}-1}=0 \qquad AR-\frac{1-\left[4\cdot AN^2+2\cdot\left(1-4\cdot AN^2\right)^{\frac{1}{2}}-1\right]^{\frac{1}{2}}}{1-\left(1-4\cdot AN^2\right)^{\frac{1}{2}}}=0$$



$$AB := 1$$

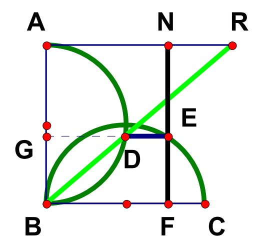
$$AN := .41$$

$$\frac{1+\sqrt{(4\cdot N^2-1)+2\cdot\sqrt{1-4\cdot N^2}}}{2} - R = 0.00000$$



$$GO := \frac{\left[4\cdot AN^2 + 2\cdot\left(1-4\cdot AN^2\right)^{\frac{1}{2}} - 1\right]^{\frac{1}{2}}}{2} \qquad BG := \frac{AB}{2} + GO$$

$$AR := BG \qquad AR - \frac{1+\left[4\cdot AN^2 + 2\cdot\left(1-4\cdot AN^2\right)^{\frac{1}{2}} - 1\right]^{\frac{1}{2}}}{2} = 0$$



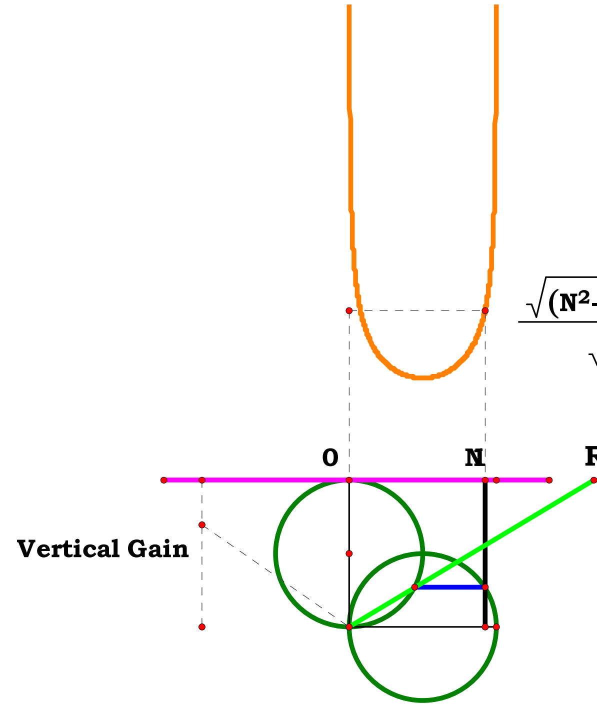
$AB := 1$

$AN := .76$

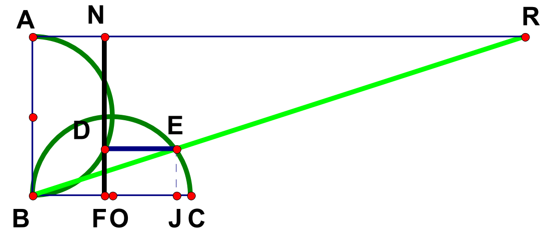
$BF := AN \quad CF := AB - BF \quad EF := \sqrt{BF \cdot CF} \quad BG := EF$

$AG := AB - BG \quad DG := \sqrt{AG \cdot BG} \quad AR := \dfrac{DG \cdot AB}{BG}$

$$AR - \frac{\left[AN^2 - AN + \left(AN - AN^2\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}}{\left(AN - AN^2\right)^{\frac{1}{2}}} = 0$$



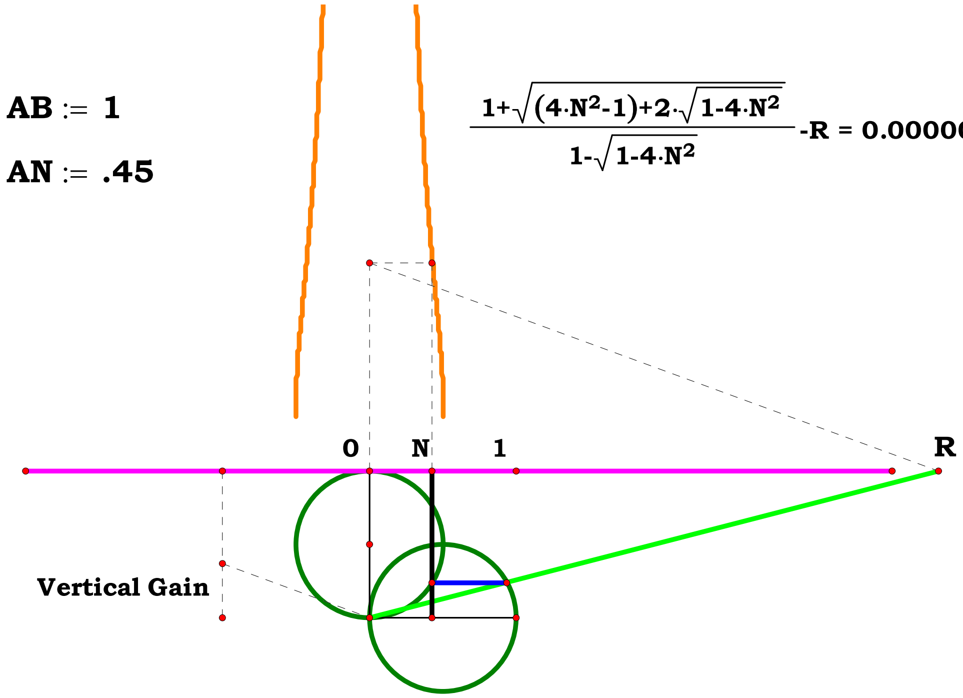
$$\frac{\sqrt{(N^2\text{-}N)+\sqrt{N\text{-}N^2}}}{\sqrt{N\text{-}N^2}}\text{-}R = 0.00000$$



$AB := 1$

$AN := .45$

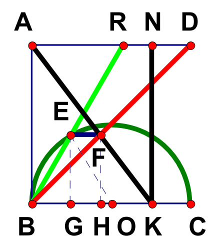
$$\frac{1+\sqrt{(4\cdot N^2-1)+2\cdot\sqrt{1-4\cdot N^2}}}{1-\sqrt{1-4\cdot N^2}}-R = 0.00000$$



$$EJ := \frac{1-\left(1-4\cdot AN^2\right)^{\frac{1}{2}}}{2} \qquad OJ := \frac{\left[4\cdot AN^2+2\cdot\left(1-4\cdot AN^2\right)^{\frac{1}{2}}-1\right]^{\frac{1}{2}}}{2}$$

$$BJ := \frac{AB}{2}+OJ \qquad AR := \frac{BJ\cdot AB}{EJ}$$

$$AR-\frac{1+\left[4\cdot AN^2+2\cdot\left(1-4\cdot AN^2\right)^{\frac{1}{2}}-1\right]^{\frac{1}{2}}}{1-\left(1-4\cdot AN^2\right)^{\frac{1}{2}}} = 0$$



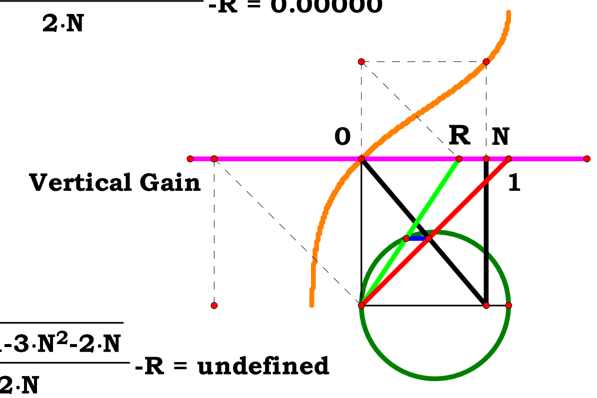
$AB := 1$

$AN := .75$

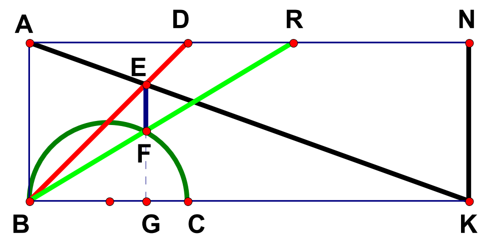
$BK := AN \quad BH := \dfrac{AB \cdot BK}{AB + BK} \quad EG := BH \quad GO := \sqrt{\left(\dfrac{AB}{2}\right)^2 - EG^2}$

$BG := \dfrac{AB}{2} - GO \quad AR := \dfrac{BG \cdot AB}{EG} \quad AR - \dfrac{1 + AN - \left(1 + 2 \cdot AN - 3 \cdot AN^2\right)^{\frac{1}{2}}}{2AN} = 0$

$\dfrac{(1+N)-\sqrt{(1-3 \cdot N^2)+2 \cdot N}}{2 \cdot N} - R = 0.00000$



$\dfrac{1-N-\sqrt{1-3 \cdot N^2-2 \cdot N}}{2 \cdot N} - R = \text{undefined}$



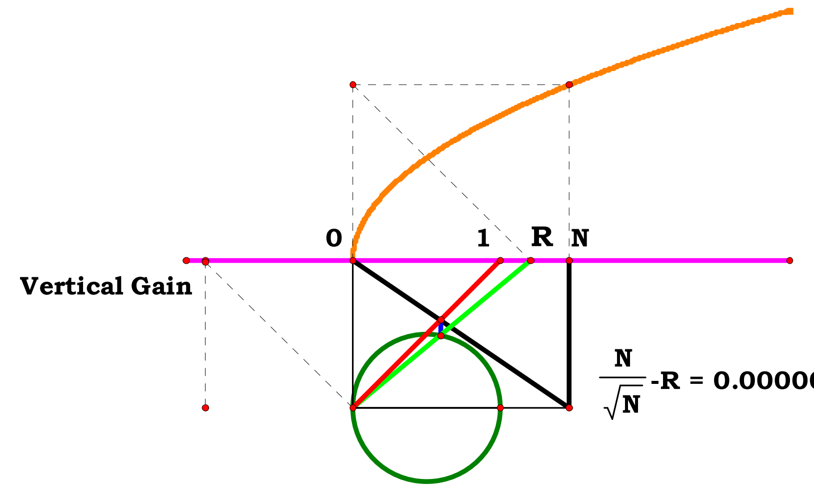
$AB := 1$

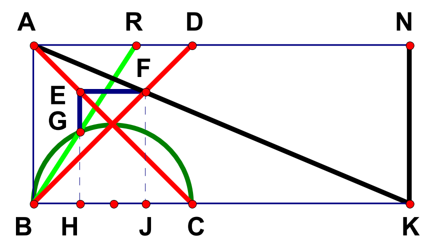
$AN := 2.7$

$BK := AN$

$$BG := \frac{AB \cdot BK}{AB + BK} \qquad CG := AB - BG \qquad FG := \sqrt{BG \cdot CG}$$

$$AR := \frac{BG \cdot AB}{FG} \qquad AR - \frac{AN}{AN^{\frac{1}{2}}} = 0$$



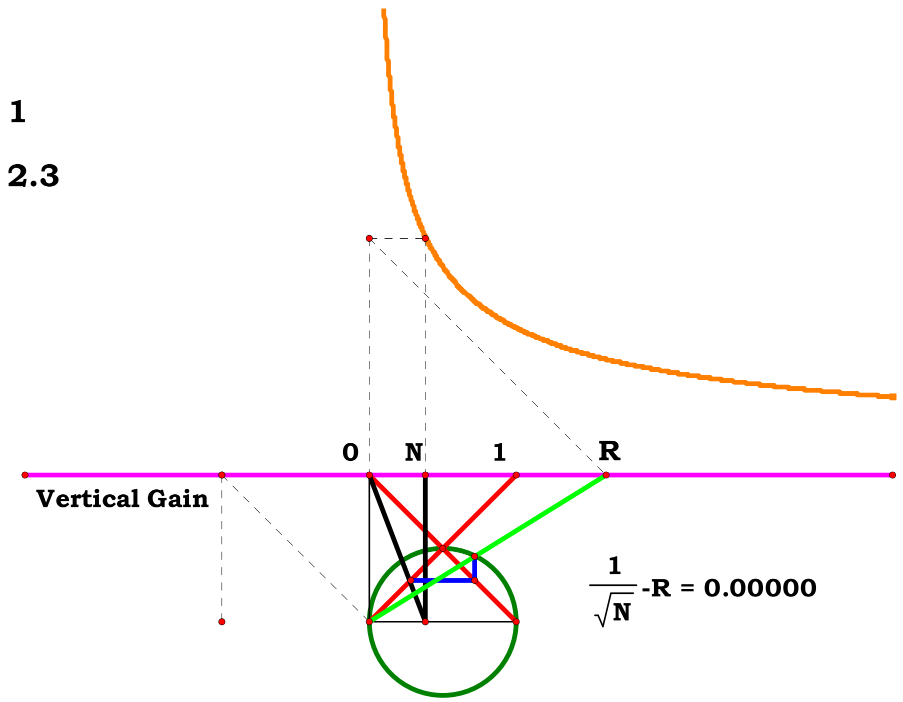


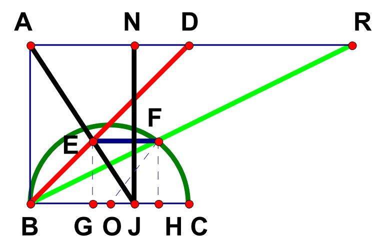
$AB := 1$

$AN := 2.3$

$$BJ := \frac{AB \cdot AN}{AB + AN} \qquad CJ := AB - BJ \qquad BH := CJ \qquad CH := AB - BH$$

$$GH := \sqrt{BH \cdot CH} \qquad AR := \frac{BH \cdot AB}{GH} \qquad AR - \frac{1}{AN^{\frac{1}{2}}} = 0$$





$AB := 1$

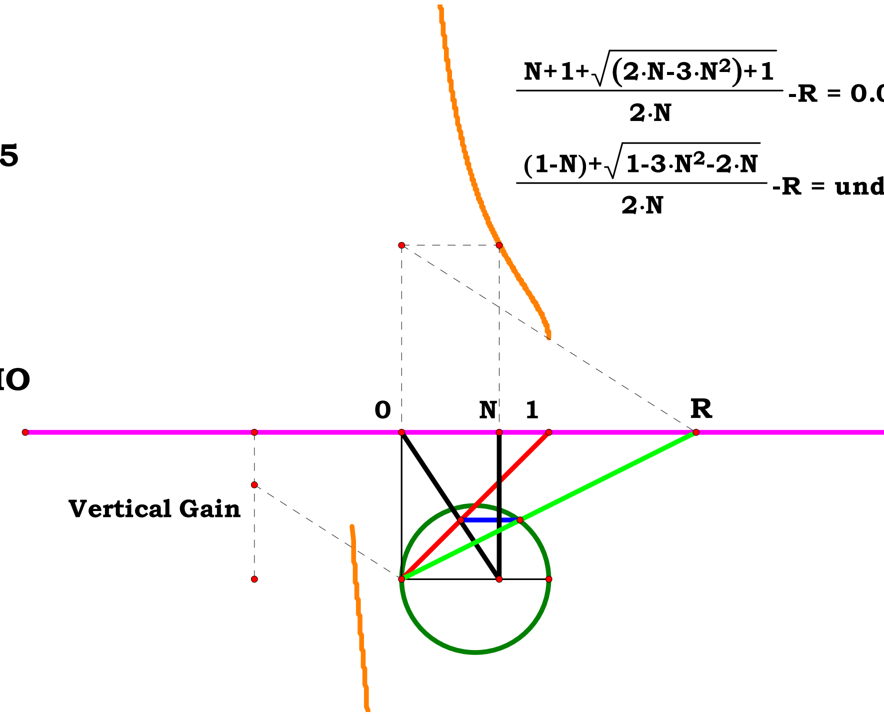
$AN := .65$

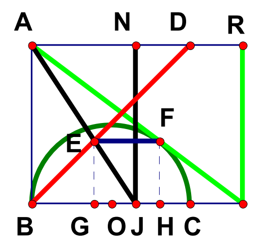
$$BG := \frac{AB \cdot AN}{AB + AN} \quad FH := BG \quad HO := \sqrt{\left(\frac{AB}{2}\right)^2 - FH^2} \quad BH := \frac{AB}{2} + HO$$

$$AR := \frac{BH \cdot AB}{FH} \quad AR - \frac{1 + AN + \left(1 + 2 \cdot AN - 3 \cdot AN^2\right)^{\frac{1}{2}}}{2AN} = 0$$

$$\frac{N+1+\sqrt{(2 \cdot N-3 \cdot N^2)+1}}{2 \cdot N} - R = 0.00000$$

$$\frac{(1-N)+\sqrt{1-3 \cdot N^2-2 \cdot N}}{2 \cdot N} - R = \text{undefined}$$



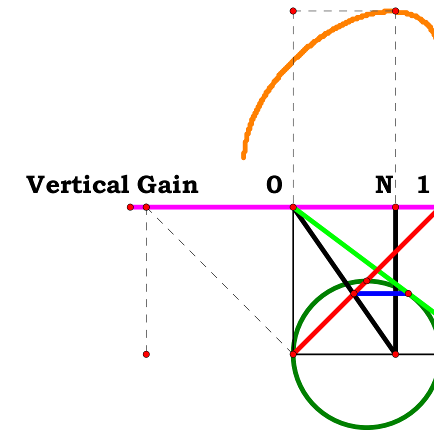


$AB := 1$

$AN := .65$

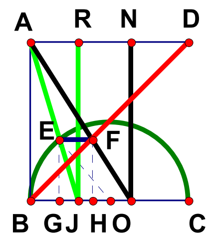
$$BG := \frac{AB \cdot AN}{AB + AN} \quad FH := BG \quad HO := \sqrt{\left(\frac{AB}{2}\right)^2 - FH^2} \quad BH := \frac{AB}{2} + HO$$

$$AR := \frac{BH \cdot AB}{AB - FH} \quad AR - \frac{1 + AN + \left(1 + 2 \cdot AN - 3 \cdot AN^2\right)^{\frac{1}{2}}}{2} = 0$$



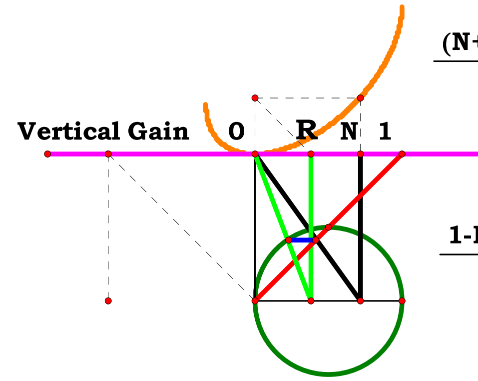
$$\frac{N+1+\sqrt{(2 \cdot N-3 \cdot N^2)+1}}{2}\text{-R} = 0.00000$$

$$\frac{(1\text{-}N)+\sqrt{1\text{-}3 \cdot N^2\text{-}2 \cdot N}}{2}\text{-R} = \text{undefined}$$



$AB := 1$

$AN := .63$



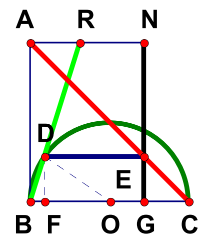
$$\frac{(N+1)-\sqrt{(2 \cdot N-3 \cdot N^2)+1}}{2}-R = 0.00000$$

$$\frac{1-N-\sqrt{1-3 \cdot N^2-2 \cdot N}}{2}-R = \text{undefined}$$

$$BH := \frac{AB \cdot AN}{AB + AN} \qquad EG := BH \qquad GO := \sqrt{\left(\frac{AB}{2}\right)^2 - EG^2}$$

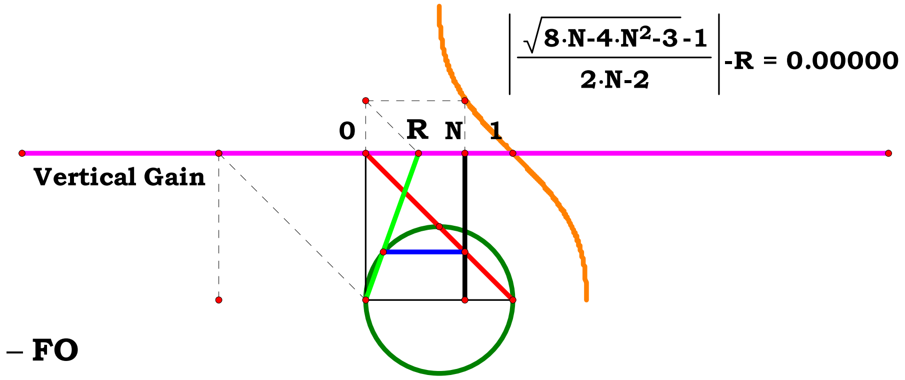
$$BG := \frac{AB}{2} - GO \qquad AR := \frac{BG \cdot AB}{AB - EG}$$

$$AR - \frac{AN + 1 - \left(1 + 2 \cdot AN - 3 \cdot AN^2\right)^{\frac{1}{2}}}{2} = 0$$



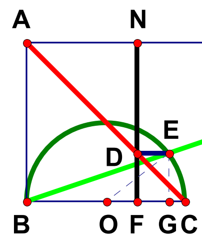
$$AB := 1$$

$$AN := .71$$



$$CG := AB - AN \qquad DF := CG \qquad FO := \sqrt{\left(\frac{AB}{2}\right)^2 - DF^2} \qquad BF := \frac{AB}{2} - FO$$

$$AR := \frac{BF \cdot AB}{DF} \qquad AR - \frac{\left(8 \cdot AN - 4 \cdot AN^2 - 3\right)^{\frac{1}{2}} - 1}{2AN - 2} = 0$$

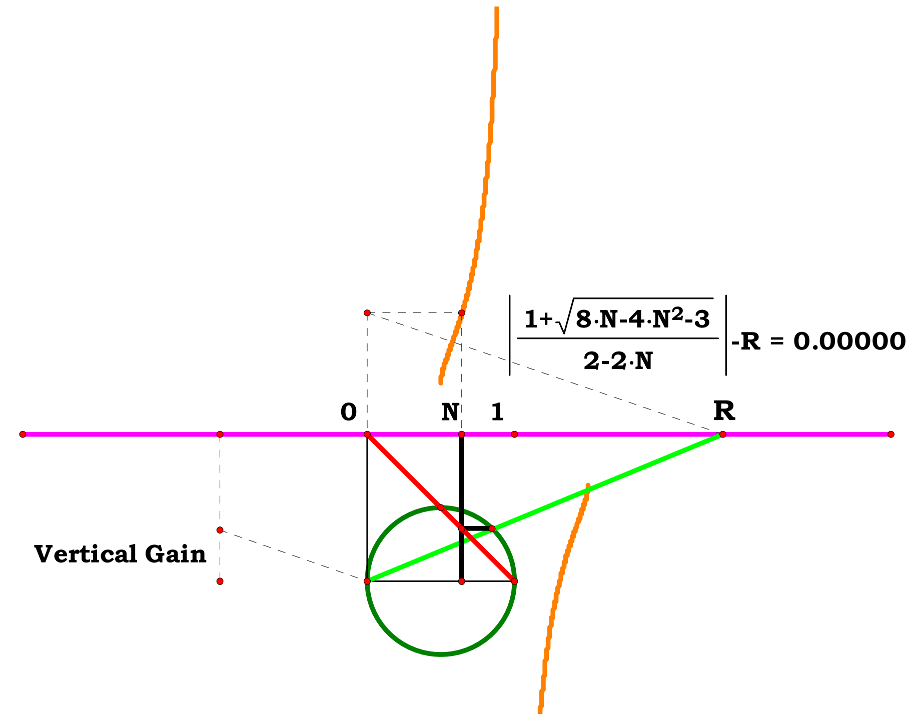


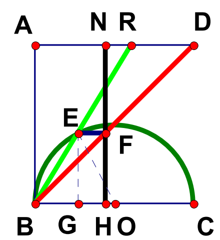
$$AB := 1$$

$$AN := .69$$

$$EG := AB - AN \qquad GO := \sqrt{\left(\frac{AB}{2}\right)^2 - EG^2} \qquad BG := \frac{AB}{2} + GO \qquad AR := \frac{BG \cdot AB}{EG}$$

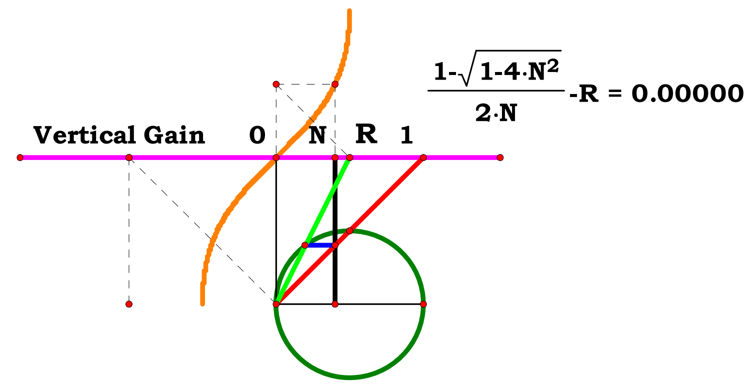
$$AR - \frac{1 + \left(8 \cdot AN - 4 \cdot AN^2 - 3\right)^{\frac{1}{2}}}{2 - 2AN} = 0$$





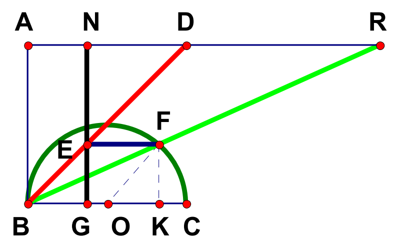
$AB := 1$

$AN := .44$



$$EG := AN \qquad GO := \sqrt{\left(\frac{AB}{2}\right)^2 - EG^2} \qquad BG := \frac{AB}{2} - GO$$

$$AR := \frac{BG \cdot AB}{EG} \qquad AR - \frac{1 - \left(1 - 4 \cdot AN^2\right)^{\frac{1}{2}}}{2AN} = 0$$

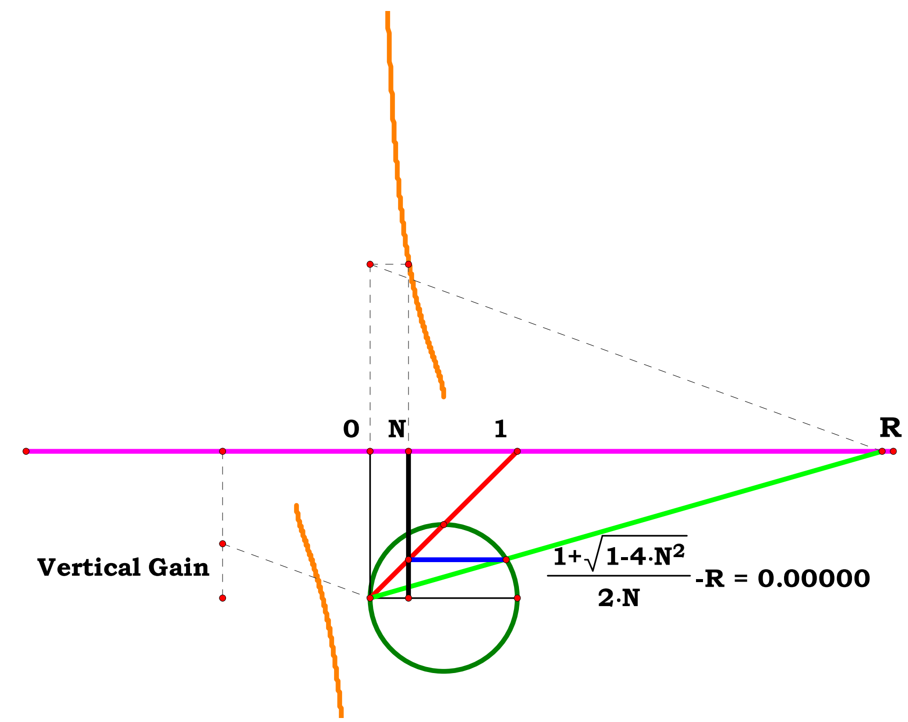


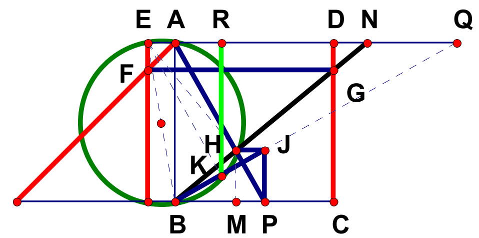
$$AB := 1$$

$$AN := .37$$

$$FK := AN \qquad KO := \sqrt{\left(\frac{AB}{2}\right)^2 - FK^2} \qquad BK := \frac{AB}{2} + KO \qquad AR := \frac{BK \cdot AB}{FK}$$

$$AR - \frac{1 + \left(1 - 4 \cdot AN^2\right)^{\frac{1}{2}}}{2AN} = 0$$

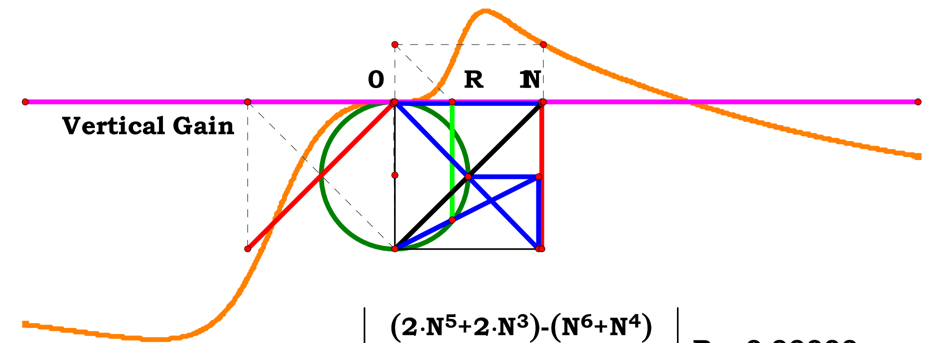




$$AB := 1$$

$$AN := 1.14165$$

$$AE := \left|\frac{1-AN}{AN}\right|$$



$$\left|\frac{(2\cdot N^5+2\cdot N^3)-(N^6+N^4)}{(N^6+3\cdot N^4+2\cdot N^3+1)-2\cdot N}\right|-R = 0.00000$$

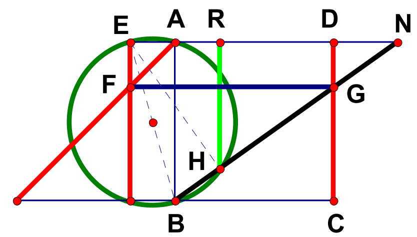
$$\frac{2\cdot N^5+2\cdot N^3+N^6+N^4}{((N^6+3\cdot N^4)-2\cdot N^3)+1+2\cdot N}-R = 0.82041$$

$$EN := AN + AE \quad BN := \sqrt{AN^2 + AB^2} \quad HN := \frac{AN\cdot EN}{BN} \quad BH := BN - HN$$

$$HM := \frac{AB\cdot BH}{BN} \quad BP := \frac{BN\cdot BH}{EN} \quad JP := HM \quad AQ := \frac{BP\cdot AB}{JP}$$

$$BQ := \sqrt{AQ^2 + AB^2} \quad EQ := AQ + AE \quad KQ := \frac{AQ\cdot EQ}{BQ} \quad BK := BQ - KQ$$

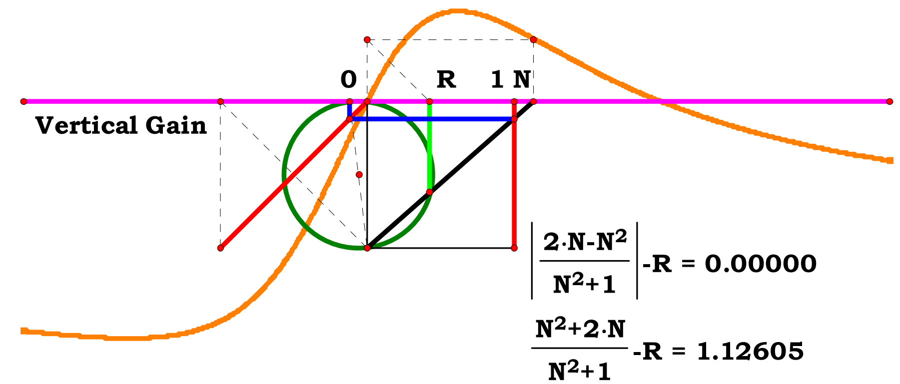
$$AR := \frac{AQ\cdot BK}{BQ} \quad AR - \frac{-AN^6 + 2\cdot AN^5 - AN^4 + 2\cdot AN^3}{AN^6 + 3\cdot AN^4 + 2\cdot AN^3 - 2\cdot AN + 1} = 0$$



$$AB := 1$$

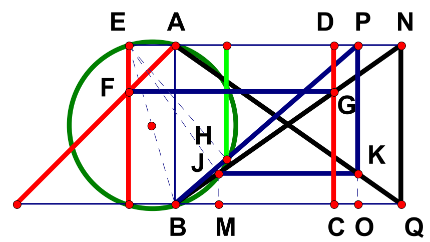
$$AN := 1.45091$$

$$AE := \left|\frac{1 - AN}{AN}\right|$$



$$EN := AN + AE \qquad BN := \sqrt{AN^2 + AB^2} \qquad HN := \frac{AN \cdot EN}{BN} \qquad RN := \frac{AN \cdot HN}{BN}$$

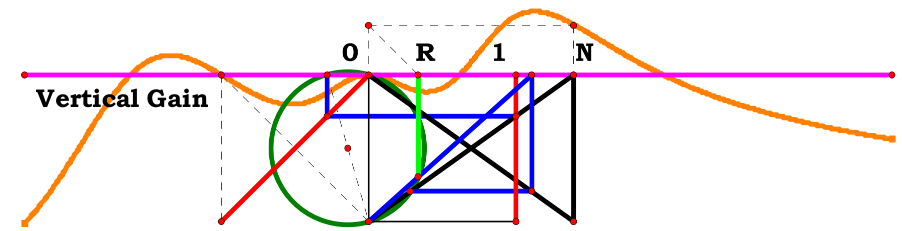
$$AR := AN - RN \qquad AR - \frac{2 \cdot AN - AN^2}{AN^2 + 1} = 0$$



$\mathbf{AB} := 1$

$\mathbf{AN} := 1.4545454$

$$\mathbf{AE} := \left|\frac{1 - \mathbf{AN}}{\mathbf{AN}}\right|$$



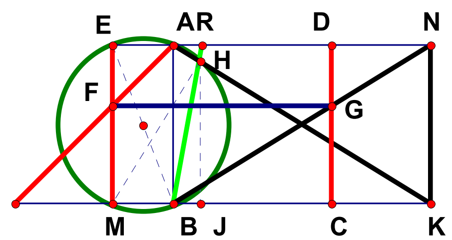
$$\left|\frac{(4\cdot N^4+N^3)-(N^6+2\cdot N^2)}{(N^6+2\cdot N^5+3\cdot N^2+1)-2\cdot N^3}\right| -R = 0.00000$$

$$\left|\frac{-4\cdot N^4+N^3+N^6+2\cdot N^2}{(N^6-2\cdot N^5)+3\cdot N^2+1+2\cdot N^3}\right| -R = -0.21171$$

$$\mathbf{BM} := \frac{2\mathbf{AN} - \mathbf{AN}^2}{\mathbf{AN}^2 + 1} \qquad \mathbf{NP} := \mathbf{BM} \quad \mathbf{AP} := \mathbf{AN} - \mathbf{NP} \quad \mathbf{BP} := \sqrt{\mathbf{AB}^2 + \mathbf{AP}^2}$$

$$\mathbf{EP} := \mathbf{AP} + \mathbf{AE} \qquad \mathbf{HP} := \frac{\mathbf{AP}\cdot\mathbf{EP}}{\mathbf{BP}} \qquad \mathbf{BH} := \mathbf{BP} - \mathbf{HP} \qquad \mathbf{AR} := \frac{\mathbf{AP}\cdot\mathbf{BH}}{\mathbf{BP}}$$

$$\mathbf{AR} - \frac{4\cdot\mathbf{AN}^4 + \mathbf{AN}^3 - 2\cdot\mathbf{AN}^2 - \mathbf{AN}^6}{\mathbf{AN}^6 + 2\cdot\mathbf{AN}^5 - 2\cdot\mathbf{AN}^3 + 3\cdot\mathbf{AN}^2 + 1} = 0$$



$$\mathbf{AB} := 1$$

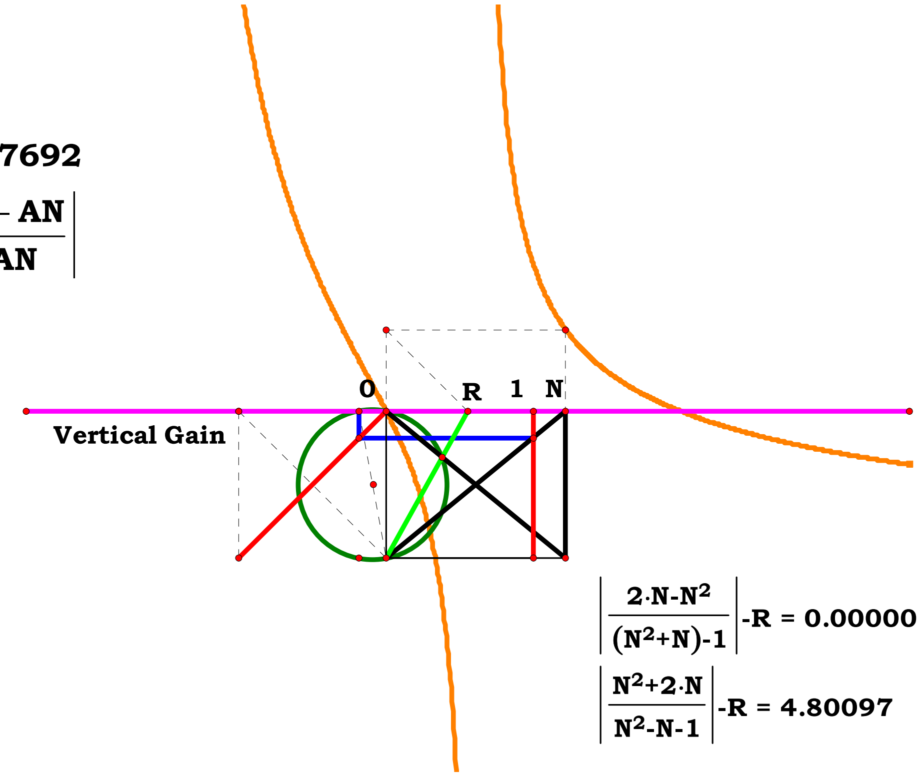
$$\mathbf{AN} := 1.27692$$

$$\mathbf{AE} := \left|\frac{1-\mathbf{AN}}{\mathbf{AN}}\right|$$

$$\mathbf{AK} := \sqrt{\mathbf{AN}^2 + \mathbf{AB}^2} \qquad \mathbf{BM} := \mathbf{AE} \qquad \mathbf{KM} := \mathbf{AN} + \mathbf{AE} \qquad \mathbf{HM} := \frac{\mathbf{AB}\cdot\mathbf{KM}}{\mathbf{AK}}$$

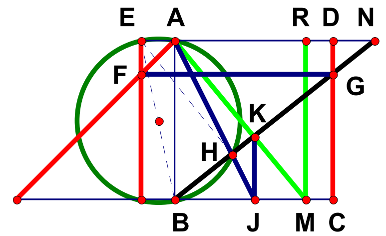
$$\mathbf{MJ} := \frac{\mathbf{AB}\cdot\mathbf{HM}}{\mathbf{AK}} \qquad \mathbf{HJ} := \frac{\mathbf{AN}\cdot\mathbf{HM}}{\mathbf{AK}} \qquad \mathbf{BJ} := \mathbf{MJ} - \mathbf{BM} \qquad \mathbf{AR} := \frac{\mathbf{BJ}\cdot\mathbf{AB}}{\mathbf{HJ}}$$

$$\mathbf{AR} - \frac{2\cdot\mathbf{AN} - \mathbf{AN}^2}{\mathbf{AN}^2 + \mathbf{AN} - 1} = 0$$



$$\left|\frac{2\cdot N - N^2}{(N^2+N)-1}\right| - R = 0.00000$$

$$\left|\frac{N^2+2\cdot N}{N^2-N-1}\right| - R = 4.80097$$

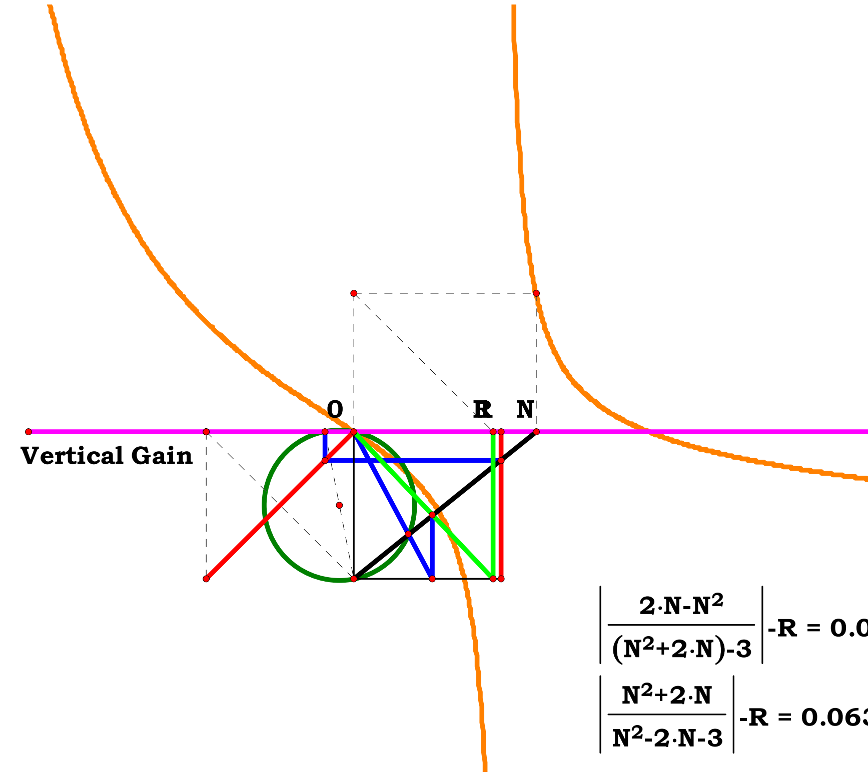


$$AB := 1$$

$$AN := 1.30769$$

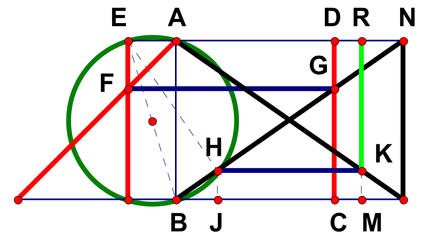
$$BJ := \frac{2 \cdot AN - AN^2}{AN^2 + AN - 1} \qquad JK := \frac{AB \cdot BJ}{AN} \qquad BM := \frac{BJ \cdot AB}{AB - JK} \qquad AR := BM$$

$$AR - \frac{2 \cdot AN - AN^2}{AN^2 + 2 \cdot AN - 3} = 0$$



$$\left|\frac{2 \cdot N - N^2}{(N^2 + 2 \cdot N) - 3}\right| - R = 0.00000$$

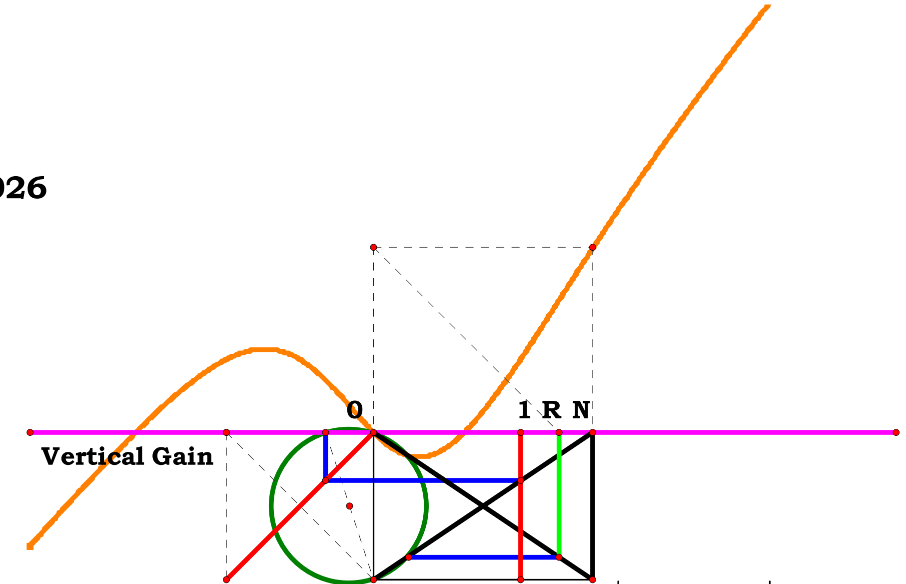
$$\left|\frac{N^2 + 2 \cdot N}{N^2 - 2 \cdot N - 3}\right| - R = 0.06368$$



$AB := 1$

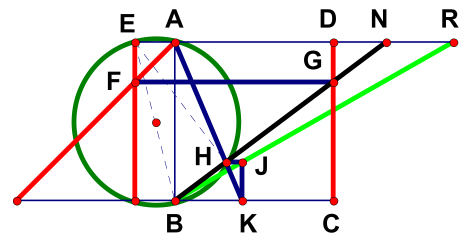
$AN := 1.21026$

$$BJ := \frac{2AN - AN^2}{AN^2 + 1} \qquad AR := AN - BJ \qquad AR - \frac{AN^3 + AN^2 - AN}{AN^2 + 1} = 0$$



$$\left|\frac{(N^3+N^2)-N}{N^2+1}\right|-R = 0.00000$$

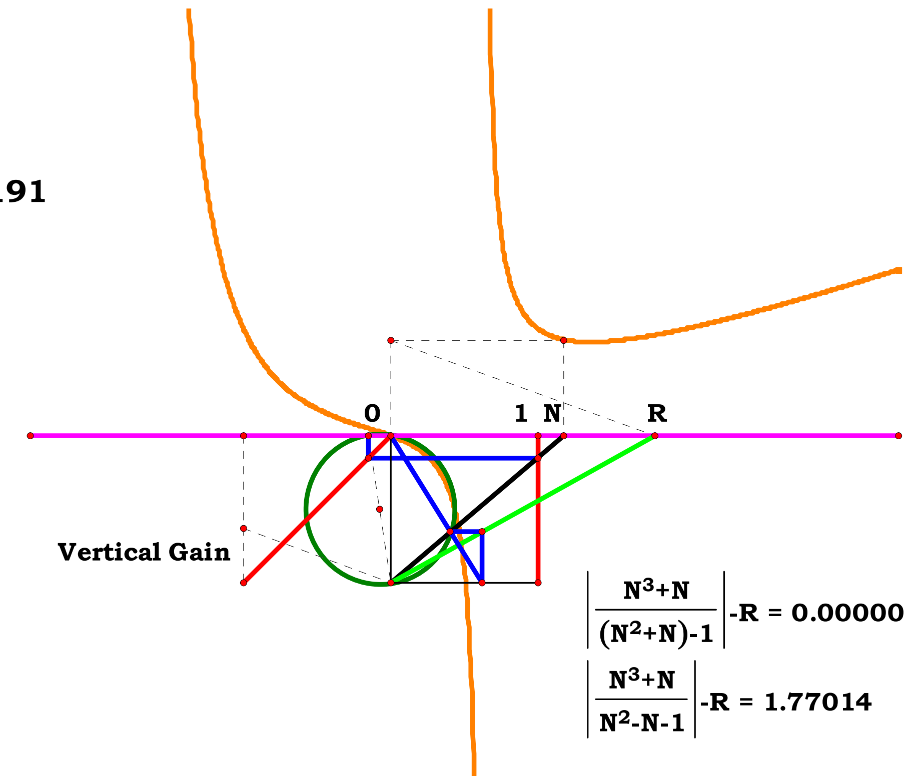
$$\left|\frac{N^3-N^2-N}{N^2+1}\right|-R = -1.13851$$

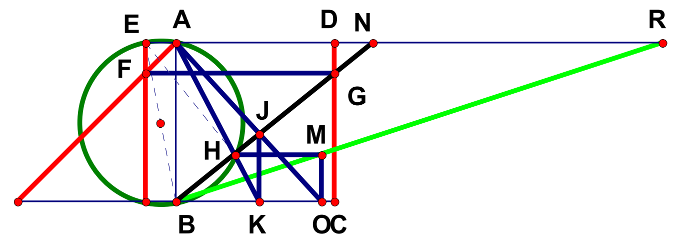


$AB := 1$

$AN := 1.21191$

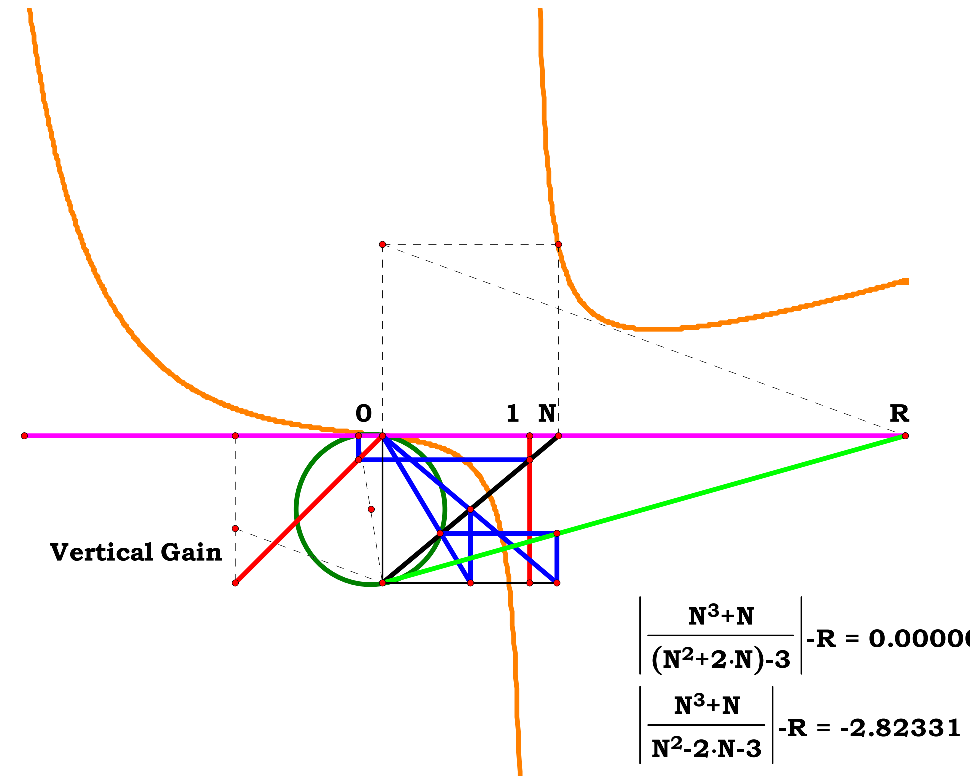
$BK := \frac{2AN - AN^2}{AN^2 + AN - 1}$ $\quad JK := \frac{2 - AN}{AN^2 + 1}$ $\quad AR := \frac{BK \cdot AB}{JK}$ $\quad AR - \frac{AN^3 + AN}{AN^2 + AN - 1} = 0$

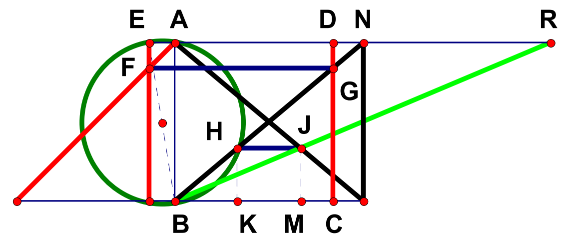




$$AB := 1 \qquad AN := 1.25645 \qquad BK := \frac{2AN - AN^2}{AN^2 + AN - 1} \qquad MO := \frac{2 - AN}{AN^2 + 1}$$

$$JK := \frac{AB \cdot BK}{AN} \qquad BO := \frac{BK \cdot AB}{AB - JK} \qquad AR := \frac{BO \cdot AB}{MO} \qquad AR - \frac{AN^3 + AN}{AN^2 + 2 \cdot AN - 3} = 0$$



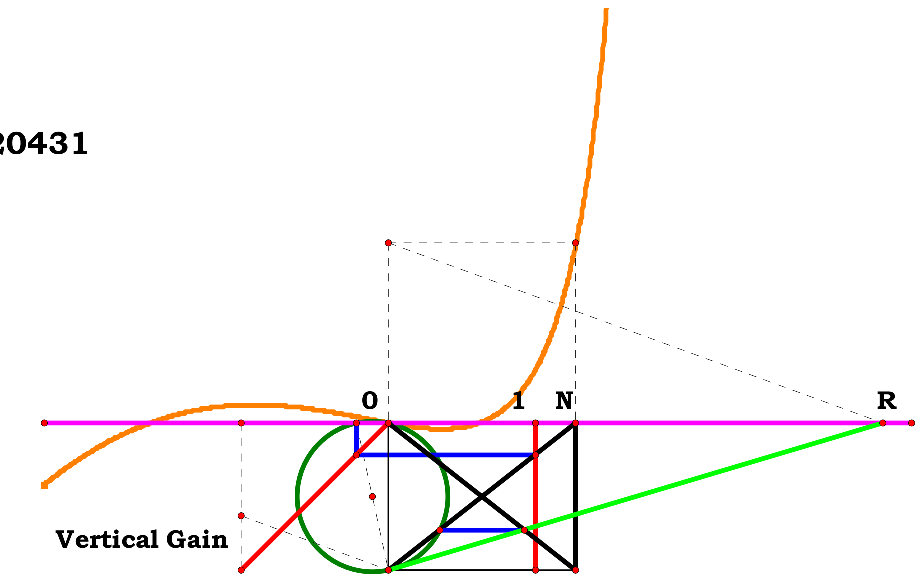


$AB := 1$

$AN := 1.20431$

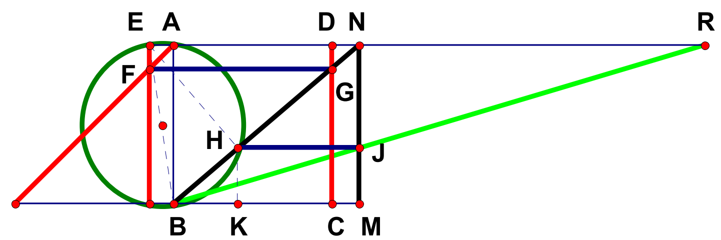
$$BK := \frac{2AN - AN^2}{AN^2 + 1} \qquad HK := \frac{2 - AN}{AN^2 + 1} \qquad BM := AN - BK \qquad AR := \frac{BM \cdot AB}{HK}$$

$$AR - \frac{AN^3 + AN^2 - AN}{2 - AN} = 0$$



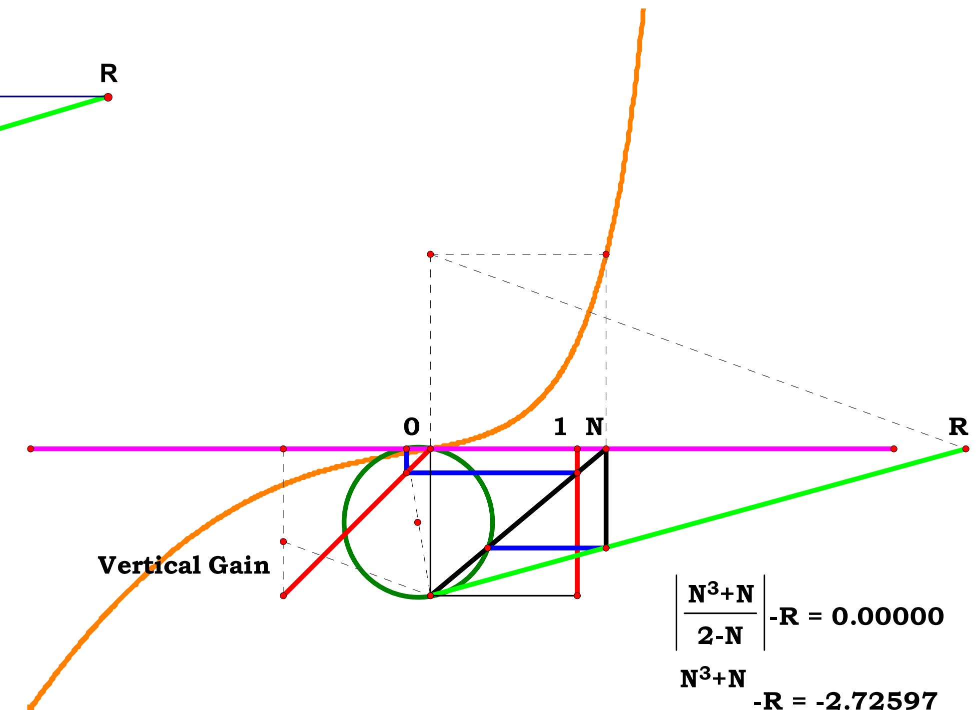
$$\left|\frac{(N^3+N^2)-N}{2-N}\right| -R = 0.00000$$

$$\left|\frac{N^3-N^2-N}{2+N}\right| -R = -3.10934$$



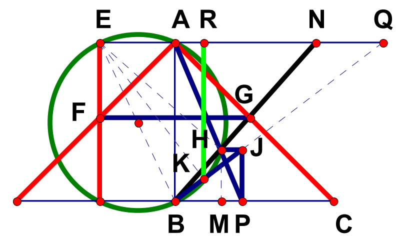
$\mathbf{AB} := 1 \quad \mathbf{AN} := 1.17022 \qquad \mathbf{HK} := \dfrac{2 - \mathbf{AN}}{\mathbf{AN}^2 + 1} \qquad \mathbf{AR} := \dfrac{\mathbf{AN} \cdot \mathbf{AB}}{\mathbf{HK}}$

$$\mathbf{AR} - \frac{\mathbf{AN}^3 + \mathbf{AN}}{2 - \mathbf{AN}} = 0$$



$\left|\dfrac{N^3+N}{2-N}\right|-R = 0.00000$

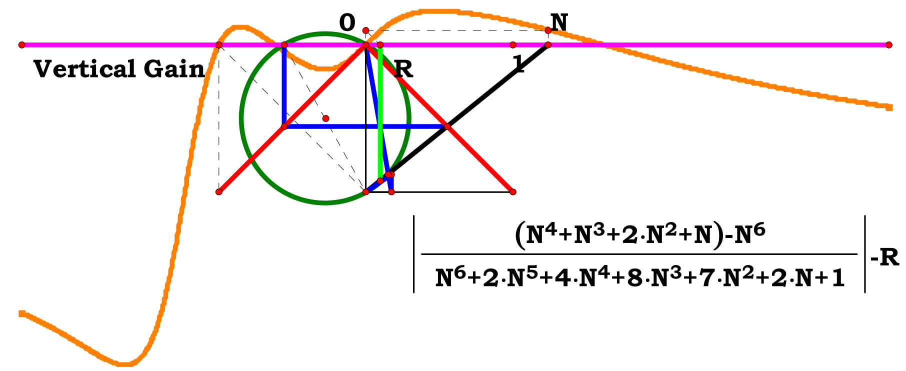
$\dfrac{N^3+N}{2+N}-R = -2.72597$



$AB := 1$

$AN := .81969$

$AE := \left| \dfrac{-AN}{AN+1} \right|$



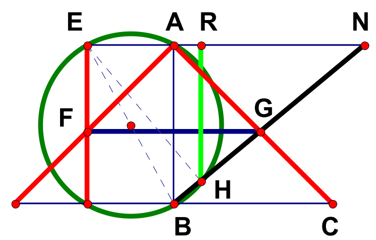
$$\left| \frac{(N^4+N^3+2\cdot N^2+N)-N^6}{N^6+2\cdot N^5+4\cdot N^4+8\cdot N^3+7\cdot N^2+2\cdot N+1} \right| -R = 0.00000$$

$EN := AN + AE \quad BN := \sqrt{AN^2 + AB^2} \quad HN := \dfrac{AN \cdot EN}{BN} \quad BH := BN - HN$

$HM := \dfrac{AB \cdot BH}{BN} \quad BP := \dfrac{BN \cdot BH}{EN} \quad JP := HM \quad AQ := \dfrac{BP \cdot AB}{JP}$

$BQ := \sqrt{AQ^2 + AB^2} \quad EQ := AQ + AE \quad KQ := \dfrac{AQ \cdot EQ}{BQ} \quad BK := BQ - KQ$

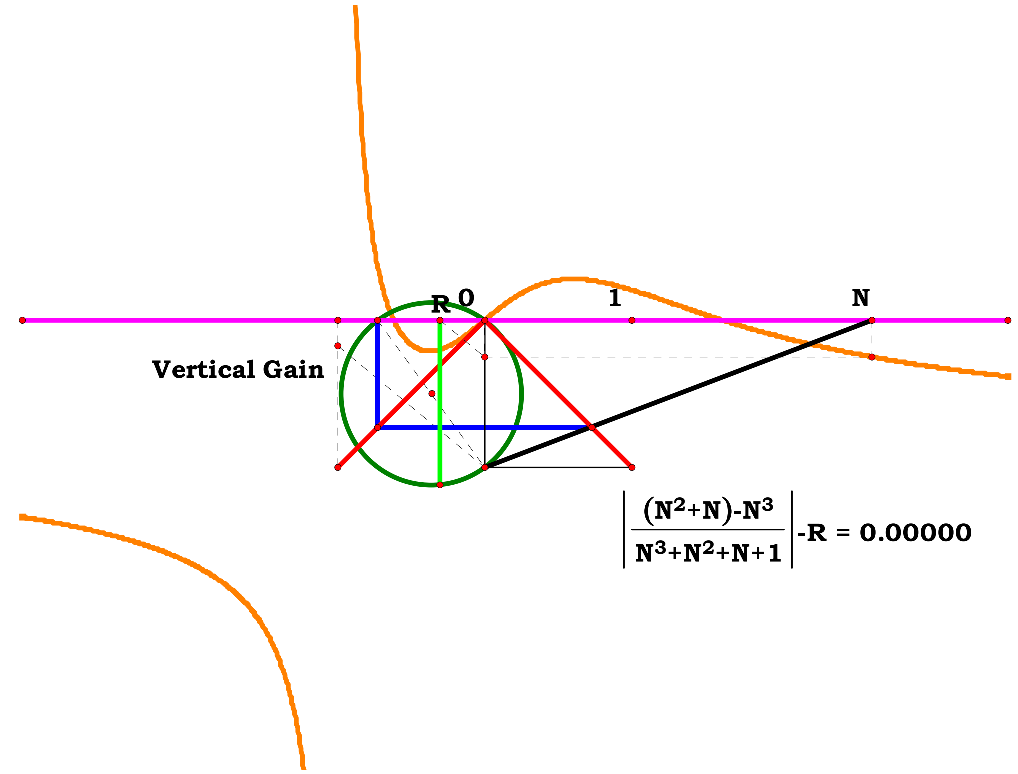
$$AR := \frac{AQ \cdot BK}{BQ} \quad AR - \frac{-AN^6 + AN^4 + AN^3 + 2 \cdot AN^2 + AN}{AN^6 + 2 \cdot AN^5 + 4 \cdot AN^4 + 8 \cdot AN^3 + 7 \cdot AN^2 + 2 \cdot AN + 1} = 0$$



$$AB := 1$$

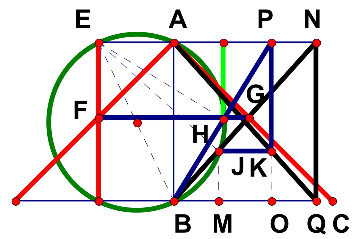
$$AN := .99821$$

$$AE := \left| \frac{-AN}{AN + 1} \right|$$



$$EN := AN + AE \qquad BN := \sqrt{AN^2 + AB^2} \qquad HN := \frac{AN \cdot EN}{BN} \qquad RN := \frac{AN \cdot HN}{BN}$$

$$AR := AN - RN \qquad AR - \frac{AN^2 + AN - AN^3}{AN^3 + AN^2 + AN + 1} = 0$$

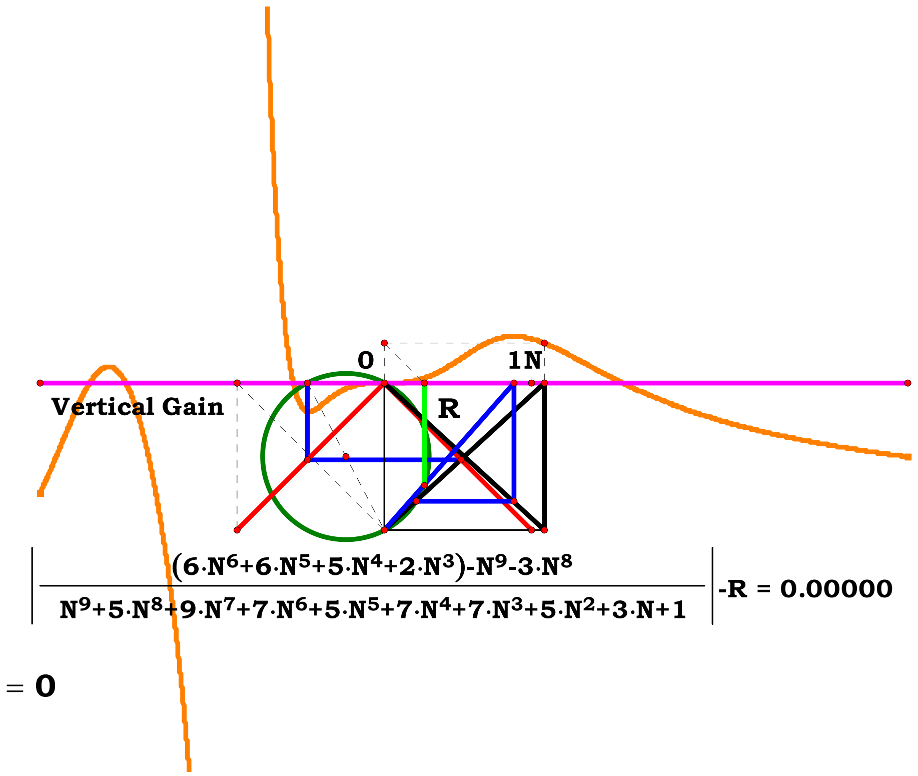


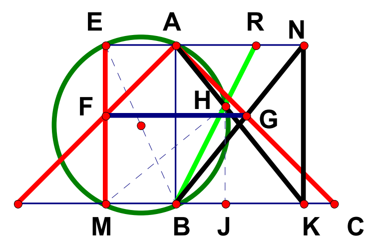
$$AB := 1$$
$$AN := 1.13710$$
$$AE := \left| \frac{-AN}{AN + 1} \right|$$

$$BM := \frac{AN^2 - AN^3 + AN}{AN^3 + AN + AN^2 + 1} \quad NP := BM \quad AP := AN - NP \quad BP := \sqrt{AB^2 + AP^2}$$

$$EP := AP + AE \quad HP := \frac{AP \cdot EP}{BP} \quad BH := BP - HP \quad AR := \frac{AP \cdot BH}{BP}$$

$$AR - \frac{-AN^9 - 3 \cdot AN^8 + 6 \cdot AN^6 + 6 \cdot AN^5 + 5 \cdot AN^4 + 2 \cdot AN^3}{AN^9 + 5 \cdot AN^8 + 9 \cdot AN^7 + 7 \cdot AN^6 + 5 \cdot AN^5 + 7 \cdot AN^4 + 7 \cdot AN^3 + 5 \cdot AN^2 + 3 \cdot AN + 1} = 0$$





$$\mathbf{AB} := \mathbf{1}$$

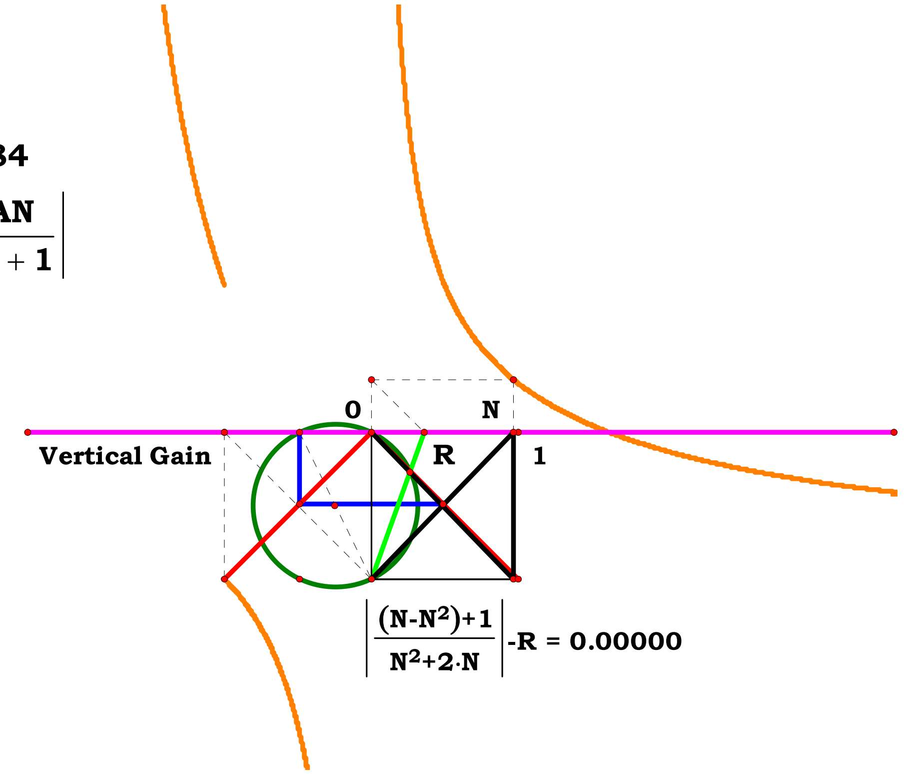
$$\mathbf{AN} := \mathbf{1.184}$$

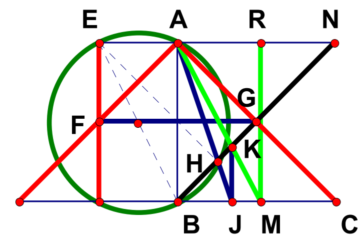
$$\mathbf{AE} := \left|\frac{-\mathbf{AN}}{\mathbf{AN}+\mathbf{1}}\right|$$

$$\mathbf{AK} := \sqrt{\mathbf{AN}^2+\mathbf{AB}^2} \qquad \mathbf{BM} := \mathbf{AE} \qquad \mathbf{KM} := \mathbf{AN}+\mathbf{AE} \qquad \mathbf{HM} := \frac{\mathbf{AB}\cdot\mathbf{KM}}{\mathbf{AK}}$$

$$\mathbf{MJ} := \frac{\mathbf{AB}\cdot\mathbf{HM}}{\mathbf{AK}} \qquad \mathbf{HJ} := \frac{\mathbf{AN}\cdot\mathbf{HM}}{\mathbf{AK}} \qquad \mathbf{BJ} := \mathbf{MJ}-\mathbf{BM} \qquad \mathbf{AR} := \frac{\mathbf{BJ}\cdot\mathbf{AB}}{\mathbf{HJ}}$$

$$\mathbf{AR}-\frac{\mathbf{AN}-\mathbf{AN}^2+\mathbf{1}}{\mathbf{AN}^2+\mathbf{2}\cdot\mathbf{AN}}=\mathbf{0}$$



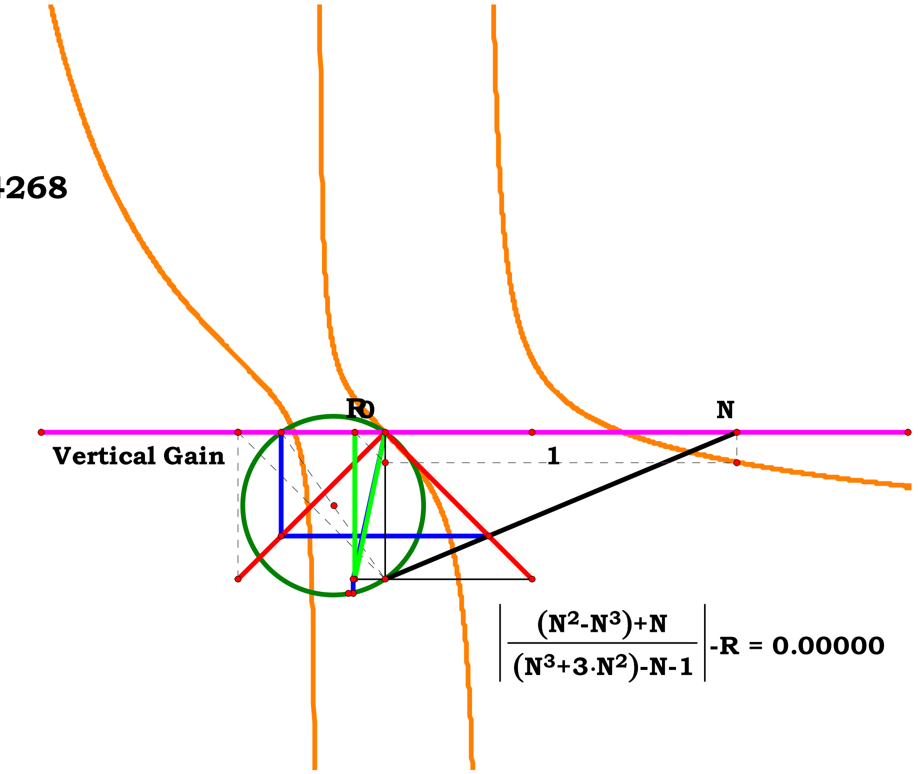


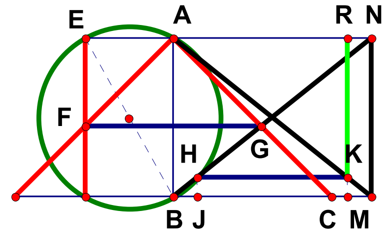
$AB := 1$

$AN := 1.04268$

$BJ := \dfrac{AN - AN^2 + 1}{AN^2 + 2AN}$ $\quad JK := \dfrac{AB \cdot BJ}{AN}$ $\quad BM := \dfrac{BJ \cdot AB}{AB - JK}$ $\quad AR := BM$

$$AR - \frac{AN^2 - AN^3 + AN}{AN^3 + 3 \cdot AN^2 - AN - 1} = 0$$

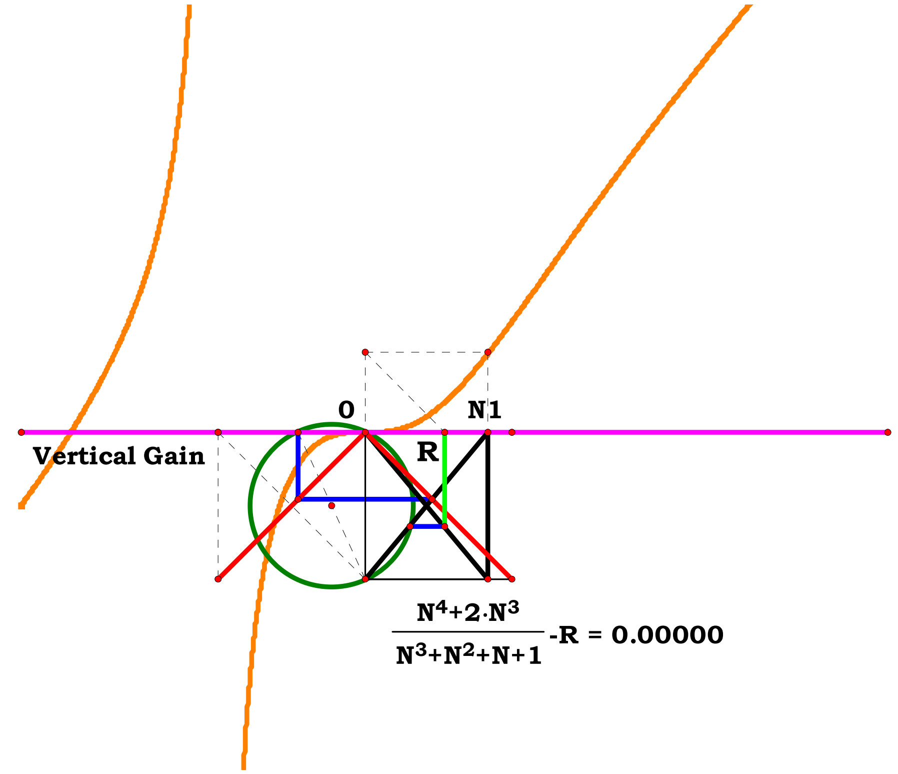




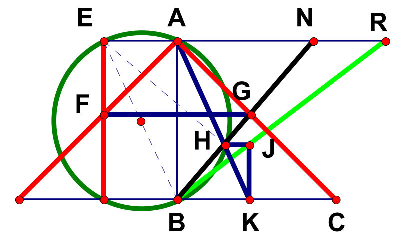
$AB := 1$

$AN := .8387$

$$BJ := \frac{AN^2 - AN^3 + AN}{AN^3 + AN^2 + AN + 1} \qquad AR := AN - BJ \qquad AR - \frac{AN^4 + 2 \cdot AN^3}{AN^3 + AN^2 + AN + 1} = 0$$

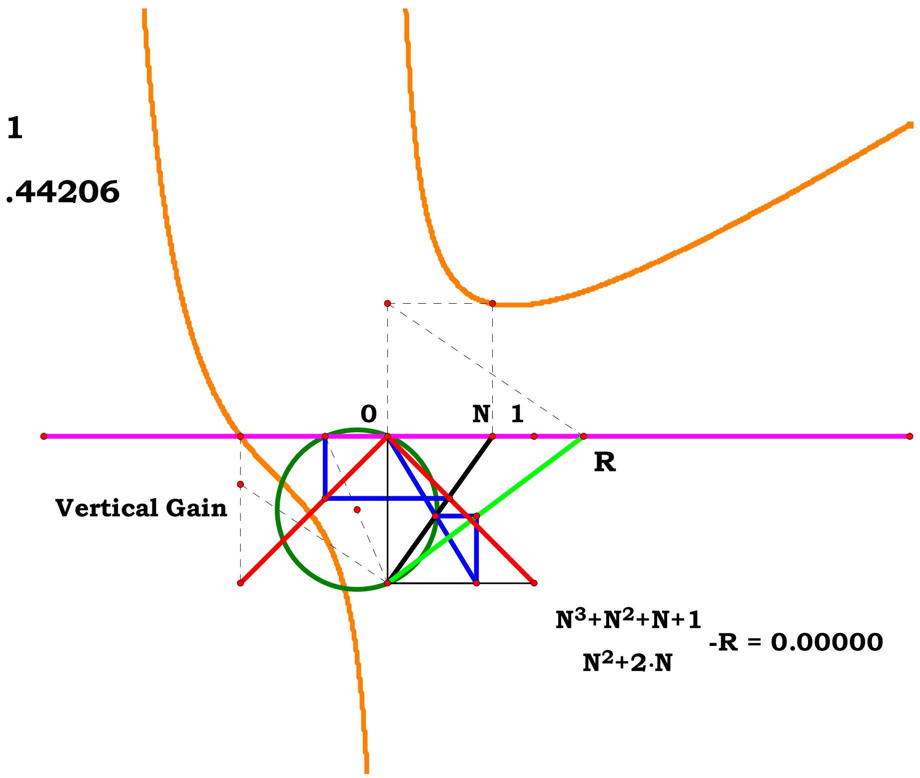


$$\frac{N^4+2\cdot N^3}{N^3+N^2+N+1} - R = 0.00000$$



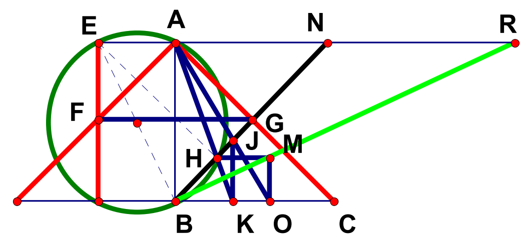
$$AB := 1$$

$$AN := .44206$$



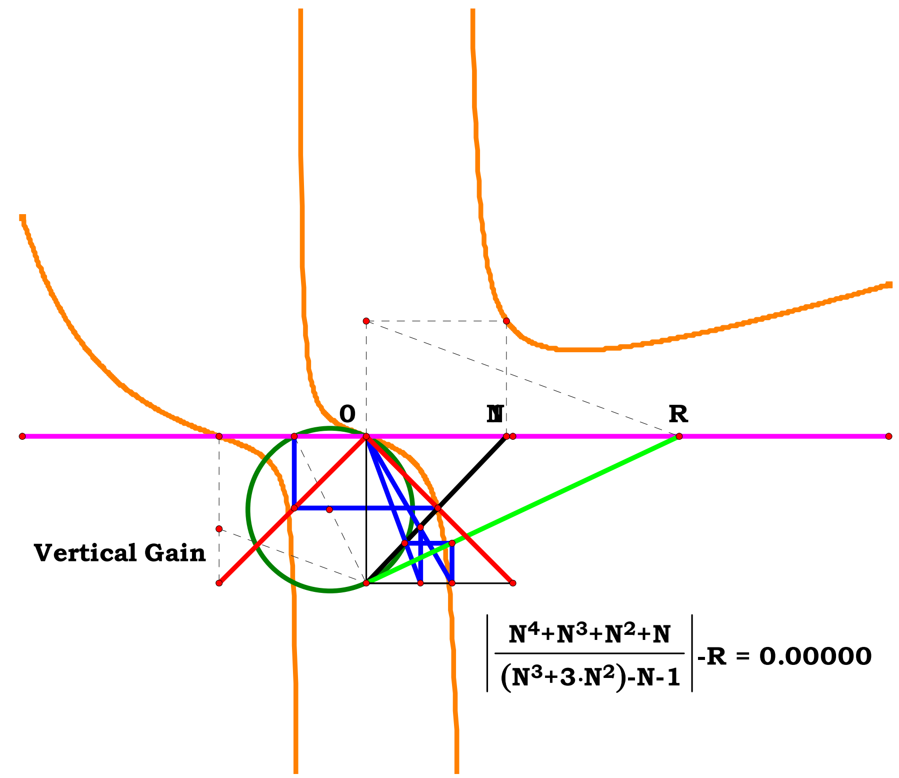
$$BK := \frac{AN - AN^2 + 1}{AN^2 + 2AN} \qquad JK := \frac{AN - AN^2 + 1}{AN^3 + AN^2 + AN + 1} \qquad AR := \frac{BK \cdot AB}{JK}$$

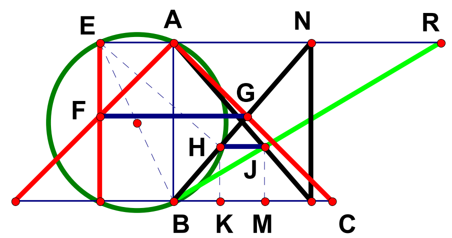
$$AR - \frac{AN^3 + AN^2 + AN + 1}{AN^2 + 2 \cdot AN} = 0$$



$$AB := 1 \qquad AN := .95697 \qquad BK := \frac{AN - AN^2 + 1}{AN^2 + 2AN} \qquad MO := \frac{AN - AN^2 + 1}{AN^3 + AN^2 + AN + 1}$$

$$JK := \frac{AB \cdot BK}{AN} \qquad BO := \frac{BK \cdot AB}{AB - JK} \qquad AR := \frac{BO \cdot AB}{MO} \qquad AR - \frac{AN^4 + AN^3 + AN^2 + AN}{AN^3 + 3 \cdot AN^2 - AN - 1} = 0$$



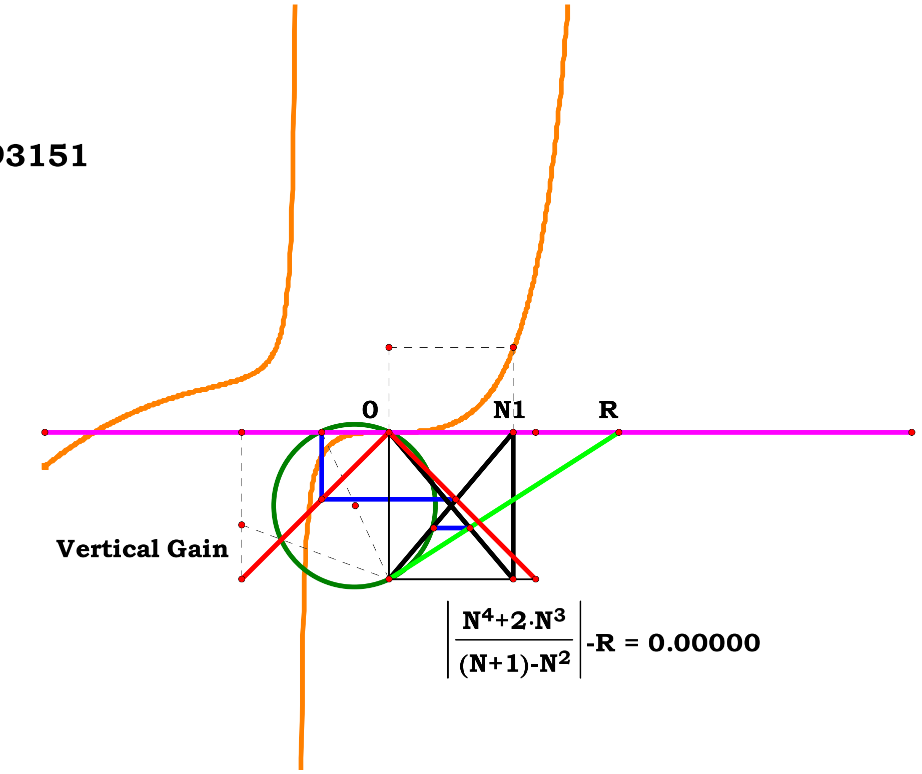


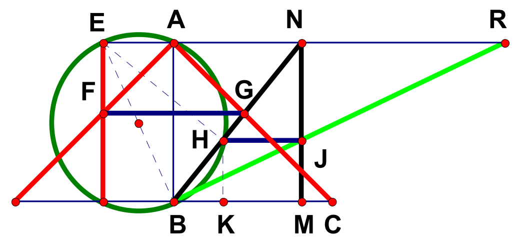
$AB := 1$

$AN := .93151$

$$BK := \frac{AN^2 - AN^3 + AN}{AN^3 + AN^2 + AN + 1} \qquad HK := \frac{AN - AN^2 + 1}{AN^3 + AN^2 + AN + 1} \qquad BM := AN - BK$$

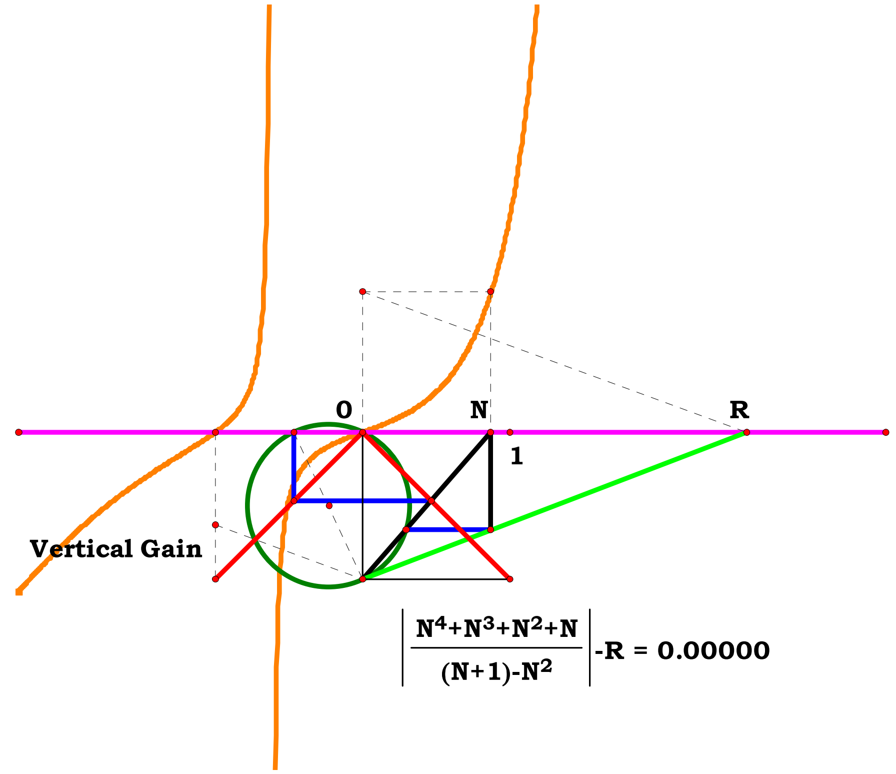
$$AR := \frac{BM \cdot AB}{HK} \qquad AR - \frac{AN^4 + 2AN^3}{AN - AN^2 + 1} = 0$$

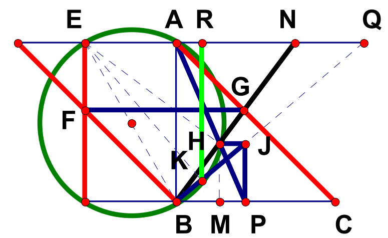




$$AB := 1 \quad AN := .9315 \quad HK := \frac{AN - AN^2 + 1}{AN^3 + AN^2 + AN + 1} \quad AR := \frac{AN \cdot AB}{HK}$$

$$AR - \frac{AN^4 + AN^3 + AN^2 + AN}{AN - AN^2 + 1} = 0$$

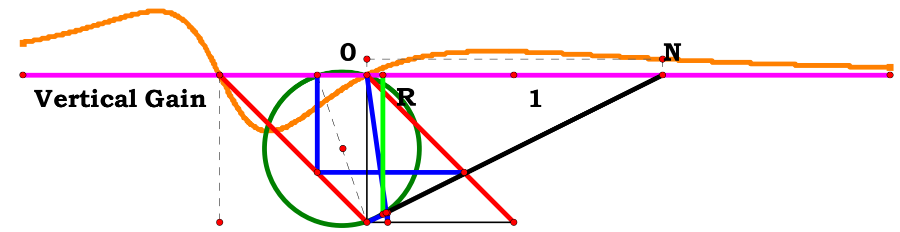




$AB := 1$

$AN := .91218$

$AE := \left|\dfrac{-1}{AN+1}\right|$



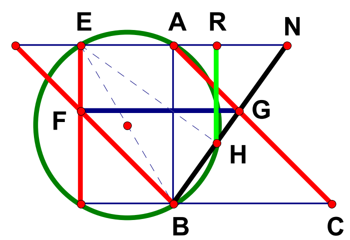
$$\frac{N^4+N^3+N^2+N}{N^6+2\cdot N^5+4\cdot N^4+6\cdot N^3+6\cdot N^2+4\cdot N+2}-R = 0.00000$$

$EN := AN + AE \quad BN := \sqrt{AN^2 + AB^2} \quad HN := \dfrac{AN \cdot EN}{BN} \quad BH := BN - HN$

$HM := \dfrac{AB \cdot BH}{BN} \quad BP := \dfrac{BN \cdot BH}{EN} \quad JP := HM \quad AQ := \dfrac{BP \cdot AB}{JP}$

$BQ := \sqrt{AQ^2 + AB^2} \quad EQ := AQ + AE \quad KQ := \dfrac{AQ \cdot EQ}{BQ} \quad BK := BQ - KQ$

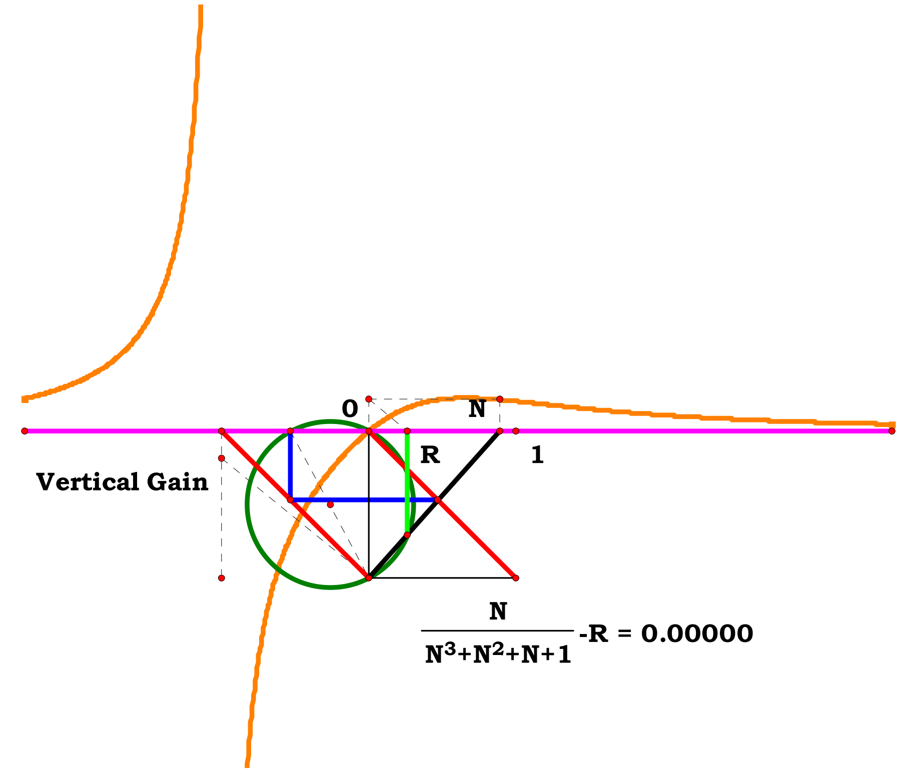
$AR := \dfrac{AQ \cdot BK}{BQ} \quad AR - \dfrac{AN^4 + AN^3 + AN^2 + AN}{AN^6 + 2 \cdot AN^5 + 4 \cdot AN^4 + 6 \cdot AN^3 + 6 \cdot AN^2 + 4 \cdot AN + 2} = 0$



$$AB := 1$$

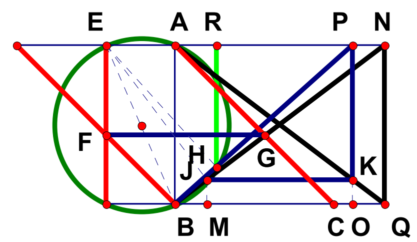
$$AN := .84610$$

$$AE := \left| \frac{-1}{AN + 1} \right|$$



$EN := AN + AE \qquad BN := \sqrt{AN^2 + AB^2} \qquad HN := \frac{AN \cdot EN}{BN} \qquad RN := \frac{AN \cdot HN}{BN}$

$AR := AN - RN \qquad AR - \frac{AN}{AN^3 + AN + AN^2 + 1} = 0$

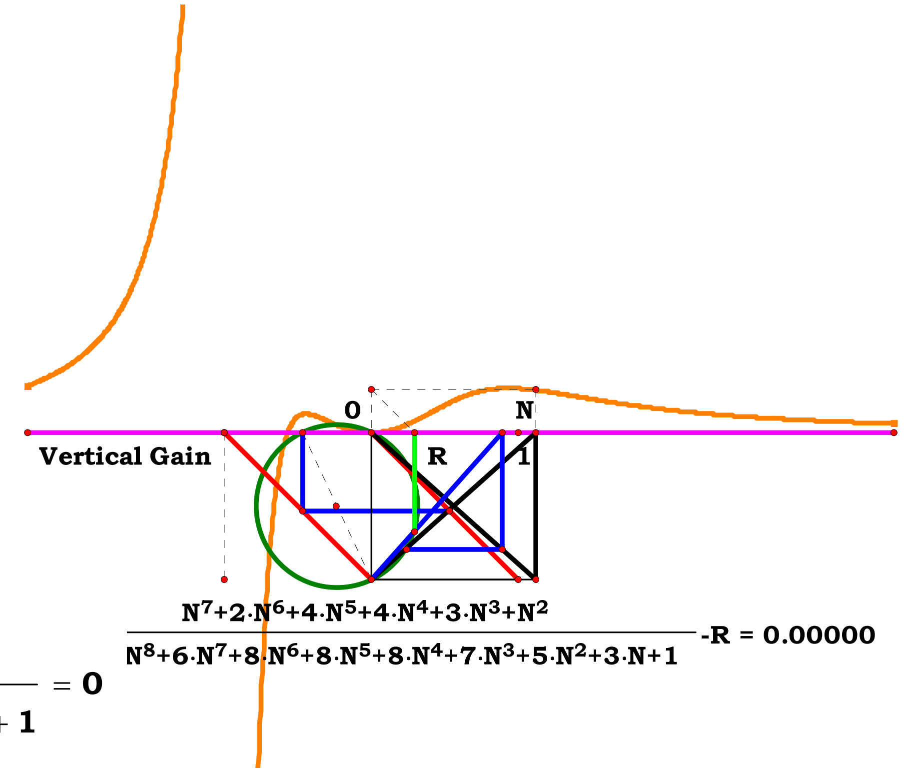


$$AB := 1$$
$$AN := 1.30599$$
$$AE := \left|\frac{-1}{AN + 1}\right|$$

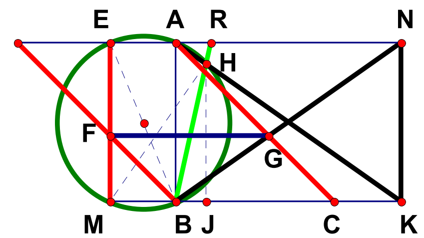
$$BM := \frac{AN}{AN^3 + AN + AN^2 + 1} \quad NP := BM \quad AP := AN - NP \quad BP := \sqrt{AB^2 + AP^2}$$

$$EP := AP + AE \quad HP := \frac{AP \cdot EP}{BP} \quad BH := BP - HP \quad AR := \frac{AP \cdot BH}{BP}$$

$$AR - \frac{AN^7 + 2 \cdot AN^6 + 4 \cdot AN^5 + 4 \cdot AN^4 + 3 \cdot AN^3 + AN^2}{AN^9 + 3 \cdot AN^8 + 6 \cdot AN^7 + 8 \cdot AN^6 + 8 \cdot AN^5 + 8 \cdot AN^4 + 7 \cdot AN^3 + 5 \cdot AN^2 + 3 \cdot AN + 1} = 0$$



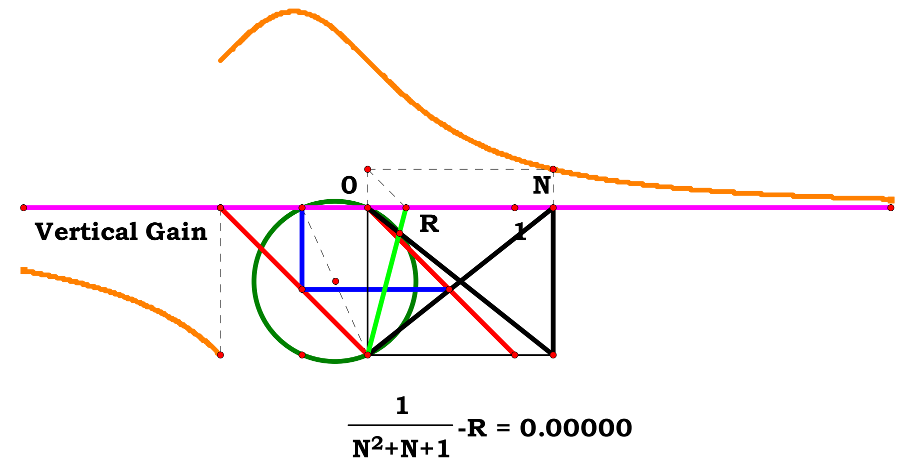
$$\frac{N^7+2 \cdot N^6+4 \cdot N^5+4 \cdot N^4+3 \cdot N^3+N^2}{N^8+6 \cdot N^7+8 \cdot N^6+8 \cdot N^5+8 \cdot N^4+7 \cdot N^3+5 \cdot N^2+3 \cdot N+1} - R = 0.00000$$



$$AB := 1$$

$$AN := .79764$$

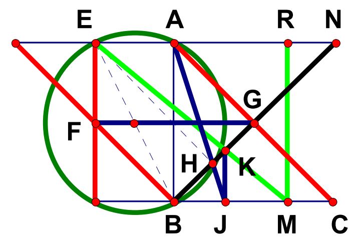
$$AE := \left| \frac{-1}{AN + 1} \right|$$



$$AK := \sqrt{AN^2 + AB^2} \qquad BM := AE \qquad KM := AN + AE \qquad HM := \frac{AB \cdot KM}{AK}$$

$$MJ := \frac{AB \cdot HM}{AK} \qquad HJ := \frac{AN \cdot HM}{AK} \qquad BJ := MJ - BM \qquad AR := \frac{BJ \cdot AB}{HJ}$$

$$AR - \frac{1}{AN^2 + AN + 1} = 0$$

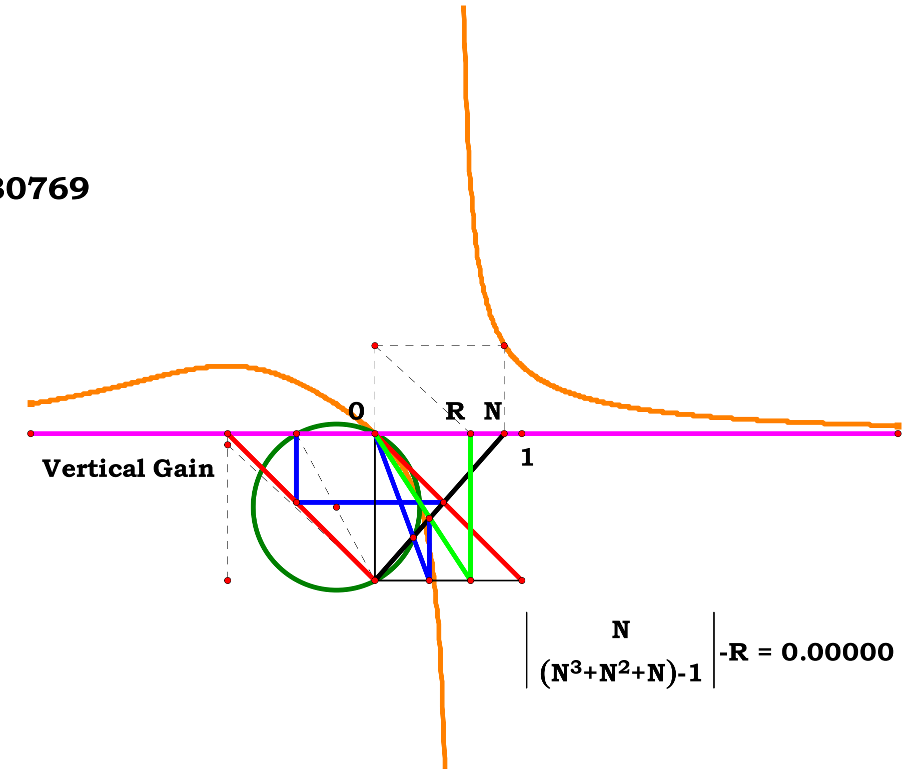


$AB := 1$

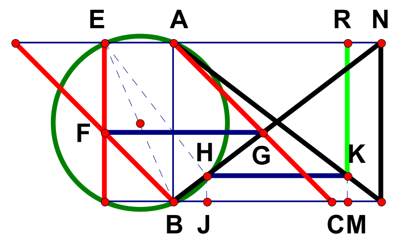
$AN := 1.30769$

$BJ := \dfrac{1}{AN^2 + AN + 1}$ $\quad JK := \dfrac{AB \cdot BJ}{AN}$ $\quad BM := \dfrac{BJ \cdot AB}{AB - JK}$ $\quad AR := BM$

$$AR - \frac{AN}{AN^3 + AN^2 + AN - 1} = 0$$



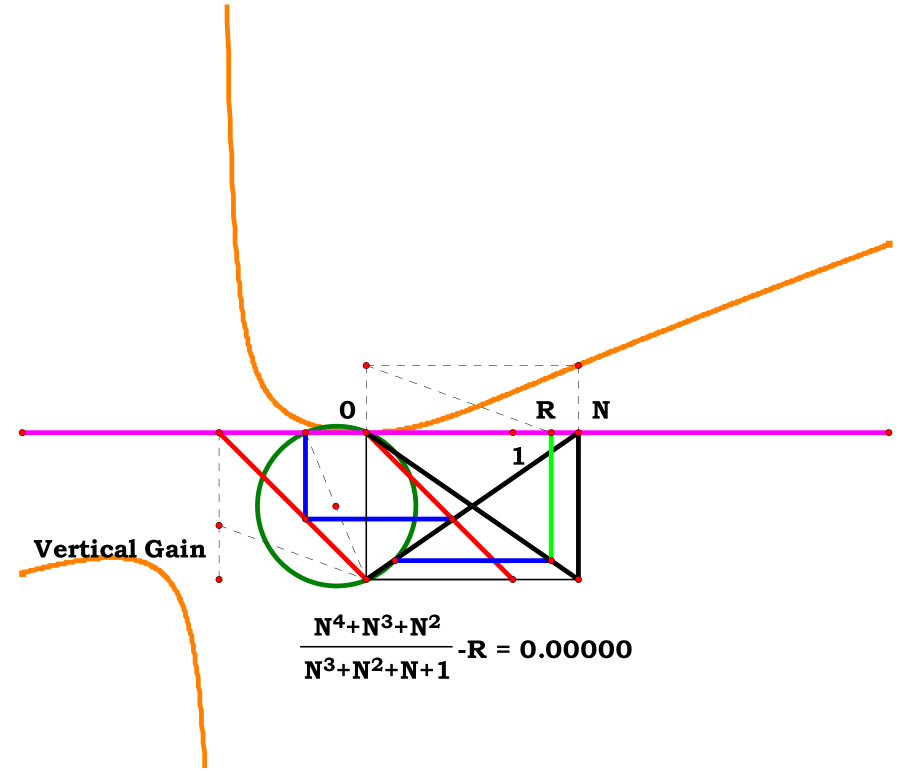
$$\left|\frac{N}{(N^3+N^2+N)-1}\right| - R = 0.00000$$

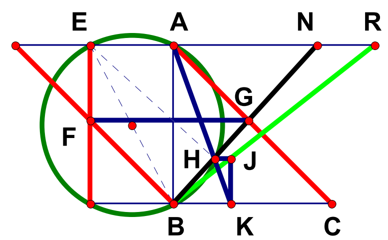


$AB := 1$

$AN := 2$

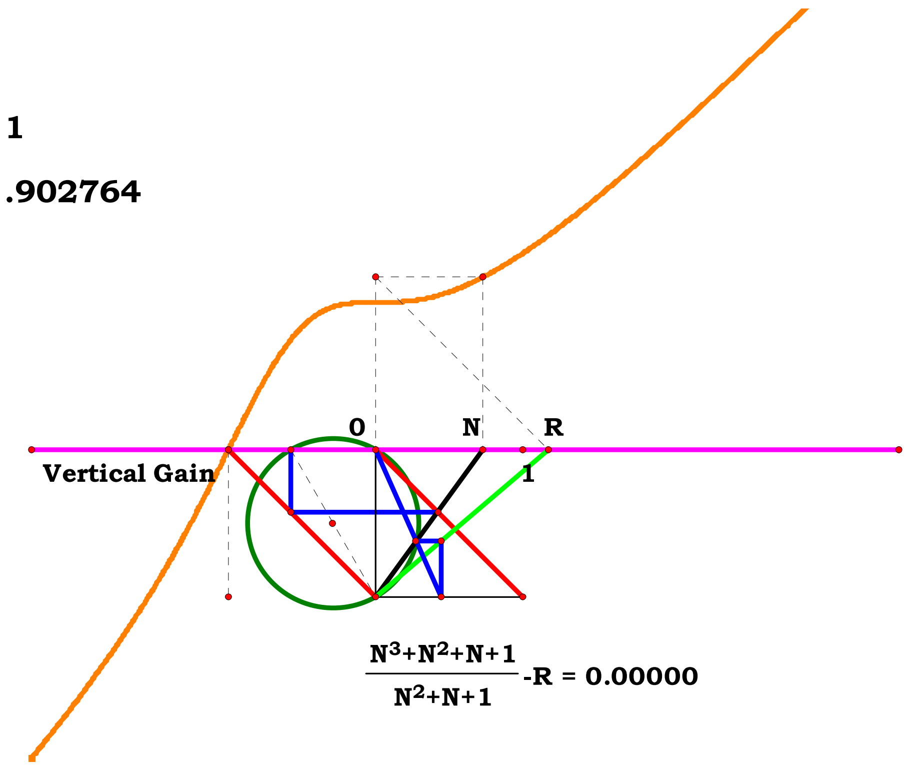
$$BJ := \frac{AN}{AN^3 + AN^2 + AN + 1} \qquad AR := AN - BJ \qquad AR - \frac{AN^4 + AN^3 + AN^2}{AN^3 + AN^2 + AN + 1} = 0$$





$AB := 1$

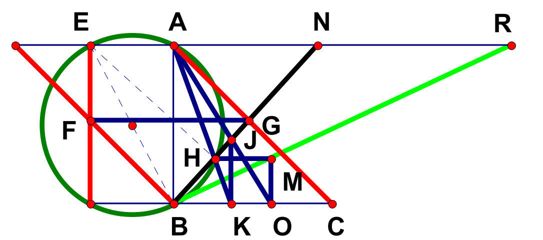
$AN := .902764$



$$BK := \frac{1}{AN^2 + AN + 1} \qquad JK := \frac{1}{AN^3 + AN^2 + AN + 1} \qquad AR := \frac{BK \cdot AB}{JK}$$

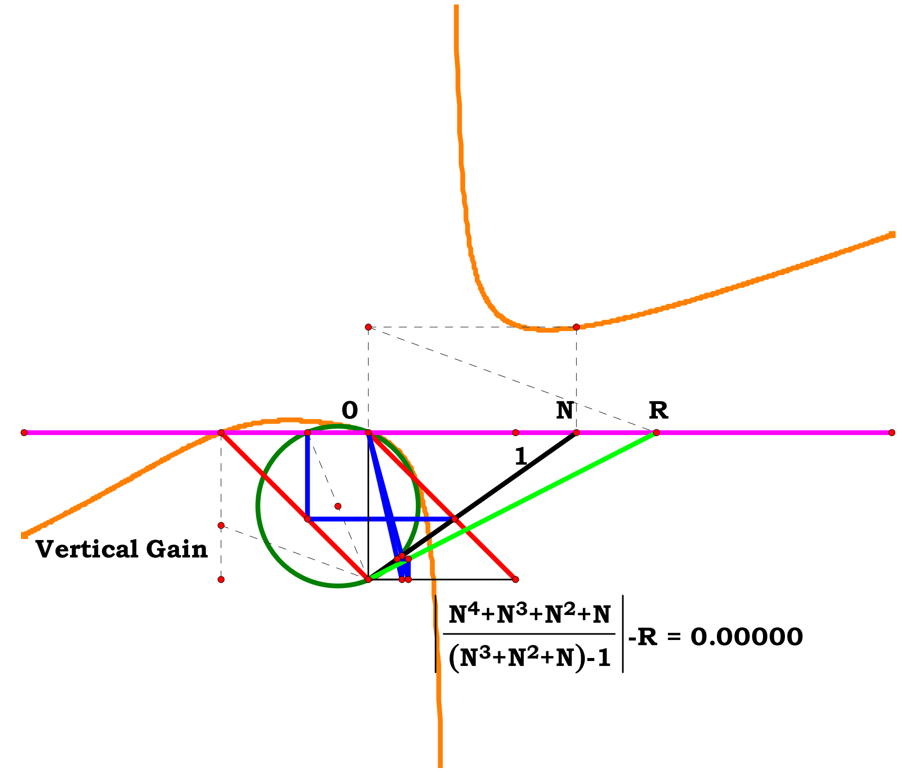
$$AR - \frac{AN^3 + AN^2 + AN + 1}{AN^2 + AN + 1} = 0$$

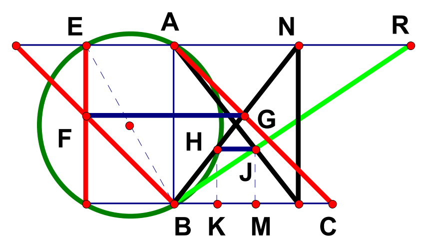
$$\frac{N^3+N^2+N+1}{N^2+N+1} - R = 0.00000$$



$$AB := 1 \qquad AN := .88433 \qquad BK := \frac{1}{AN^2 + AN + 1} \qquad MO := \frac{1}{AN^3 + AN^2 + AN + 1}$$

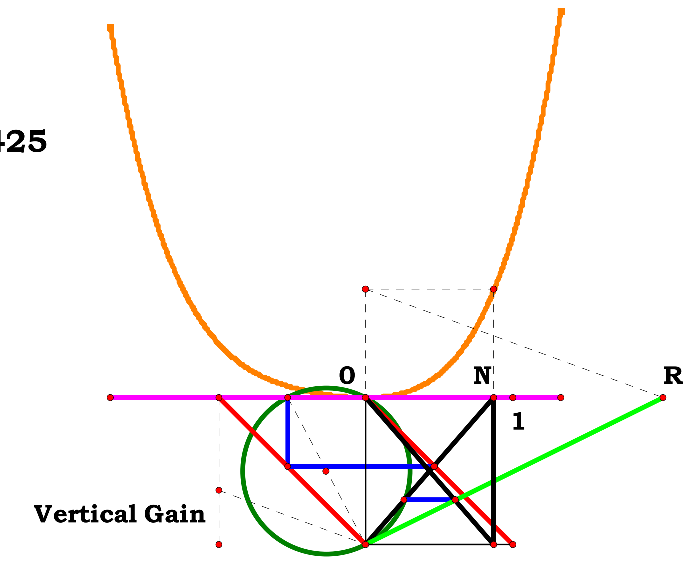
$$JK := \frac{AB \cdot BK}{AN} \qquad BO := \frac{BK \cdot AB}{AB - JK} \qquad AR := \frac{BO \cdot AB}{MO} \qquad AR - \frac{AN^4 + AN^3 + AN^2 + AN}{AN^3 + AN^2 + AN - 1} = 0$$





$$AB := 1$$

$$AN := .76425$$

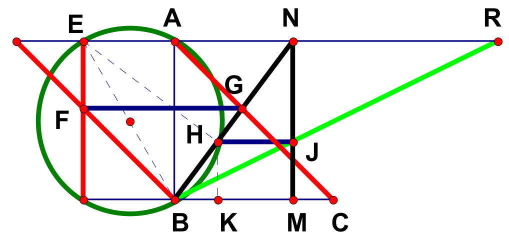


$$BK := \frac{AN}{AN^3 + AN^2 + AN + 1} \qquad HK := \frac{1}{AN^3 + AN^2 + AN + 1}$$

$$(N^4+N^3+N^2)\text{-}R = 0.00000$$

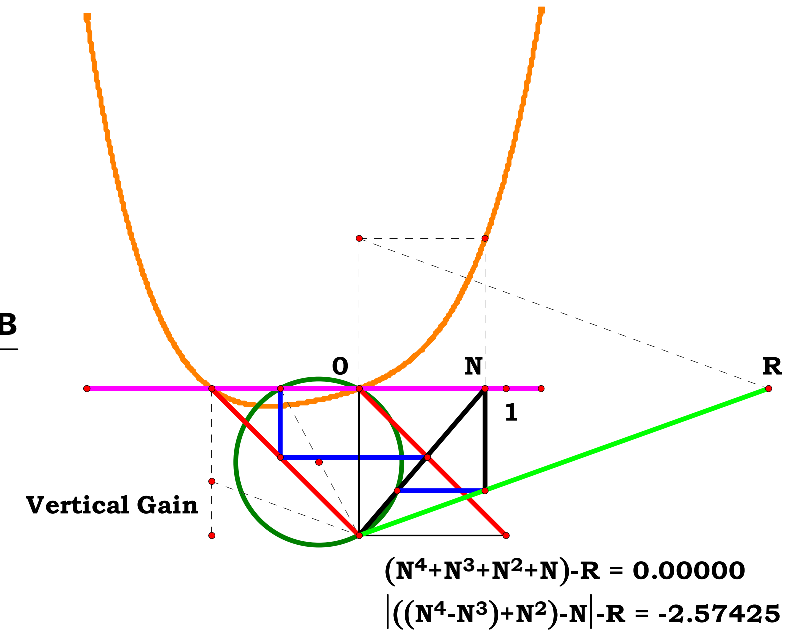
$$((N^4\text{-}N^3)+N^2)\text{-}R = -1.33943$$

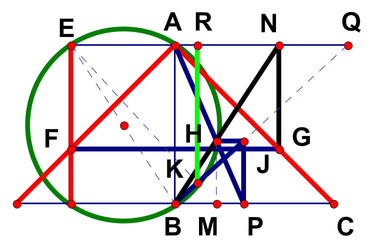
$$BM := AN - BK \qquad AR := \frac{BM \cdot AB}{HK} \qquad AR - \left(AN^4 + AN^3 + AN^2\right) = 0$$



$$AB := 1 \quad AN := .75454 \qquad HK := \frac{1}{AN^3 + AN^2 + AN + 1} \qquad AR := \frac{AN \cdot AB}{HK}$$

$$AR - \left(AN^4 + AN^3 + AN^2 + AN\right) = 0$$

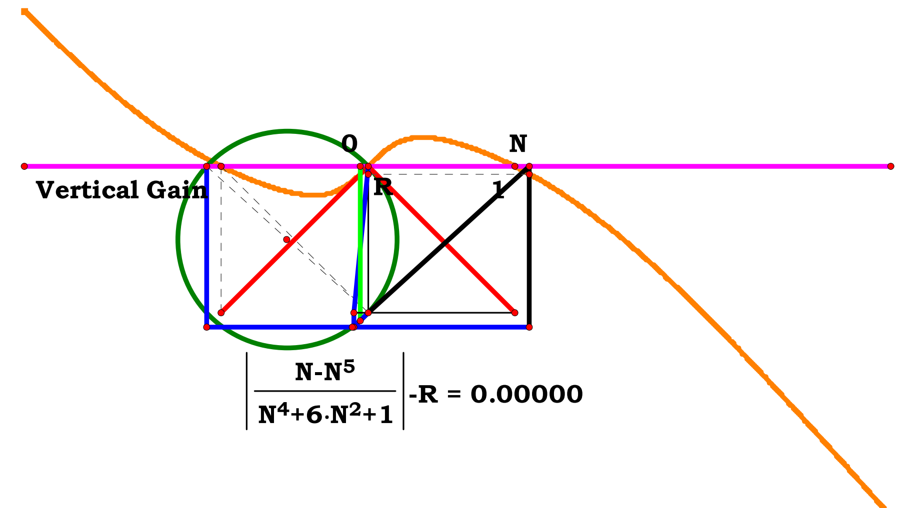




$$AB := 1$$

$$AN := .626355$$

$$AE := |-AN|$$

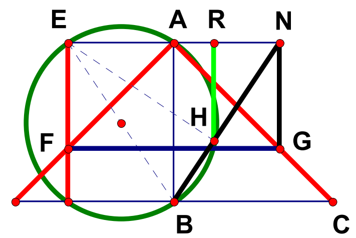


$$EN := AN + AE \quad BN := \sqrt{AN^2 + AB^2} \quad HN := \frac{AN \cdot EN}{BN} \quad BH := BN - HN$$

$$HM := \frac{AB \cdot BH}{BN} \quad BP := \frac{BN \cdot BH}{EN} \quad JP := HM \quad AQ := \frac{BP \cdot AB}{JP}$$

$$BQ := \sqrt{AQ^2 + AB^2} \quad EQ := AQ + AE \quad KQ := \frac{AQ \cdot EQ}{BQ} \quad BK := BQ - KQ$$

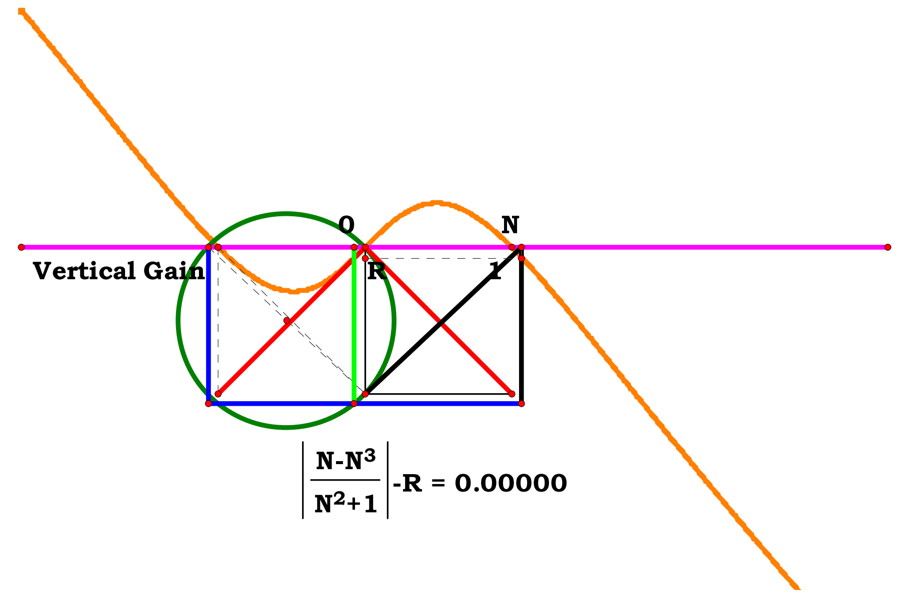
$$AR := \frac{AQ \cdot BK}{BQ} \quad AR - \frac{AN - AN^5}{AN^4 + 6 \cdot AN^2 + 1} = 0$$



$$AB := 1$$

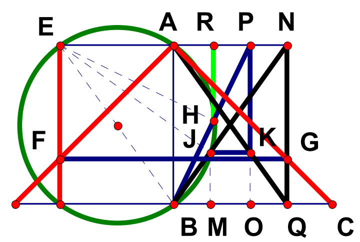
$$AN := .59100$$

$$AE := |-AN|$$



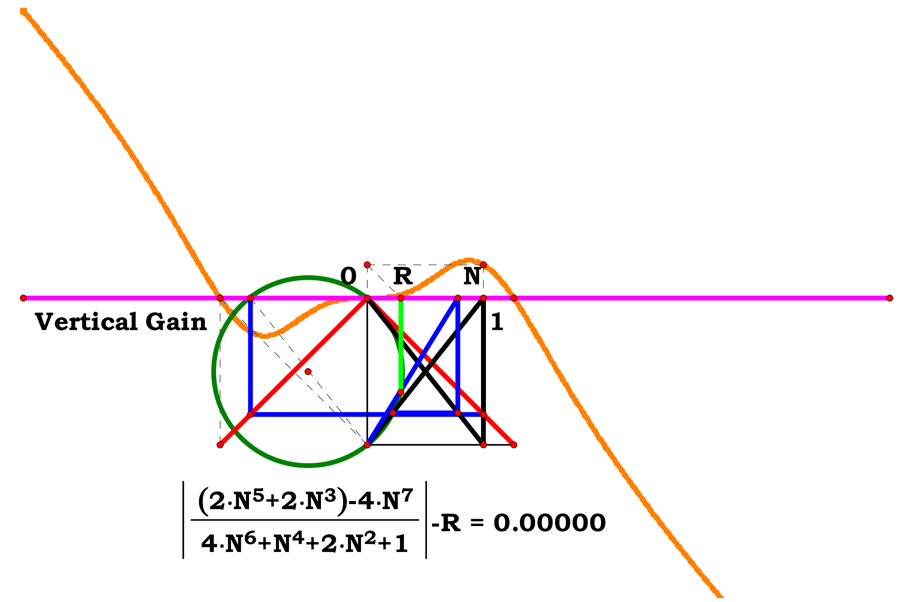
$$EN := AN + AE \qquad BN := \sqrt{AN^2 + AB^2} \qquad HN := \frac{AN \cdot EN}{BN} \qquad RN := \frac{AN \cdot HN}{BN}$$

$$AR := AN - RN \qquad AR - \frac{AN - AN^3}{AN^2 + 1} = 0$$



$$AB := 1$$
$$AN := .72012$$

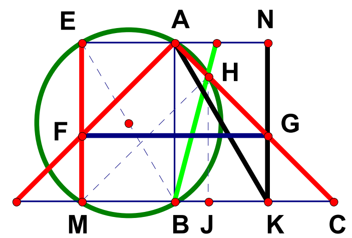
$$AE := |-AN|$$



$$BM := \frac{AN - AN^3}{AN^2 + 1} \qquad NP := BM \quad AP := AN - NP \quad BP := \sqrt{AB^2 + AP^2}$$

$$EP := AP + AE \qquad HP := \frac{AP \cdot EP}{BP} \qquad BH := BP - HP \qquad AR := \frac{AP \cdot BH}{BP}$$

$$AR - \frac{-4AN^7 + 2AN^5 + 2AN^3}{4AN^6 + AN^4 + 2AN^2 + 1} = 0$$



$$AB := 1$$

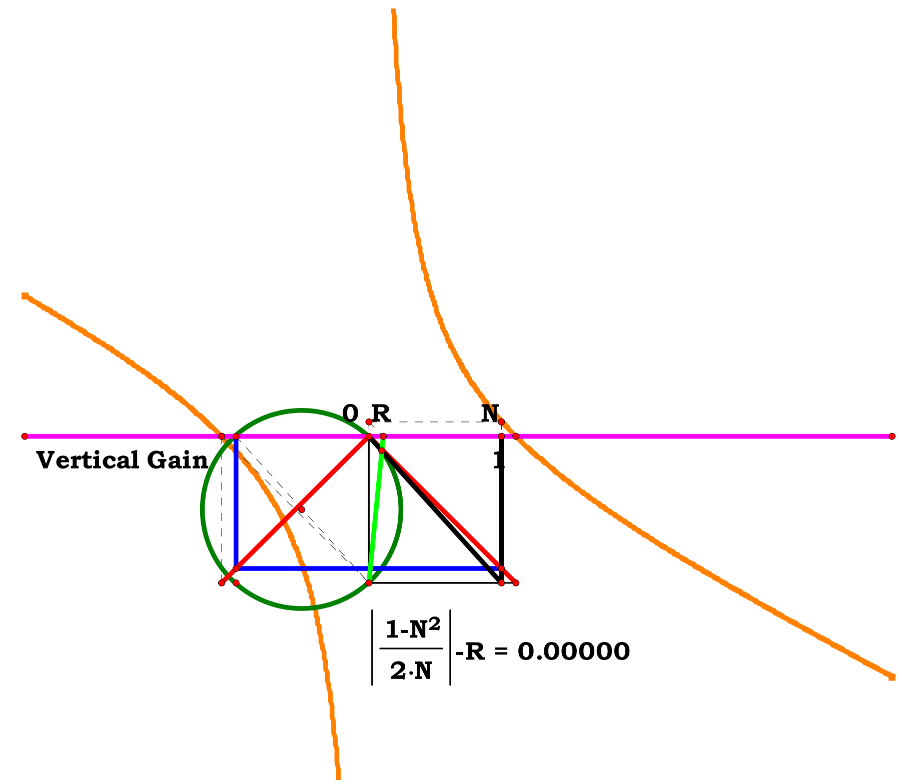
$$AN := .43444$$

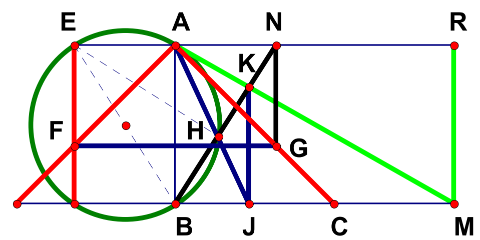
$$AE := \left|-AN\right|$$

$$AK := \sqrt{AN^2 + AB^2} \qquad BM := AE \qquad KM := AN + AE \qquad HM := \frac{AB \cdot KM}{AK}$$

$$MJ := \frac{AB \cdot HM}{AK} \qquad HJ := \frac{AN \cdot HM}{AK} \qquad BJ := MJ - BM \qquad AR := \frac{BJ \cdot AB}{HJ}$$

$$AR - \frac{1 - AN^2}{2AN} = 0$$



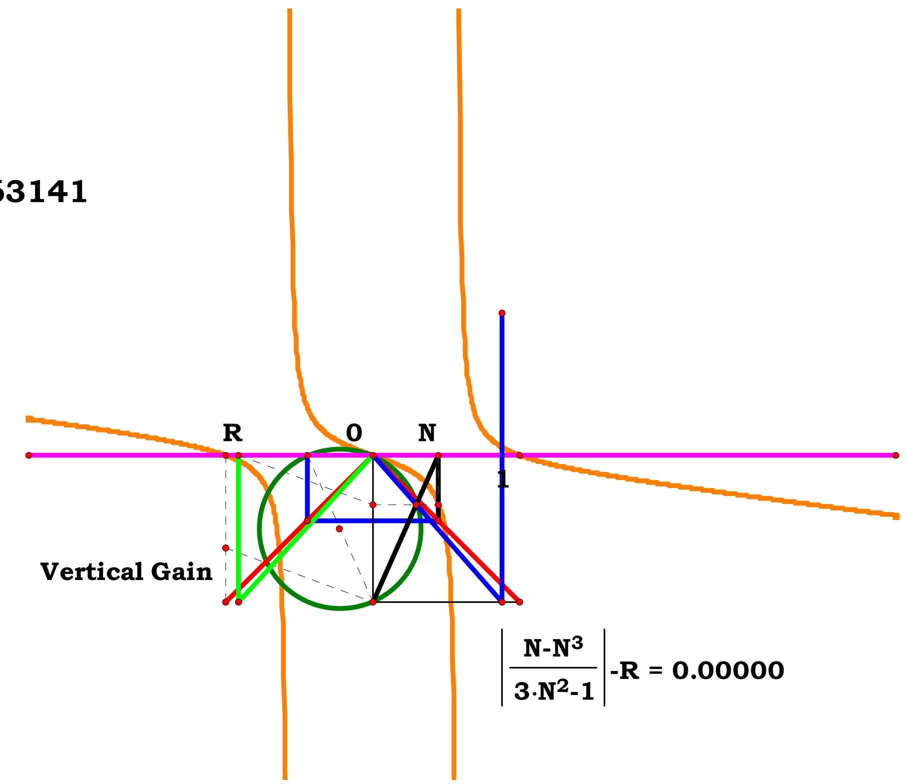


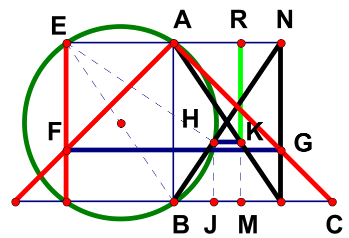
$$AB := 1$$

$$AN := .63141$$

$$BJ := \frac{1 - AN^2}{2AN} \qquad JK := \frac{AB \cdot BJ}{AN} \qquad BM := \frac{BJ \cdot AB}{AB - JK} \qquad AR := BM$$

$$AR - \frac{AN - AN^3}{3 \cdot AN^2 - 1} = 0$$

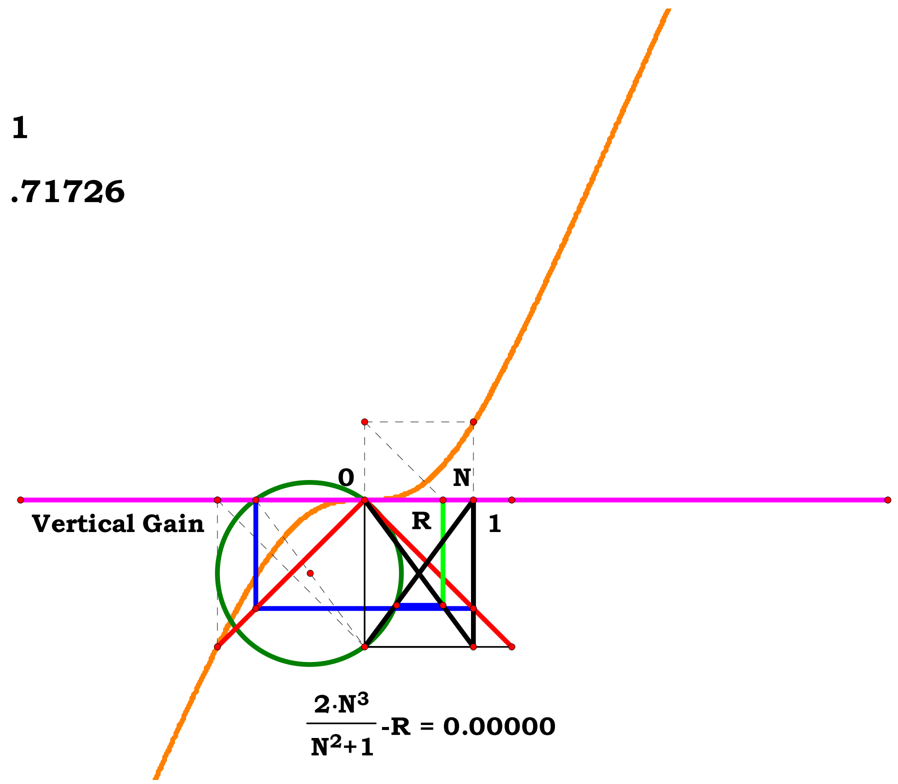




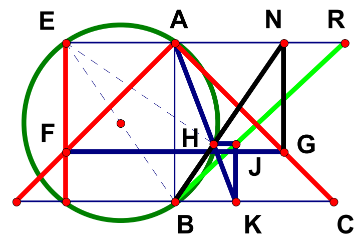
$$AB := 1$$

$$AN := .71726$$

$$BJ := \frac{AN - AN^3}{AN^2 + 1} \qquad AR := AN - BJ \qquad AR - \frac{2 \cdot AN^3}{AN^2 + 1} = 0$$



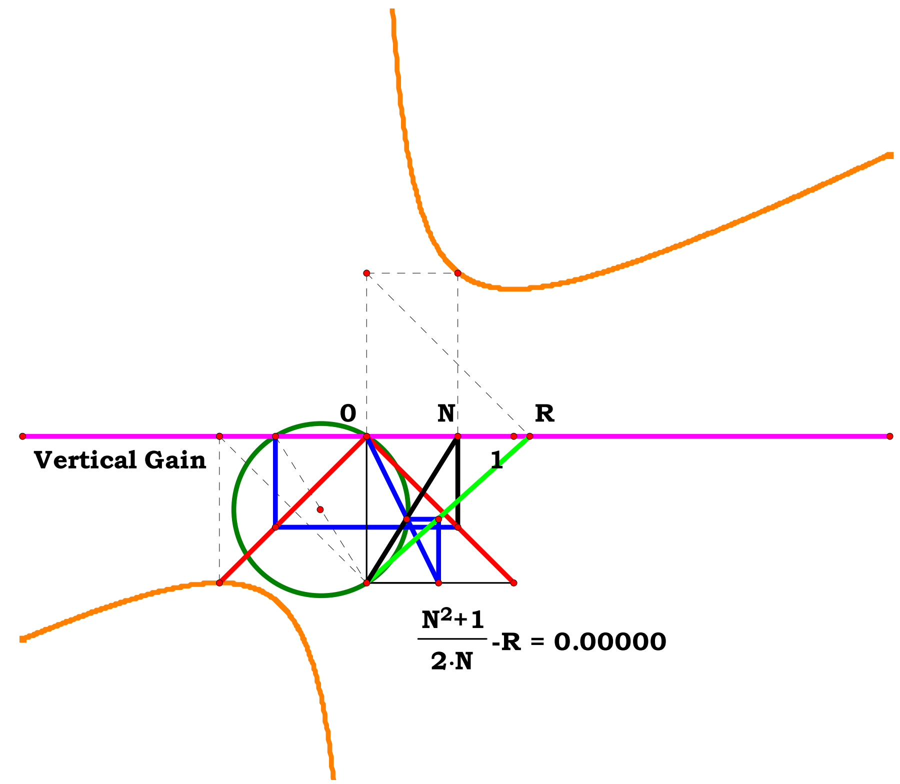
$$\frac{2 \cdot N^3}{N^2+1} - R = 0.00000$$

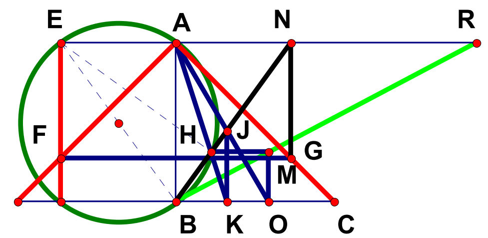


$$AB := 1$$

$$AN := .63646$$

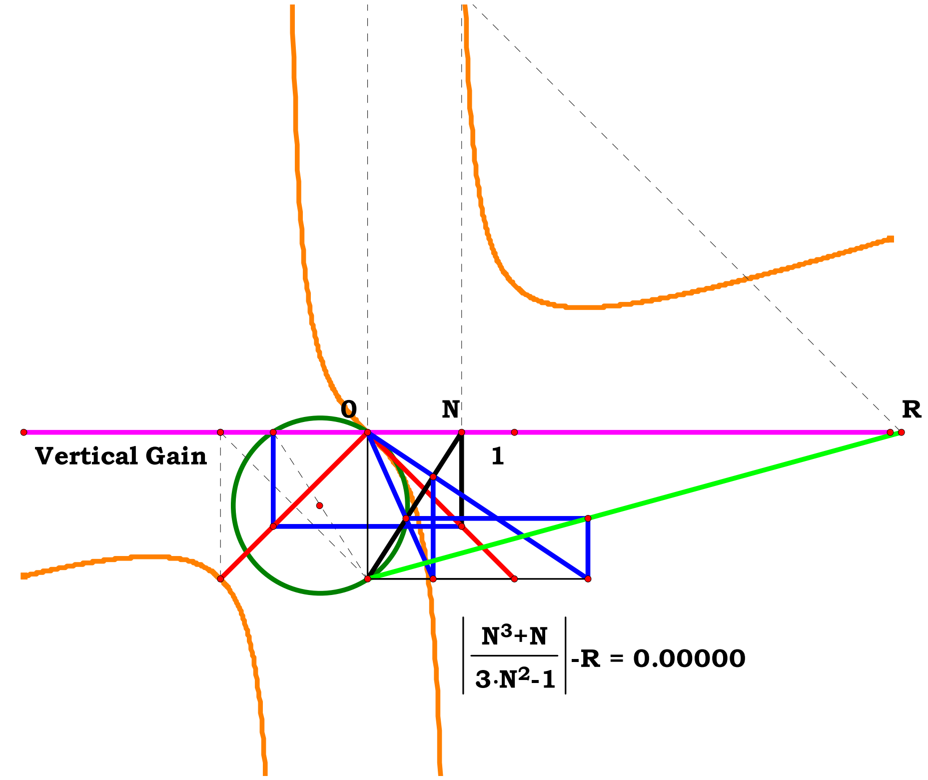
$$BK := \frac{1 - AN^2}{2AN} \qquad JK := \frac{1 - AN^2}{AN^2 + 1} \qquad AR := \frac{BK \cdot AB}{JK} \qquad AR - \frac{AN^2 + 1}{2 \cdot AN} = 0$$

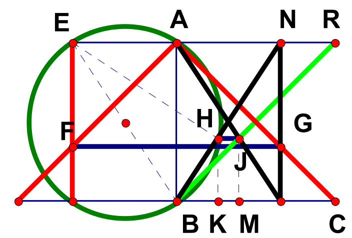




$$AB := 1 \qquad AN := 1.25645 \qquad BK := \frac{1 - AN^2}{2AN} \qquad MO := \frac{1 - AN^2}{AN^2 + 1}$$

$$JK := \frac{AB \cdot BK}{AN} \qquad BO := \frac{BK \cdot AB}{AB - JK} \qquad AR := \frac{BO \cdot AB}{MO} \qquad AR - \frac{AN^3 + AN}{3 \cdot AN^2 - 1} = 0$$



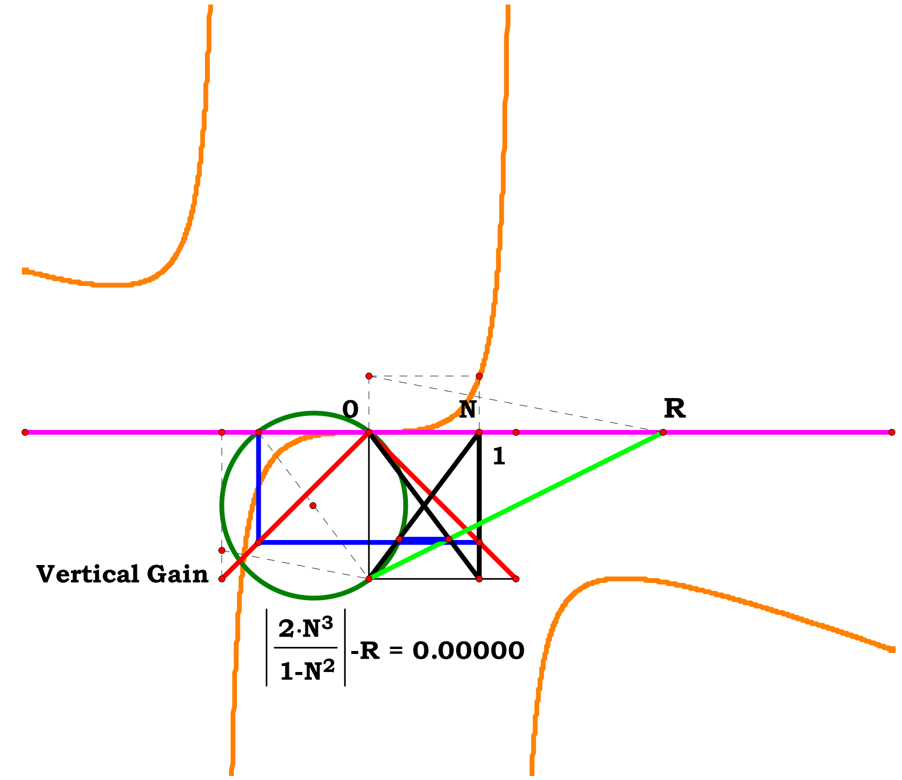


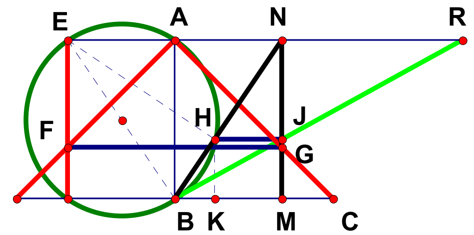
$$AB := 1$$

$$AN := .77282$$

$$BK := \frac{AN - AN^3}{AN^2 + 1} \qquad HK := \frac{1 - AN^2}{AN^2 + 1} \qquad BM := AN - BK \qquad AR := \frac{BM \cdot AB}{HK}$$

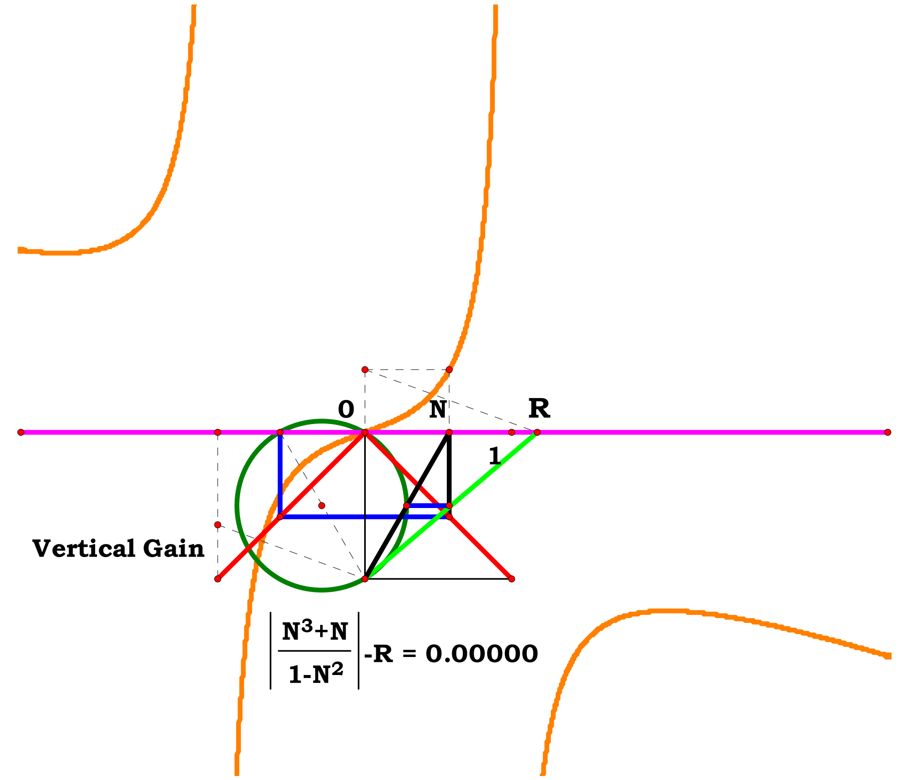
$$AR - \frac{2AN^3}{1 - AN^2} = 0$$

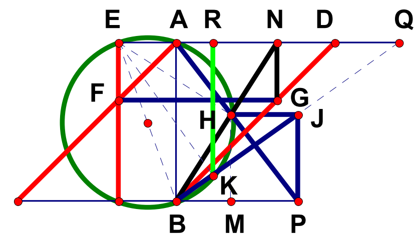




$$AB := 1 \quad AN := 3 \quad HK := \frac{1 - AN^2}{AN^2 + 1} \quad AR := \frac{AN \cdot AB}{HK}$$

$$AR - \frac{AN^3 + AN}{1 - AN^2} = 0$$





$AB := 1$

$AN := .74847$

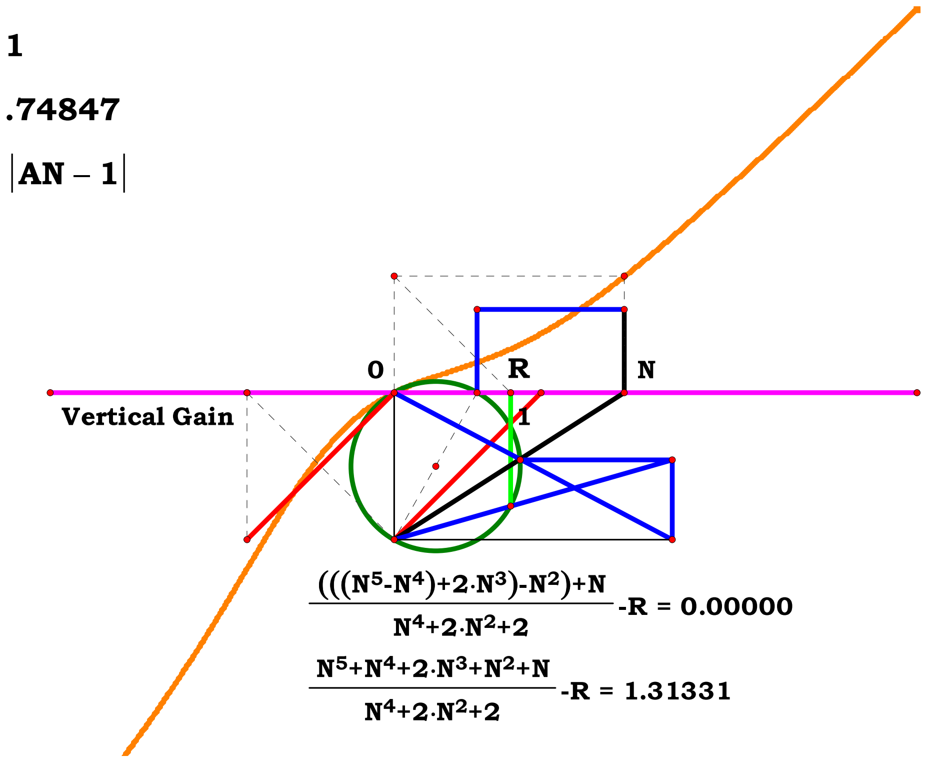
$AE := |AN - 1|$

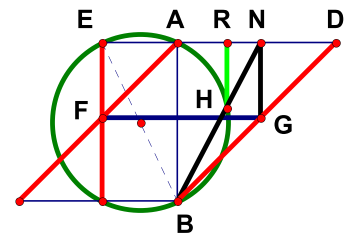
$EN := AN + AE \quad BN := \sqrt{AN^2 + AB^2} \quad HN := \frac{AN \cdot EN}{BN} \quad BH := BN - HN$

$HM := \frac{AB \cdot BH}{BN} \quad BP := \frac{BN \cdot BH}{EN} \quad JP := HM \quad AQ := \frac{BP \cdot AB}{JP}$

$BQ := \sqrt{AQ^2 + AB^2} \quad EQ := AQ + AE \quad KQ := \frac{AQ \cdot EQ}{BQ} \quad BK := BQ - KQ$

$AR := \frac{AQ \cdot BK}{BQ} \quad AR - \frac{AN^5 - AN^4 + 2 \cdot AN^3 - AN^2 + AN}{AN^4 + 2 \cdot AN^2 + 2} = 0$





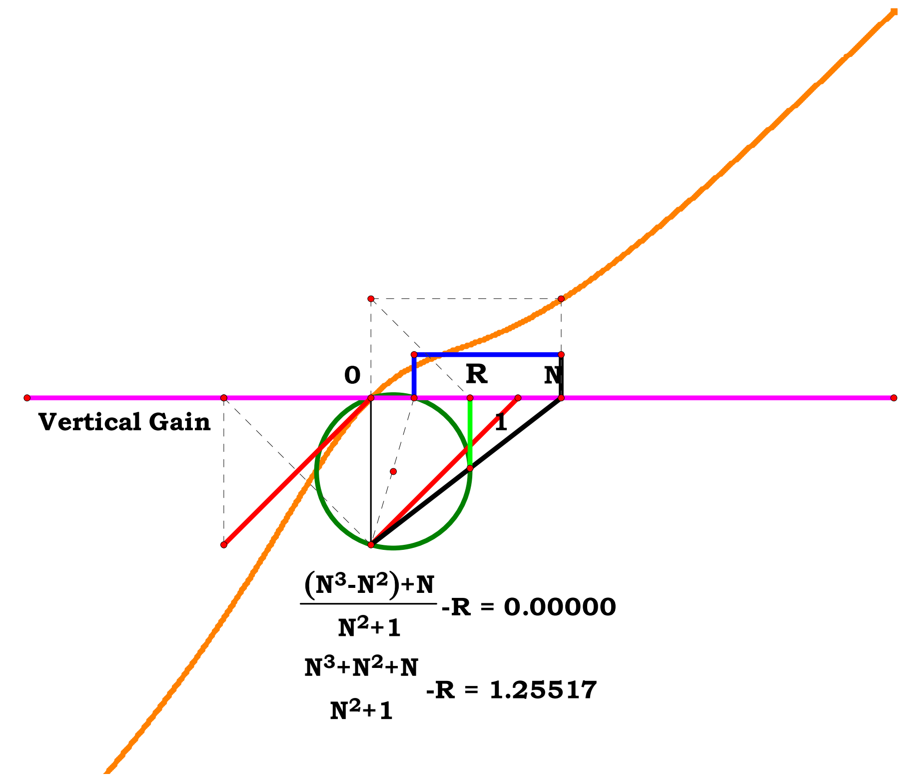
$$AB := 1$$

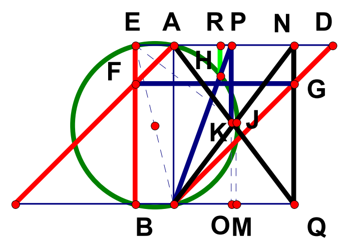
$$AN := .64930$$

$$AE := 1 - AN$$

$$EN := AN + AE \qquad BN := \sqrt{AN^2 + AB^2} \qquad HN := \frac{AN \cdot EN}{BN} \qquad RN := \frac{AN \cdot HN}{BN}$$

$$AR := AN - RN \qquad AR - \frac{AN^3 - AN^2 + AN}{AN^2 + 1} = 0$$

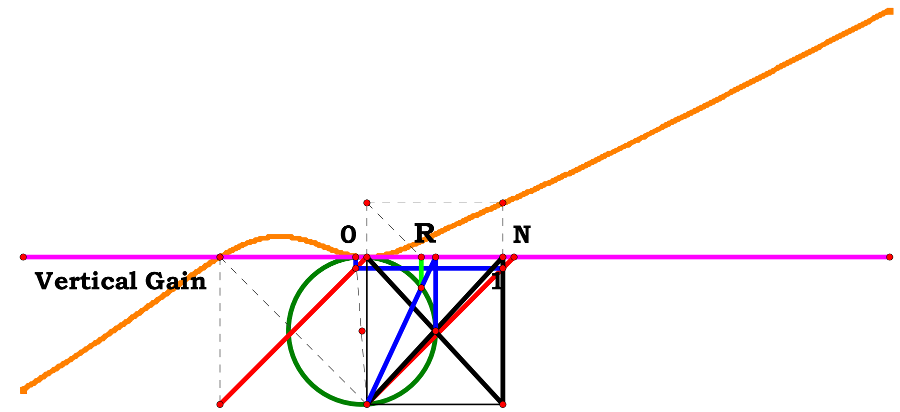




$$AB := 1$$

$$AN := .63005$$

$$AE := |AN - 1|$$



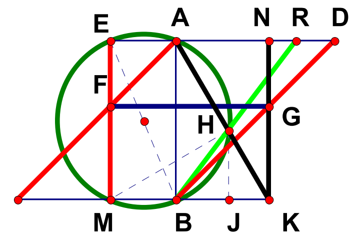
$$\frac{N^5+N^2}{2\cdot N^4+2\cdot N^2+1}-R = 0.00000$$

$$\frac{N^5-N^2}{2\cdot N^4+2\cdot N^2+1}-R = -0.40948$$

$$BM := \frac{AN^3 - AN^2 + AN}{AN^2 + 1} \quad NP := BM \quad AP := AN - NP \quad BP := \sqrt{AB^2 + AP^2}$$

$$EP := AP + AE \quad HP := \frac{AP \cdot EP}{BP} \quad BH := BP - HP \quad AR := \frac{AP \cdot BH}{BP}$$

$$AR - \frac{AN^5 + AN^2}{2AN^4 + 2AN^2 + 1} = 0$$



$AB := 1$

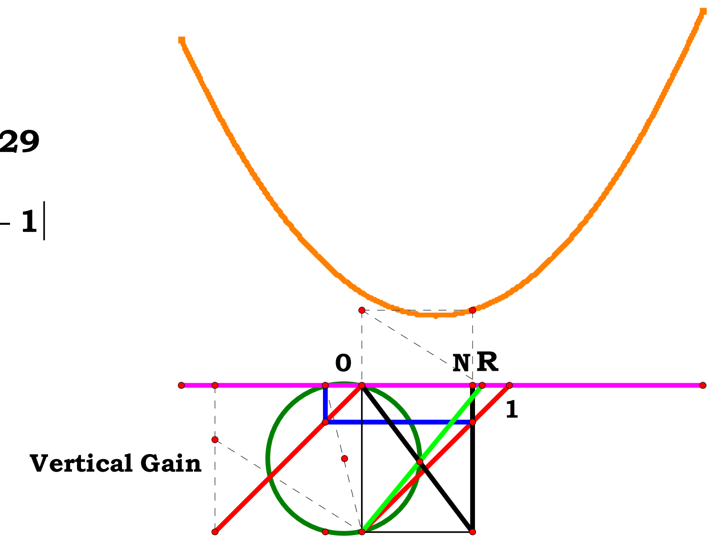
$AN := .64429$

$AE := |AN - 1|$

$AK := \sqrt{AN^2 + AB^2}$ $\quad BM := AE$ $\quad KM := AN + AE$ $\quad HM := \dfrac{AB \cdot KM}{AK}$

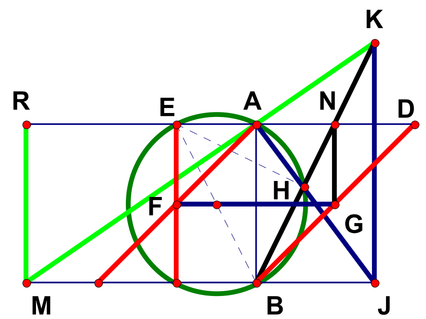
$MJ := \dfrac{AB \cdot HM}{AK}$ $\quad HJ := \dfrac{AN \cdot HM}{AK}$ $\quad BJ := MJ - BM$ $\quad AR := \dfrac{BJ \cdot AB}{HJ}$

$AR - \left(AN^2 - AN + 1\right) = 0$



$((N^2-N)+1)-R = 0.00000$

$(N^2+N+1)-R = 1.50120$

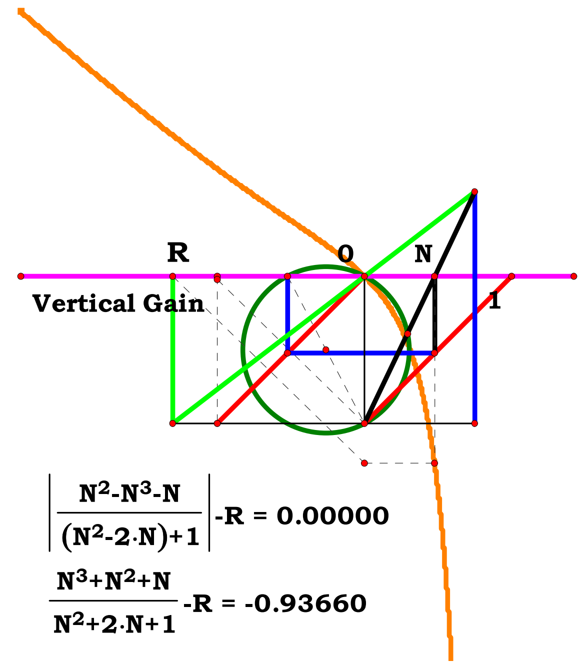


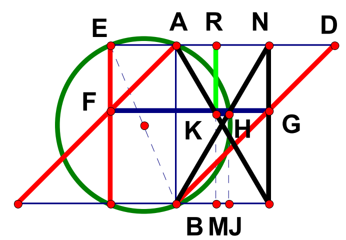
$$AB := 1$$

$$AN := .3957$$

$$BJ := AN^2 - AN + 1 \qquad JK := \frac{AB \cdot BJ}{AN} \qquad BM := \frac{BJ \cdot AB}{AB - JK} \qquad AR := BM$$

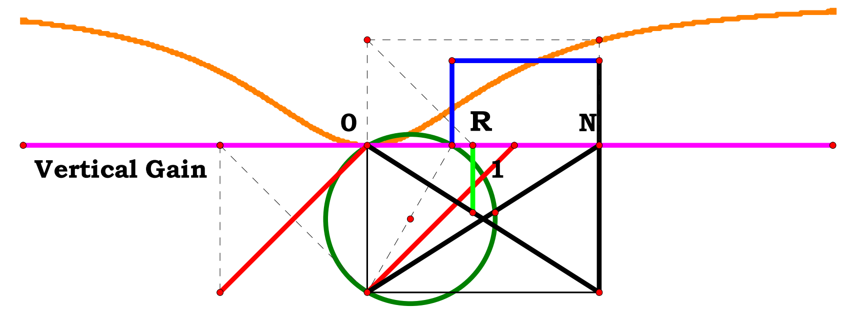
$$AR - \frac{AN^2 - AN^3 - AN}{AN^2 - 2AN + 1} = 0$$





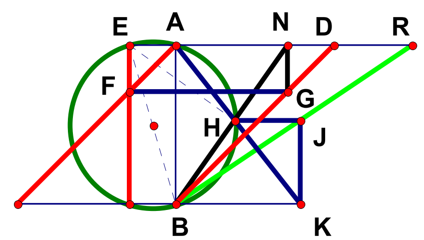
$$AB := 1$$

$$AN := .51896$$



$$\frac{N^2}{N^2+1} - R = 0.00000$$

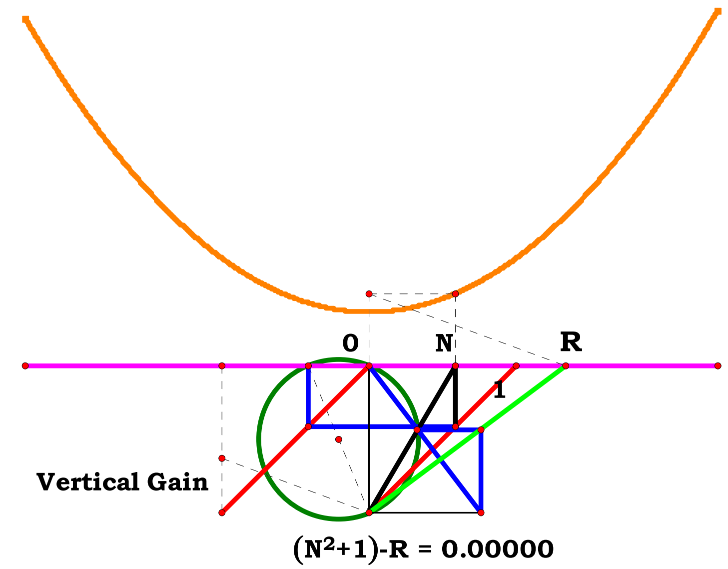
$$BJ := \frac{AN^3 - AN^2 + AN}{AN^2 + 1} \qquad AR := AN - BJ \qquad AR - \frac{AN^2}{AN^2 + 1} = 0$$

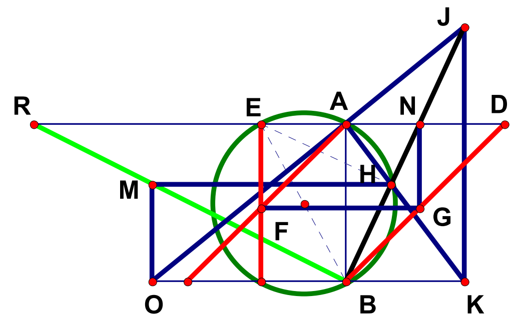


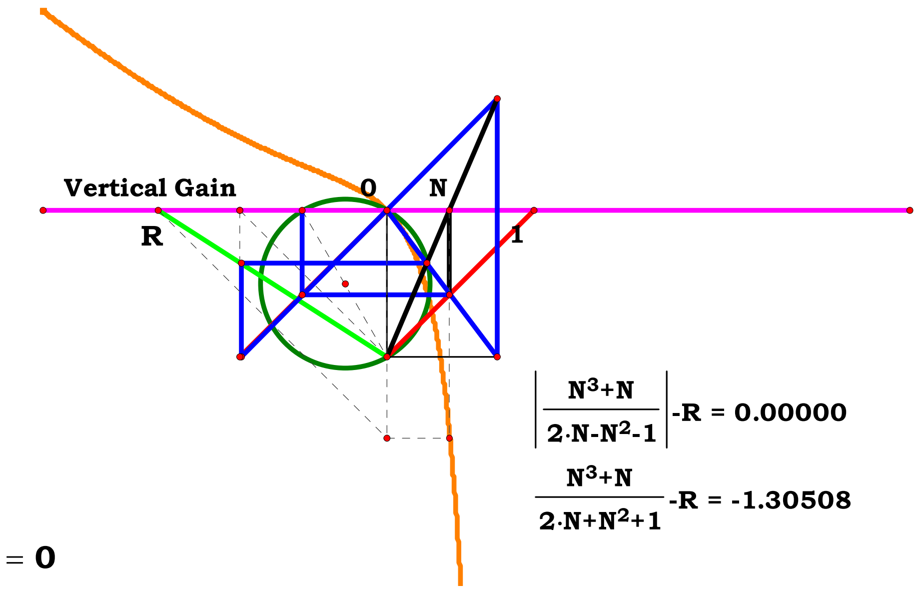
$AB := 1$

$AN := .61225$

$BK := AN^2 - AN + 1 \qquad JK := \dfrac{AN^2 - AN + 1}{AN^2 + 1} \qquad AR := \dfrac{BK \cdot AB}{JK} \qquad AR - \left(AN^2 + 1\right) = 0$





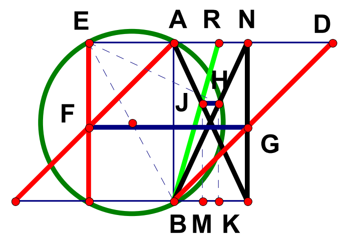


$$\left|\frac{N^3+N}{2\cdot N-N^2-1}\right|-R = 0.00000$$

$$\frac{N^3+N}{2\cdot N+N^2+1}-R = -1.30508$$

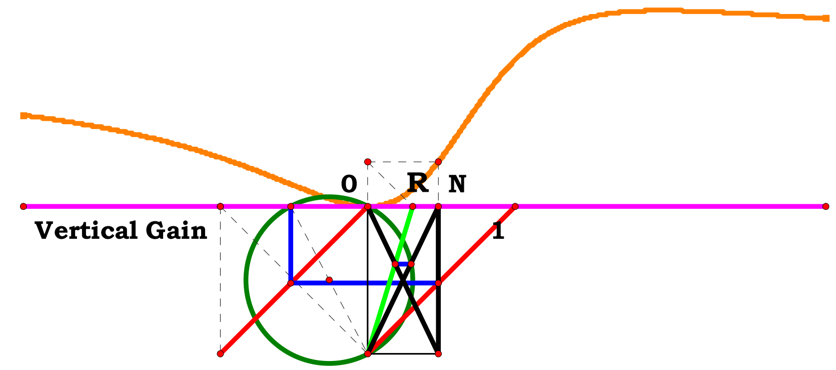
$AB := 1$ $\quad AN := .38816$ $\quad BK := AN^2 - AN + 1$ $\quad MO := \dfrac{AN^2 - AN + 1}{AN^2 + 1}$

$JK := \dfrac{AB \cdot BK}{AN}$ $\quad BO := \dfrac{BK \cdot AB}{AB - JK}$ $\quad AR := \dfrac{BO \cdot AB}{MO}$ $\quad AR - \dfrac{AN^3 + AN}{2 \cdot AN - AN^2 - 1} = 0$



$$AB := 1$$

$$AN := .48387$$

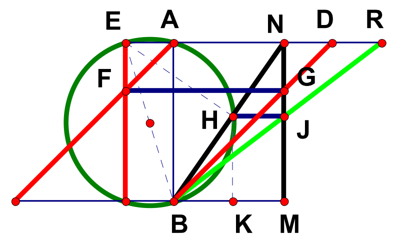


$$\frac{N^2}{(N^2-N)+1} - R = 0.00000$$

$$\frac{N^2}{N^2+N+1} - R = -0.16980$$

$$BK := \frac{AN^3 - AN^2 + AN}{AN^2 + 1} \qquad HK := \frac{AN^2 - AN + 1}{AN^2 + 1}$$

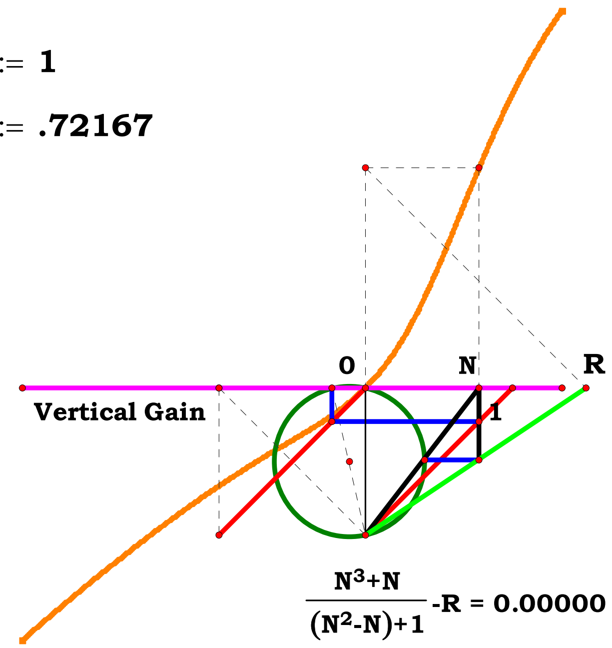
$$BM := AN - BK \qquad AR := \frac{BM \cdot AB}{HK} \qquad AR - \frac{AN^2}{AN^2 - AN + 1} = 0$$



$AB := 1$

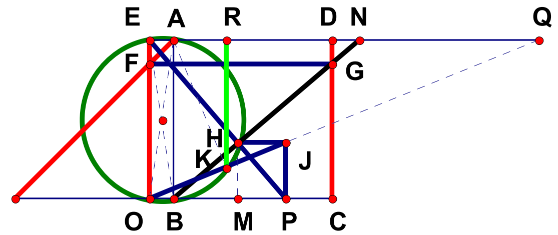
$AN := .72167$

$$HK := \frac{AN^2 - AN + 1}{AN^2 + 1} \qquad AR := \frac{AN \cdot AB}{HK} \qquad AR - \frac{AN^3 + AN}{AN^2 - AN + 1} = 0$$



$$\frac{N^3+N}{(N^2-N)+1} - R = 0.00000$$

$$\frac{N^3+N}{N^2+N+1} - R = -0.98196$$



$AB := 1$

$AN := 1.23239$

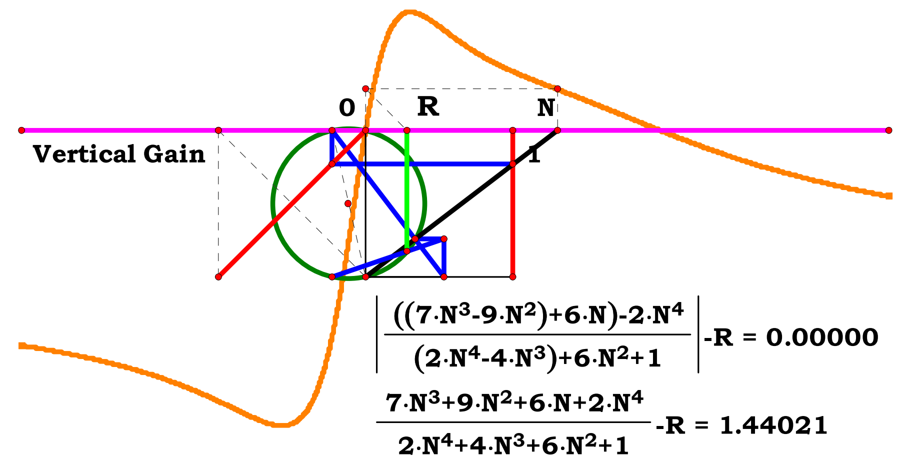
$AE := \left| \frac{1 - AN}{AN} \right|$

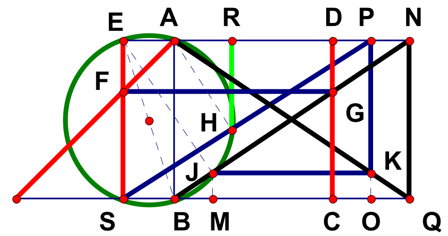
$EN := AN + AE \quad BN := \sqrt{AN^2 + AB^2} \quad HN := \frac{AN \cdot EN}{BN} \quad BH := BN - HN$

$HM := \frac{AB \cdot BH}{BN} \quad JP := HM \quad OP := \frac{AB^2}{AN} \quad EQ := \frac{OP \cdot AB}{JP} \quad AQ := EQ - AE$

$OQ := \sqrt{EQ^2 + AB^2} \quad KQ := \frac{EQ \cdot AQ}{OQ} \quad KO := OQ - KQ \quad ER := \frac{EQ \cdot KO}{OQ}$

$AR := ER - AE \qquad AR - \frac{7 \cdot AN^3 - 9 \cdot AN^2 + 6 \cdot AN - 2 \cdot AN^4}{2 \cdot AN^4 - 4 \cdot AN^3 + 6 \cdot AN^2 + 1} = 0$

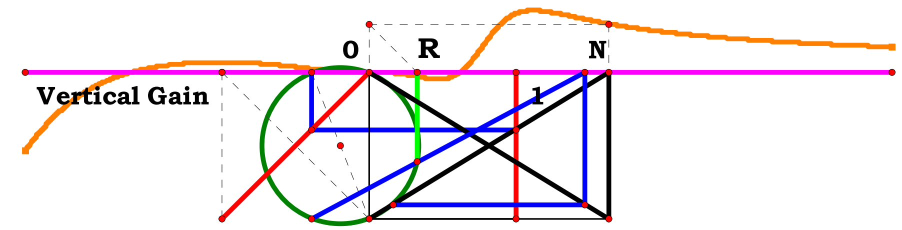




$$AB := 1$$

$$AN := 1.52113$$

$$AE := \left|\frac{1 - AN}{AN}\right|$$

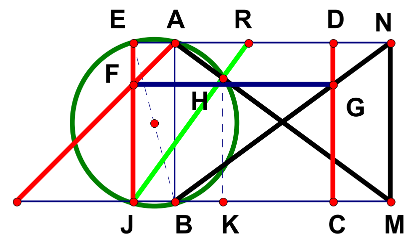


$$\left|\frac{(N^7+N^6+N^4)-N^3}{((((((N^8+4\cdot N^7+N^6)-6\cdot N^5)+8\cdot N^4)-8\cdot N^3)+6\cdot N^2)-2\cdot N)+1}\right| - R = 0.00000$$

$$BM := \frac{2AN - AN^2}{AN^2 + 1} \qquad PN := BM \qquad AP := AN - PN \qquad EP := AP + AE$$

$$PS := \sqrt{EP^2 + AB^2} \qquad HP := \frac{EP \cdot AP}{PS} \qquad RP := \frac{EP \cdot HP}{PS} \qquad AR := AP - RP$$

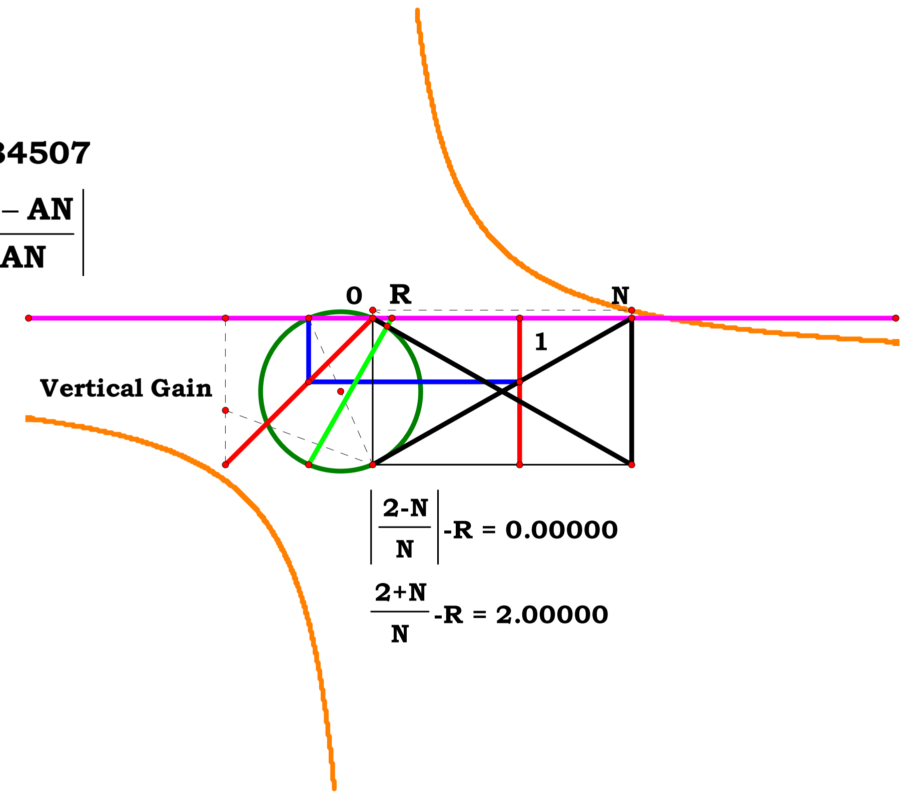
$$AR - \frac{AN^7 + AN^6 + AN^4 - AN^3}{AN^8 + 4\cdot AN^7 + AN^6 - 6\cdot AN^5 + 8\cdot AN^4 - 8\cdot AN^3 + 6\cdot AN^2 - 2\cdot AN + 1} = 0$$



$$AB := 1$$

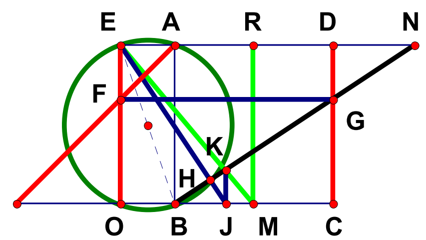
$$AN := 1.34507$$

$$AE := \left|\frac{1 - AN}{AN}\right|$$



$$AM := \sqrt{AN^2 + AB^2} \qquad JM := AN + AE \qquad HJ := \frac{AB \cdot JM}{AM} \qquad HK := \frac{AN \cdot HJ}{AM}$$

$$JK := \frac{AB \cdot HJ}{AM} \qquad ER := \frac{JK \cdot AB}{HK} \qquad AR := ER - AE \qquad AR - \frac{2 - AN}{AN} = 0$$



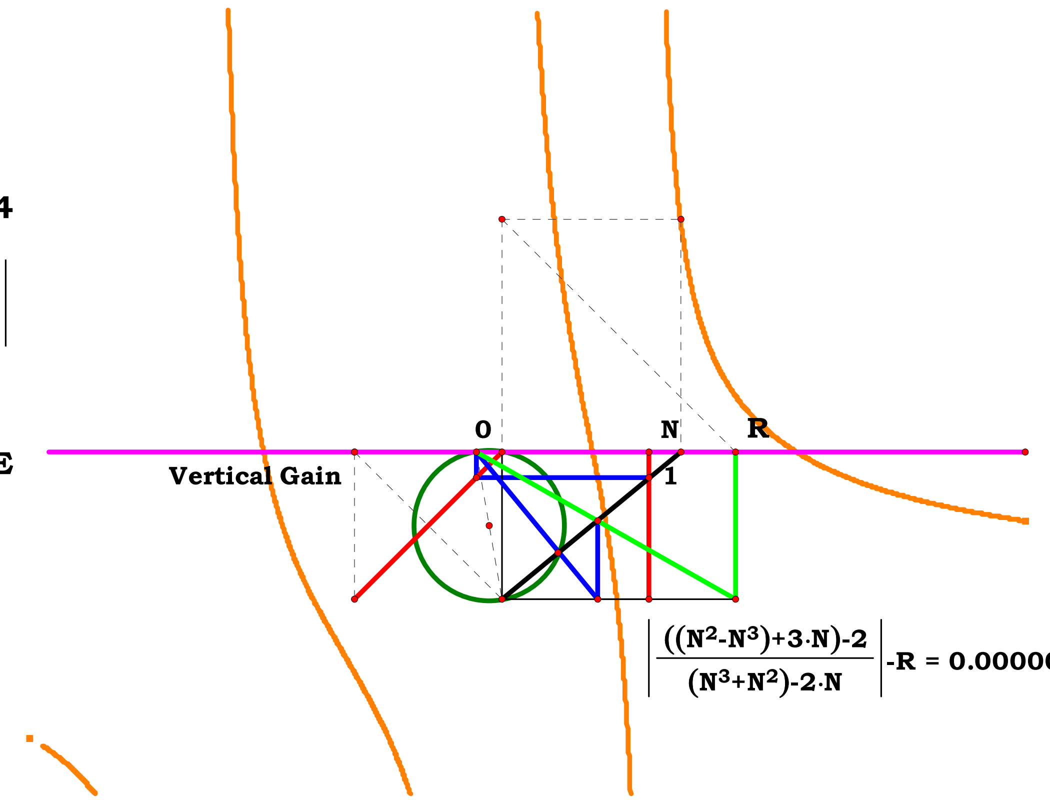
$$AB := 1$$

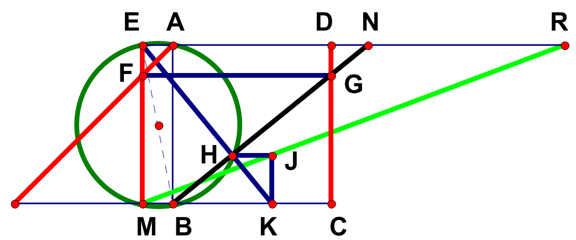
$$AN := 1.55634$$

$$AE := \left| \frac{1 - AN}{AN} \right|$$

$$BJ := \frac{2 - AN}{AN} \quad KJ := \frac{AB \cdot BJ}{AN} \quad JO := \frac{AB^2}{AN} \quad OM := \frac{JO \cdot AB}{AB - KJ} \quad AR := OM - AE$$

$$AR - \frac{AN^2 - AN^3 + 3 \cdot AN - 2}{AN^3 + AN^2 - 2 \cdot AN} = 0$$





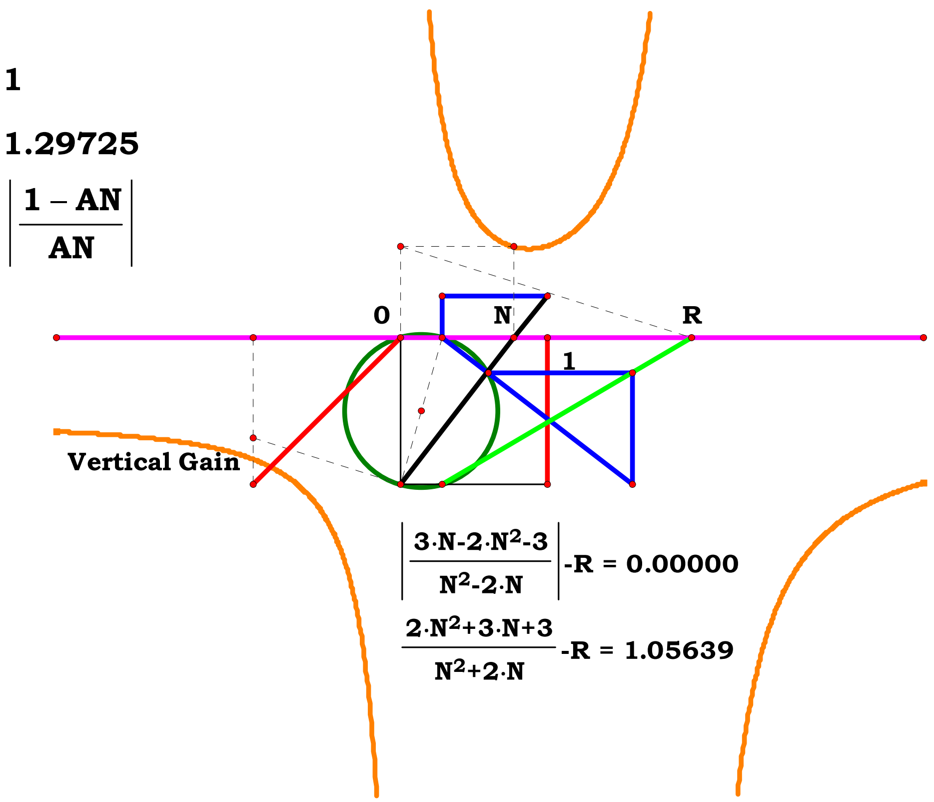
$$AB := 1$$

$$AN := 1.29725$$

$$AE := \left|\frac{1 - AN}{AN}\right|$$

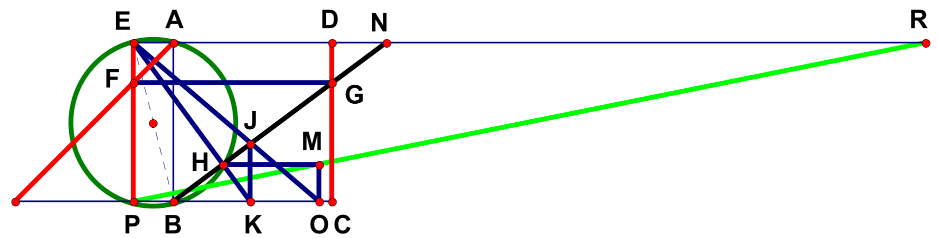
$$JK := \frac{2 - AN}{AN^2 + 1} \qquad MK := \frac{AB^2}{AN} \qquad ER := \frac{MK \cdot AB}{JK} \qquad AR := ER - AE$$

$$AR - \frac{3 \cdot AN - 2 \cdot AN^2 - 3}{AN^2 - 2 \cdot AN} = 0$$



$$\left|\frac{3 \cdot N - 2 \cdot N^2 - 3}{N^2 - 2 \cdot N}\right| - R = 0.00000$$

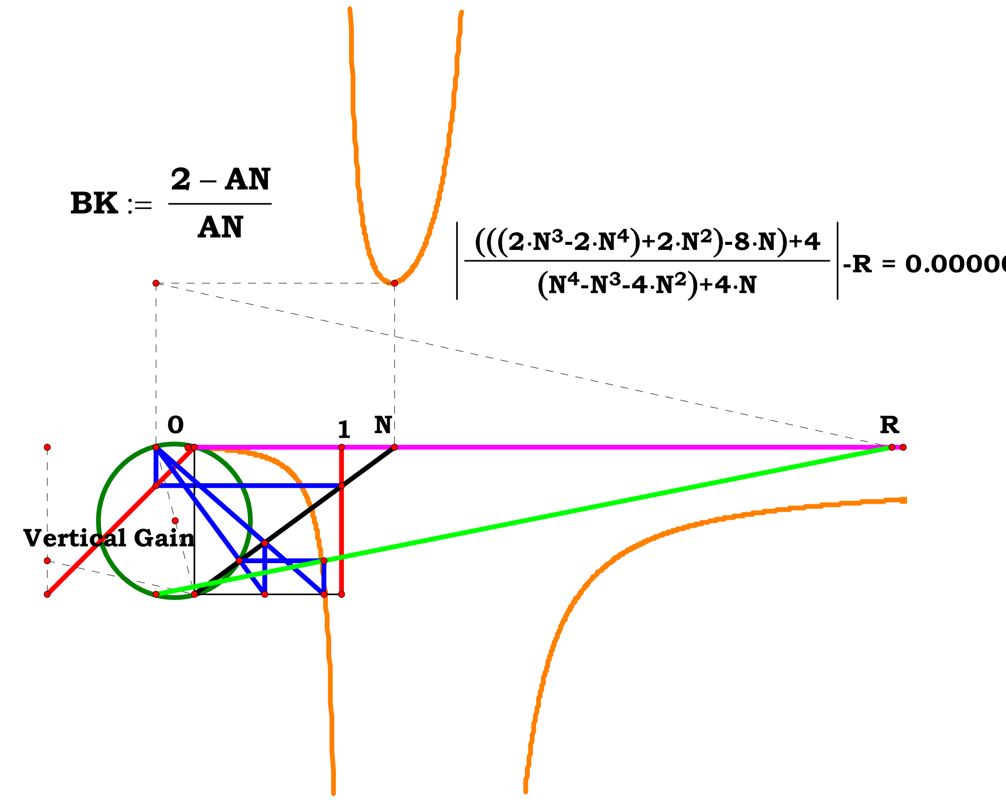
$$\frac{2 \cdot N^2 + 3 \cdot N + 3}{N^2 + 2 \cdot N} - R = 1.05639$$

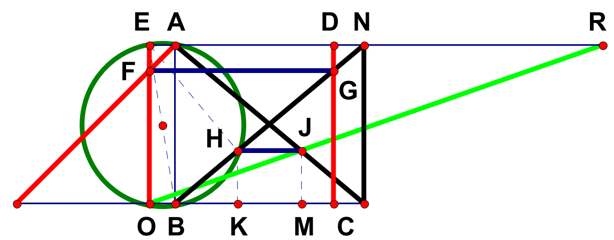


$$AB := 1 \quad AN := 1.34825 \quad AE := \left|\frac{1-AN}{AN}\right| \quad PK := \frac{AB^2}{AN} \quad MO := \frac{2-AN}{AN^2+1} \quad BK := \frac{2-AN}{AN}$$

$$JK := \frac{AB \cdot BK}{AN} \quad PO := \frac{PK \cdot AB}{AB-JK} \quad ER := \frac{PO \cdot AB}{MO} \quad AR := ER - AE$$

$$AR - \frac{2AN^3 - 2AN^4 + 2AN^2 - 8AN + 4}{AN^4 - AN^3 - 4AN^2 + 4AN} = 0$$

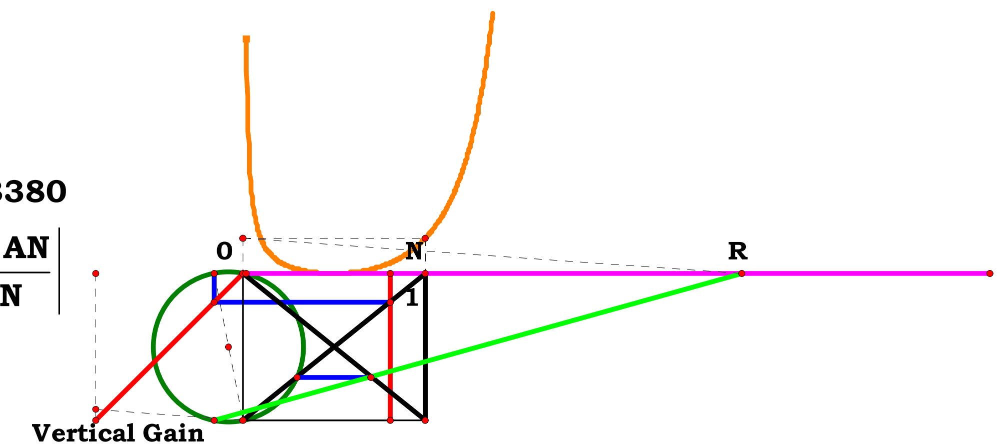




$$AB := 1$$

$$AN := 1.13380$$

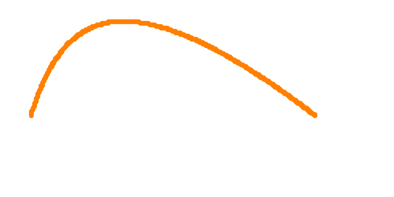
$$AE := \left|\frac{1 - AN}{AN}\right|$$

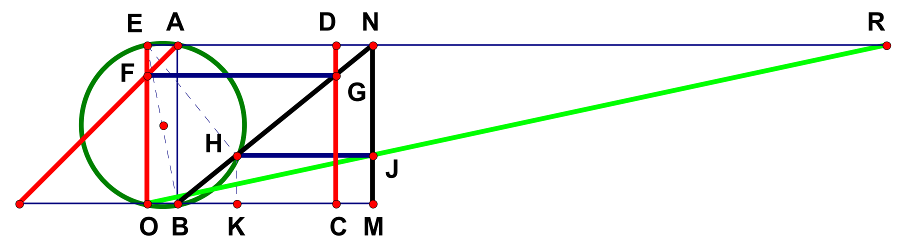


$$\left|\frac{((N^4+2\cdot N^3)-N^2-2\cdot N)+1}{2\cdot N-N^2}\right|-R = 0.00000$$

$$BK := \frac{2AN - AN^2}{AN^2 + 1} \qquad HK := \frac{2 - AN}{AN^2 + 1} \qquad MO := AE + AN - BK$$

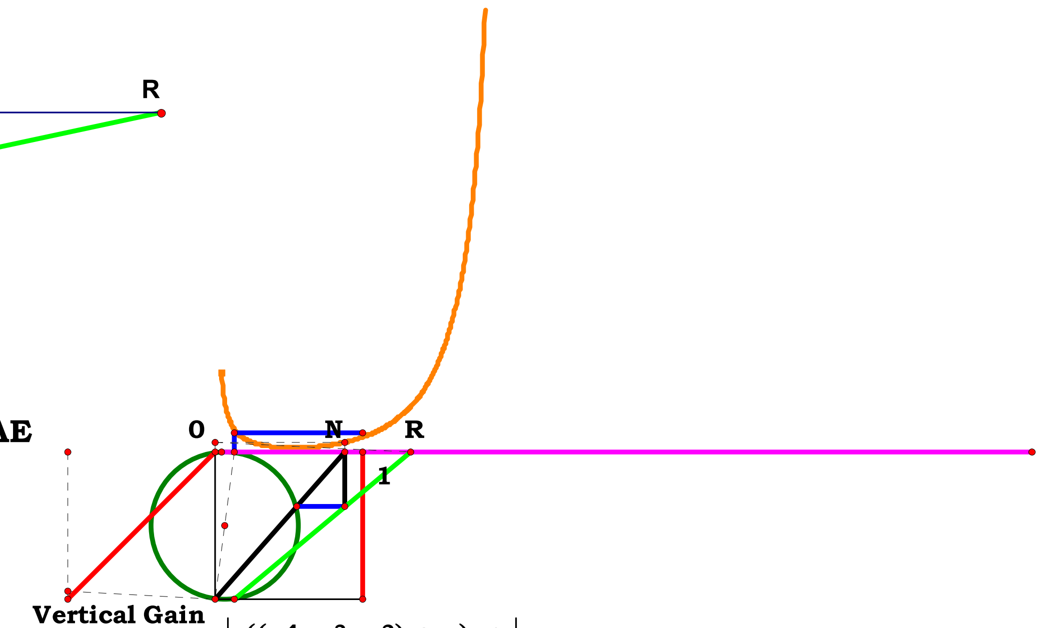
$$ER := \frac{MO \cdot AB}{HK} \qquad AR := ER - AE \qquad AR - \frac{AN^4 + 2AN^3 - AN^2 - 2AN + 1}{2AN - AN^2} = 0$$



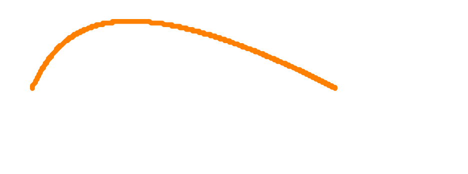


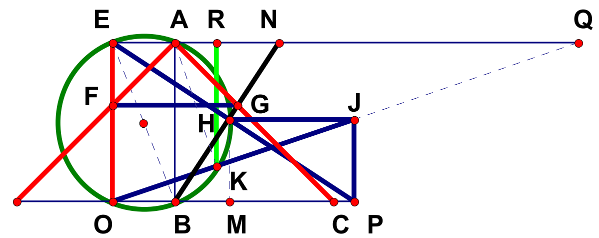
$$AB := 1 \quad AN := 1.14982 \quad AE := \left|\frac{1-AN}{AN}\right| \quad HK := \frac{2-AN}{AN^2+1} \quad MO := AN + AE$$

$$AR := \frac{MO \cdot AB}{HK} - AE \quad AR - \frac{AN^4 + AN^3 + AN^2 - 2 \cdot AN + 1}{2AN - AN^2} = 0$$



$$\left|\frac{((N^4+N^3+N^2)-2\cdot N)+1}{2\cdot N-N^2}\right| \text{-R} = 0.00000$$

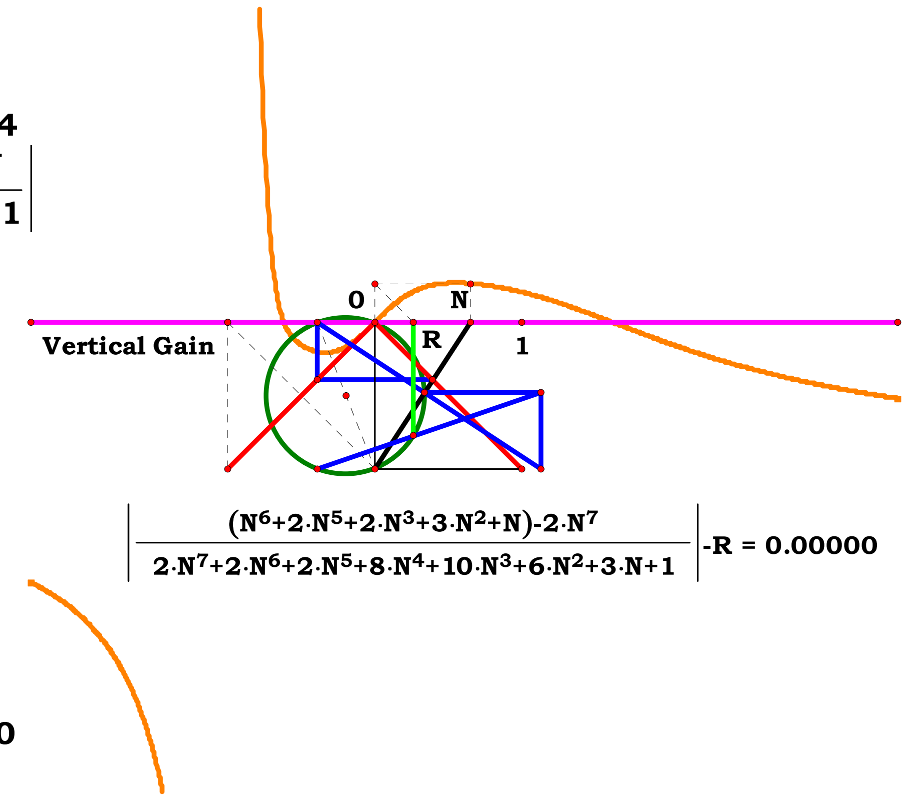




$AB := 1$

$AN := .98404$

$AE := \left|\frac{-AN}{AN+1}\right|$

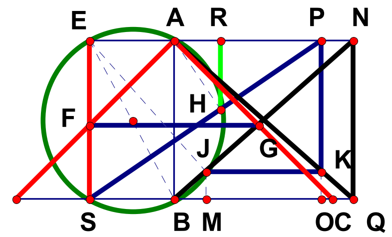


$EN := AN + AE \quad BN := \sqrt{AN^2 + AB^2} \quad HN := \frac{AN \cdot EN}{BN} \quad BH := BN - HN$

$HM := \frac{AB \cdot BH}{BN} \quad JP := HM \quad OP := \frac{AB^2}{AN} \quad EQ := \frac{OP \cdot AB}{JP} \quad AQ := EQ - AE$

$OQ := \sqrt{EQ^2 + AB^2} \quad KQ := \frac{EQ \cdot AQ}{OQ} \quad KO := OQ - KQ \quad ER := \frac{EQ \cdot KO}{OQ}$

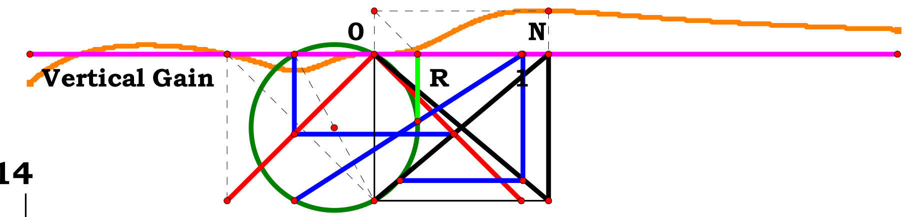
$AR := ER - AE \quad AR - \frac{AN^6 + 2\cdot AN^5 + 2\cdot AN^3 + 3\cdot AN^2 + AN - 2\cdot AN^7}{2\cdot AN^7 + 2\cdot AN^6 + 2\cdot AN^5 + 8\cdot AN^4 + 10\cdot AN^3 + 6\cdot AN^2 + 3\cdot AN + 1} = 0$



$$AB := 1$$

$$AN := 1.15014$$

$$AE := \left|\frac{-AN}{AN+1}\right|$$

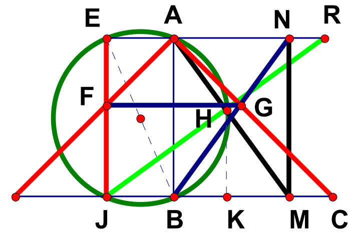


$$\frac{N^7+3\cdot N^6+3\cdot N^5+3\cdot N^4+2\cdot N^3}{N^8+6\cdot N^7+10\cdot N^6+4\cdot N^5+9\cdot N^4+4\cdot N^3+4\cdot N^2+2\cdot N+1}-R = 0.00000$$

$$BM := \frac{AN^2 - AN^3 + AN}{AN^3 + AN^2 + AN + 1} \quad PN := BM \quad AP := AN - PN \quad EP := AP + AE$$

$$PS := \sqrt{EP^2 + AB^2} \quad HP := \frac{EP \cdot AP}{PS} \quad RP := \frac{EP \cdot HP}{PS} \quad AR := AP - RP$$

$$AR - \frac{AN^7 + 3\cdot AN^6 + 3\cdot AN^5 + 3\cdot AN^4 + 2\cdot AN^3}{AN^8 + 6\cdot AN^7 + 10\cdot AN^6 + 4\cdot AN^5 + 9\cdot AN^4 + 4\cdot AN^3 + 4\cdot AN^2 + 2\cdot AN + 1} = 0$$



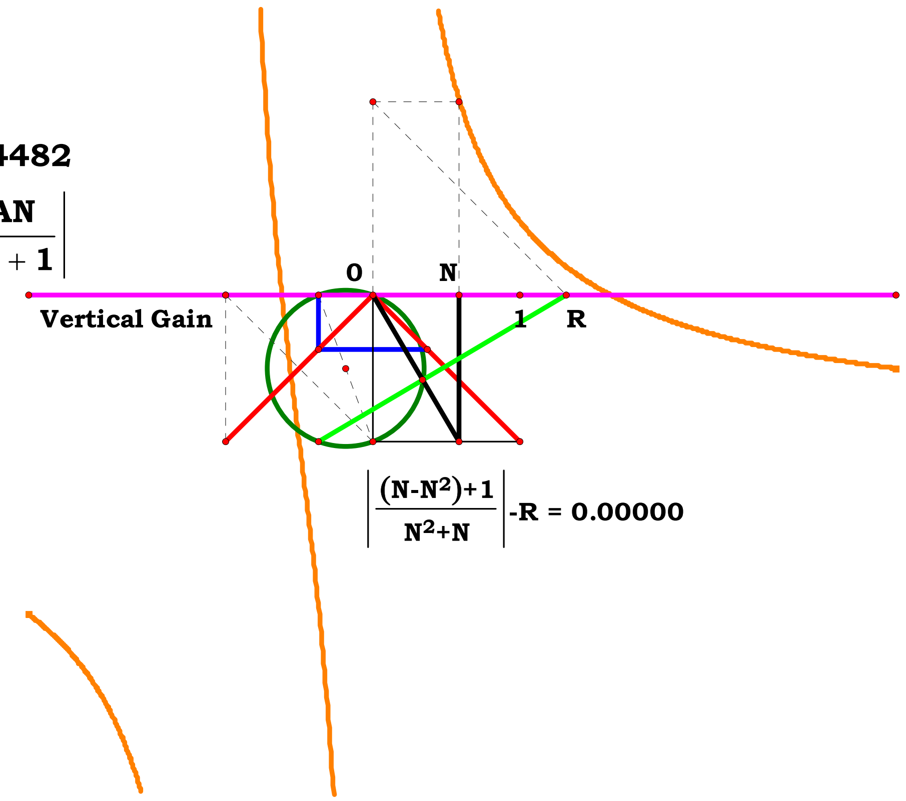
$$AB := 1$$

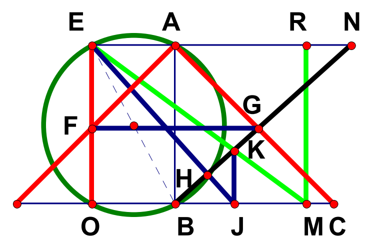
$$AN := 2.64482$$

$$AE := \left|\frac{-AN}{AN+1}\right|$$

$$AM := \sqrt{AN^2 + AB^2} \quad JM := AN + AE \quad HJ := \frac{AB \cdot JM}{AM} \quad HK := \frac{AN \cdot HJ}{AM}$$

$$JK := \frac{AB \cdot HJ}{AM} \quad ER := \frac{JK \cdot AB}{HK} \quad AR := ER - AE \quad AR - \frac{AN - AN^2 + 1}{AN^2 + AN} = 0$$





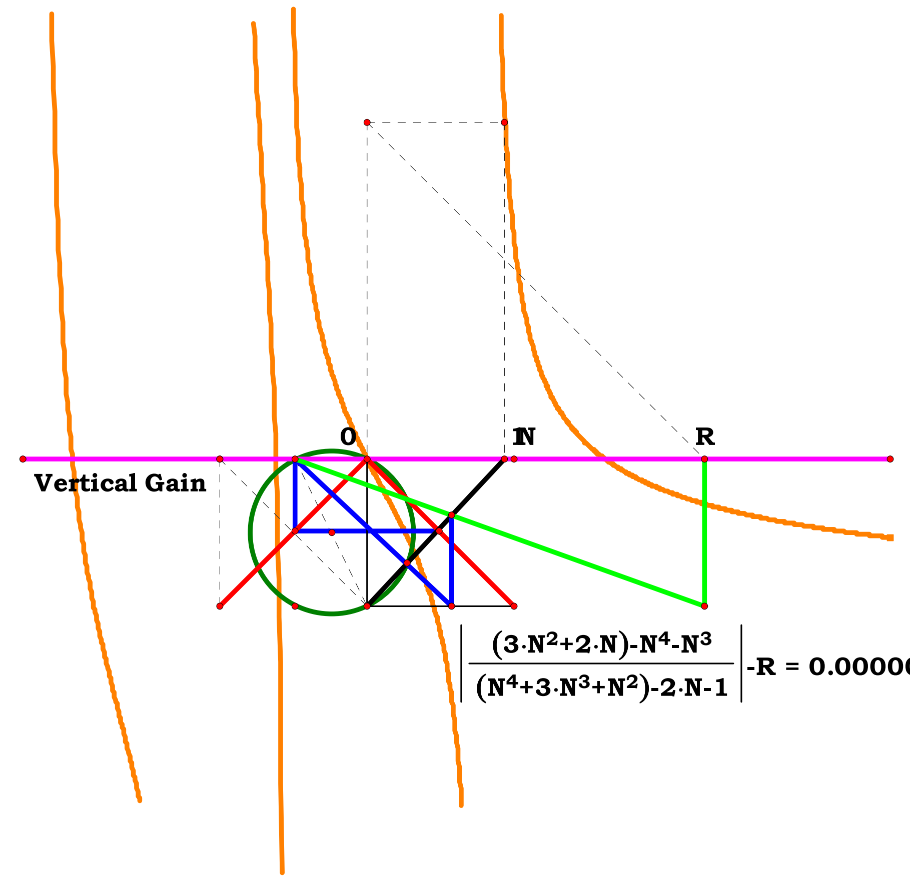
$$AB := 1$$

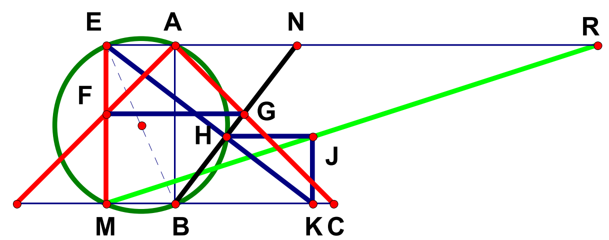
$$AN := 1.2234$$

$$AE := \left| \frac{-AN}{AN + 1} \right|$$

$$BJ := \frac{AN - AN^2 + 1}{AN^2 + AN} \qquad KJ := \frac{AB \cdot BJ}{AN} \quad JO := \frac{AB^2}{AN} \qquad OM := \frac{JO \cdot AB}{AB - KJ}$$

$$AR := OM - AE \qquad AR - \frac{3 \cdot AN^2 - AN^4 - AN^3 + 2 \cdot AN}{AN^4 + 3 \cdot AN^3 + AN^2 - 2 \cdot AN - 1} = 0$$





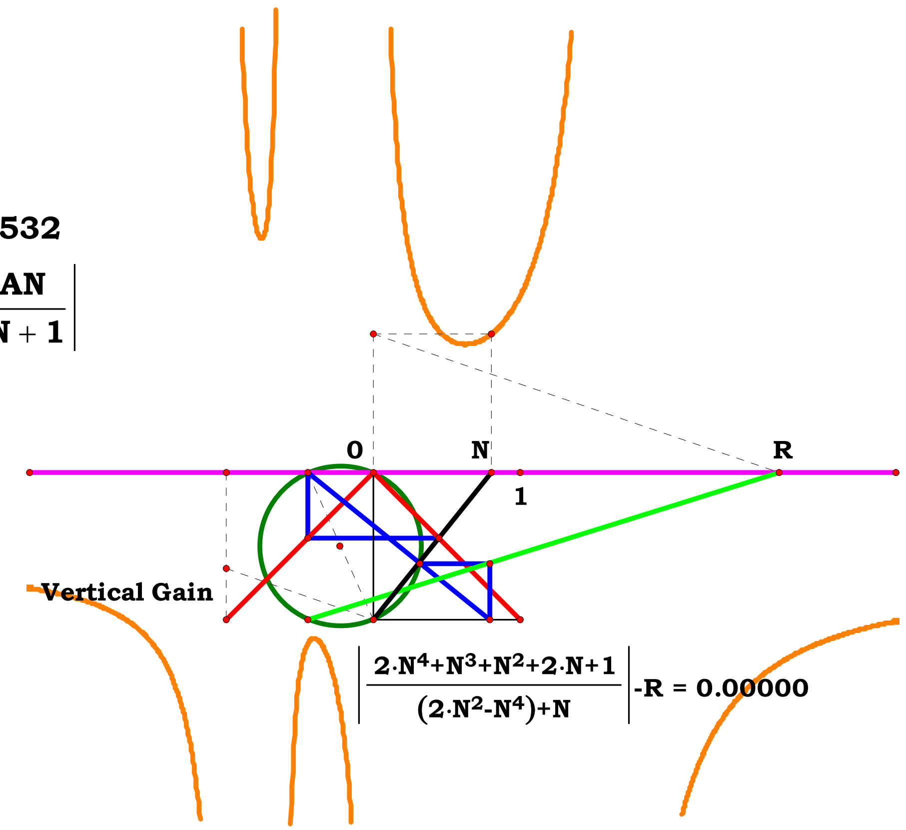
$$AB := 1$$

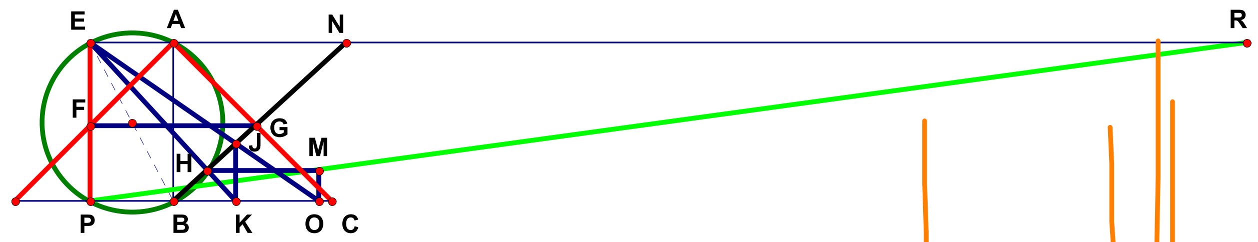
$$AN := .50532$$

$$AE := \left|\frac{-AN}{AN+1}\right|$$

$$JK := \frac{AN - AN^2 + 1}{AN^3 + AN^2 + AN + 1} \quad MK := \frac{AB^2}{AN} \quad ER := \frac{MK \cdot AB}{JK} \quad AR := ER - AE$$

$$AR - \frac{2 \cdot AN^4 + AN^3 + AN^2 + 2 \cdot AN + 1}{2 \cdot AN^2 - AN^4 + AN} = 0$$

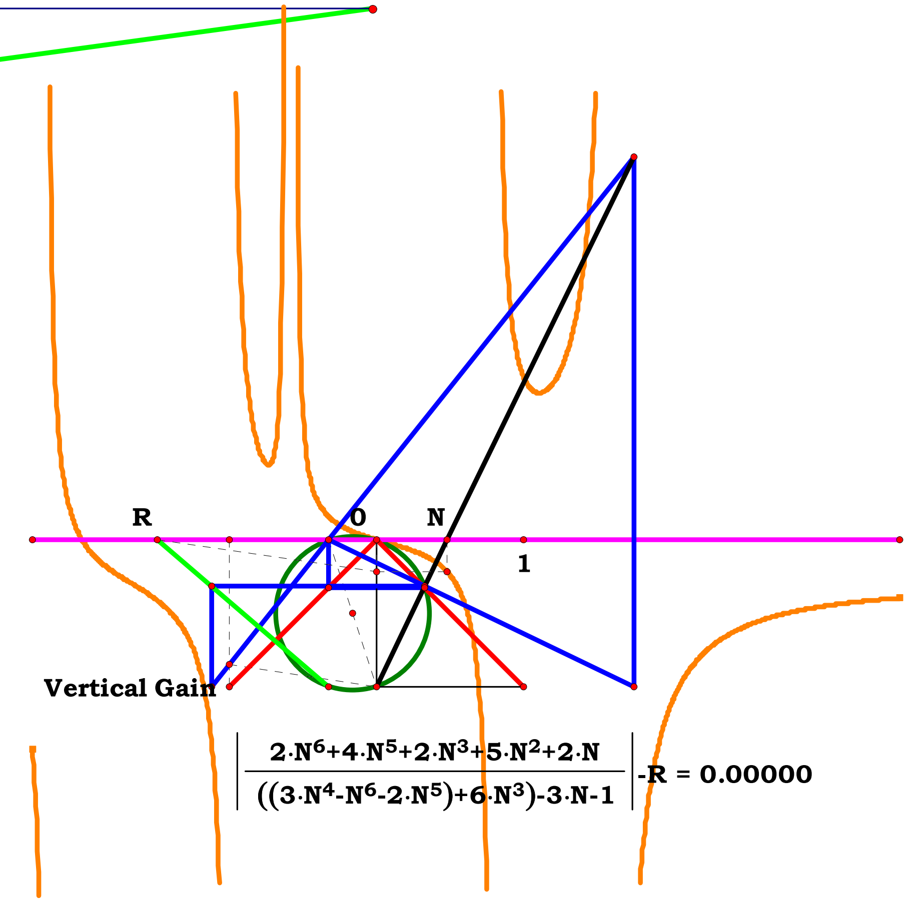


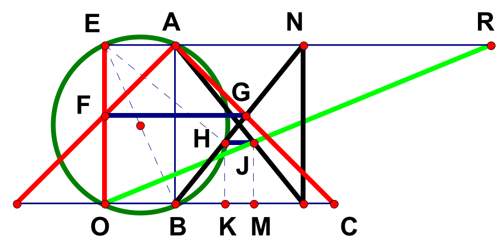


$$AB := 1 \quad AN := 1.04564 \quad AE := \left|\frac{-AN}{AN+1}\right| \quad PK := \frac{AB^2}{AN} \quad MO := \frac{AN - AN^2 + 1}{AN^3 + AN^2 + AN + 1}$$

$$BK := \frac{AN - AN^2 + 1}{AN^2 + AN} \quad JK := \frac{AB \cdot BK}{AN} \quad PO := \frac{PK \cdot AB}{AB - JK} \quad ER := \frac{PO \cdot AB}{MO}$$

$$AR := ER - AE \quad AR - \frac{2AN^6 + 4AN^5 + 2AN^3 + 5AN^2 + 2AN}{3AN^4 - AN^6 - 2AN^5 + 6AN^3 - 3AN - 1} = 0$$





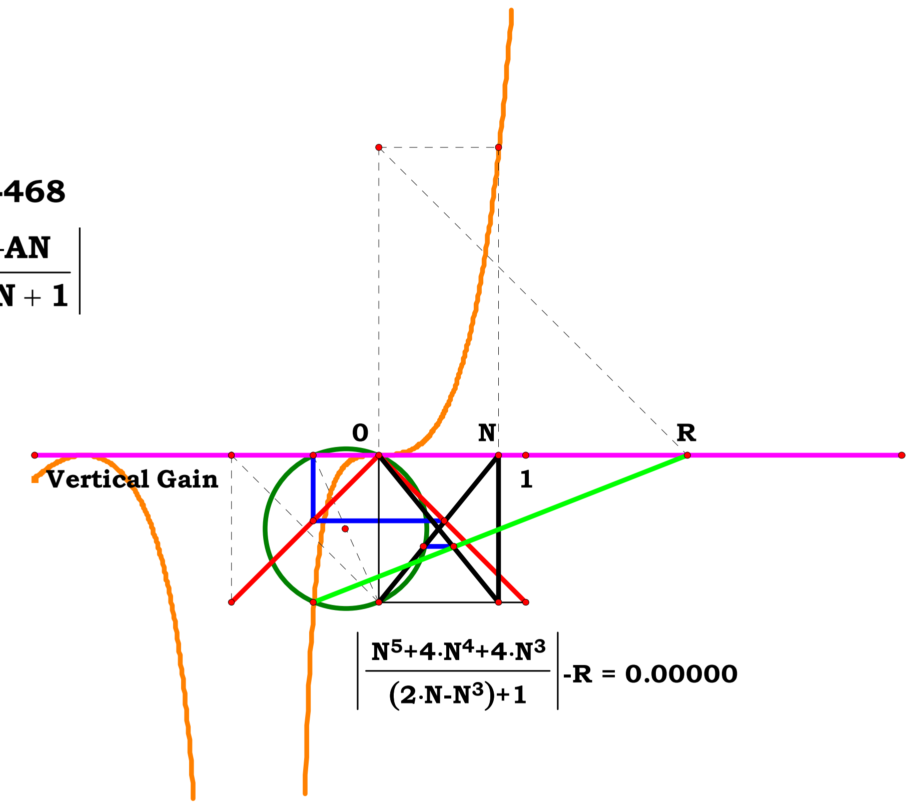
$AB := 1$

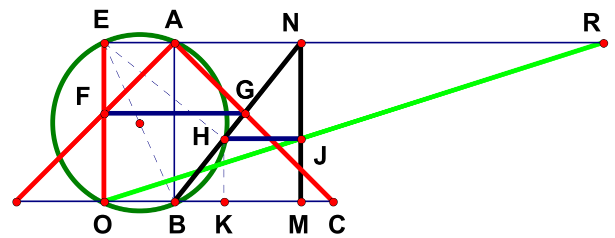
$AN := .74468$

$AE := \left| \dfrac{-AN}{AN + 1} \right|$

$BK := \dfrac{AN^2 - AN^3 + AN}{AN^3 + AN^2 + AN + 1}$ $\quad HK := \dfrac{AN - AN^2 + 1}{AN^3 + AN^2 + AN + 1}$ $\quad MO := AE + AN - BK$

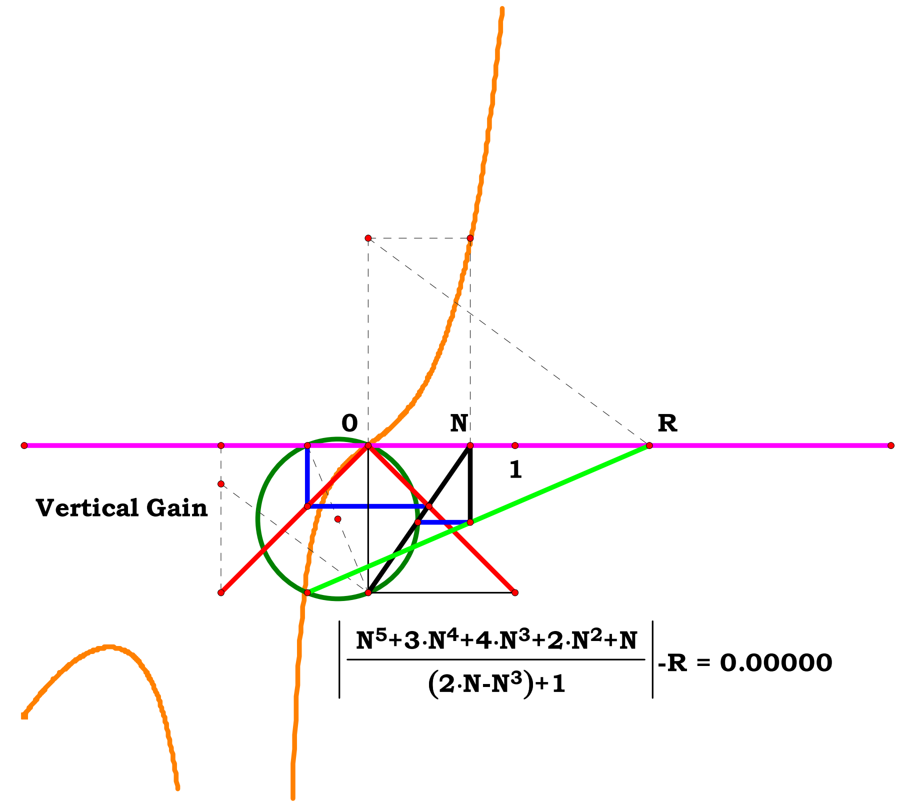
$ER := \dfrac{MO \cdot AB}{HK}$ $\quad AR := ER - AE$ $\quad AR - \dfrac{AN^5 + 4 \cdot AN^4 + 4 \cdot AN^3}{2 \cdot AN - AN^3 + 1} = 0$

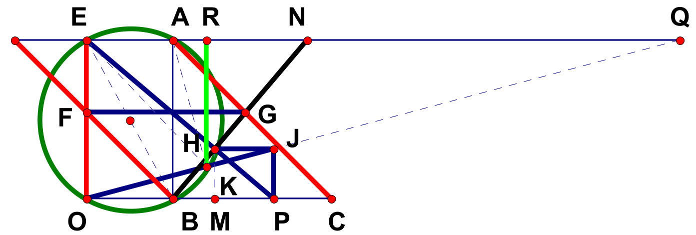




$$AB := 1 \qquad AN := .46809 \qquad AE := \left|\frac{-AN}{AN+1}\right| \qquad HK := \frac{AN - AN^2 + 1}{AN^3 + AN^2 + AN + 1}$$

$$MO := AN + AE \qquad AR := \frac{MO \cdot AB}{HK} - AE \qquad AR - \frac{AN^5 + 3\cdot AN^4 + 4\cdot AN^3 + 2\cdot AN^2 + AN}{2\cdot AN - AN^3 + 1} = 0$$

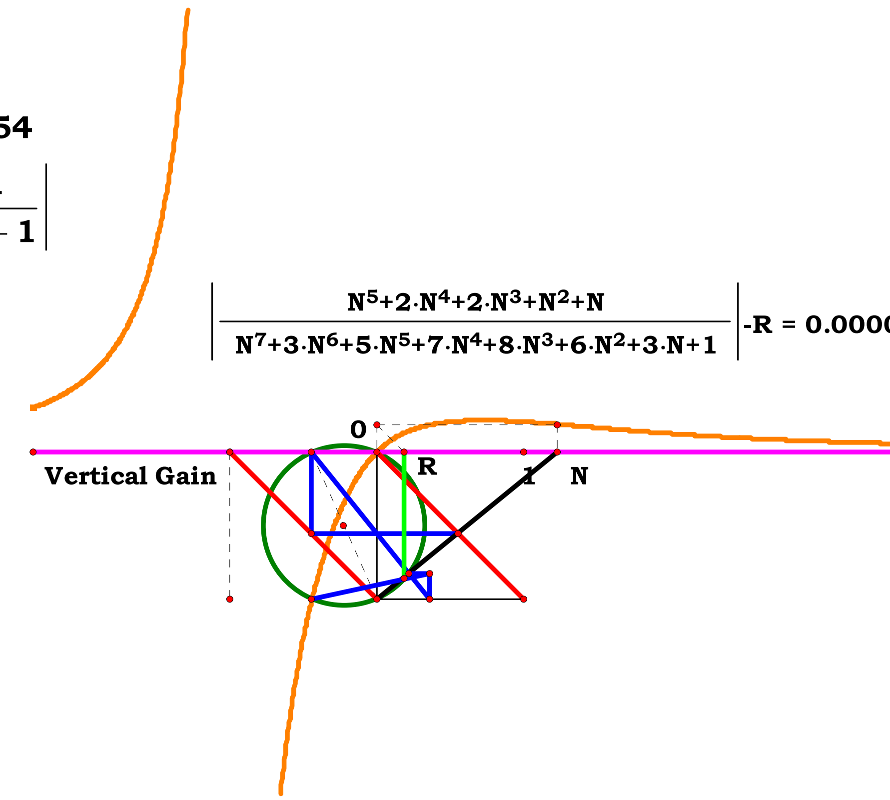




$$AB := 1$$

$$AN := .88854$$

$$AE := \left| \frac{-1}{AN + 1} \right|$$



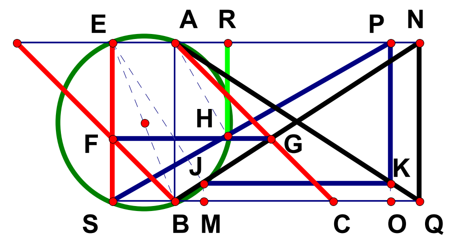
$$\left| \frac{N^5+2 \cdot N^4+2 \cdot N^3+N^2+N}{N^7+3 \cdot N^6+5 \cdot N^5+7 \cdot N^4+8 \cdot N^3+6 \cdot N^2+3 \cdot N+1} \right| \text{-} R = 0.00000$$

$$EN := AN + AE \quad BN := \sqrt{AN^2 + AB^2} \quad HN := \frac{AN \cdot EN}{BN} \quad BH := BN - HN$$

$$HM := \frac{AB \cdot BH}{BN} \quad JP := HM \quad OP := \frac{AB^2}{AN} \quad EQ := \frac{OP \cdot AB}{JP} \quad AQ := EQ - AE$$

$$OQ := \sqrt{EQ^2 + AB^2} \quad KQ := \frac{EQ \cdot AQ}{OQ} \quad KO := OQ - KQ \quad ER := \frac{EQ \cdot KO}{OQ}$$

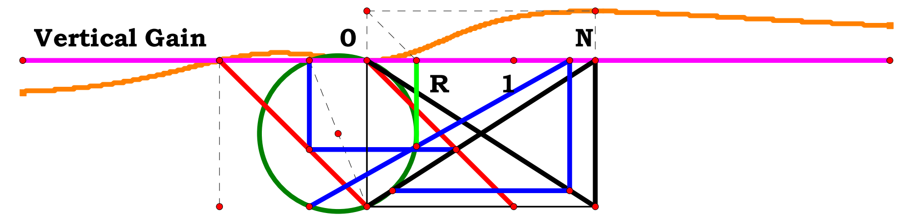
$$AR := ER - AE \quad AR - \frac{AN^5 + 2 \cdot AN^4 + 2 \cdot AN^3 + AN^2 + AN}{AN^7 + 3 \cdot AN^6 + 5 \cdot AN^5 + 7 \cdot AN^4 + 8 \cdot AN^3 + 6 \cdot AN^2 + 3 \cdot AN + 1} = 0$$



$$AB := 1$$

$$AN := 1.12676$$

$$AE := \left|\frac{-1}{AN+1}\right|$$

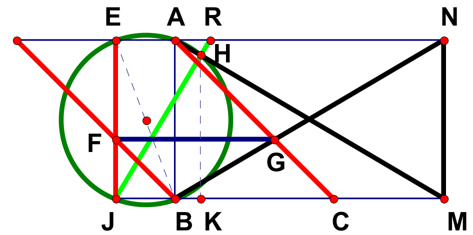


$$\frac{N^7+2\cdot N^6+3\cdot N^5+3\cdot N^4+2\cdot N^3+N^2}{N^8+2\cdot N^7+6\cdot N^6+6\cdot N^5+9\cdot N^4+6\cdot N^3+7\cdot N^2+2\cdot N+2}-R = 0.00000$$

$$BM := \frac{AN}{AN^3+AN^2+AN+1} \quad PN := BM \quad AP := AN - PN \quad EP := AP + AE$$

$$PS := \sqrt{EP^2+AB^2} \quad HP := \frac{EP\cdot AP}{PS} \quad RP := \frac{EP\cdot HP}{PS} \quad AR := AP - RP$$

$$AR - \frac{AN^7+2\cdot AN^6+3\cdot AN^5+3\cdot AN^4+2\cdot AN^3+AN^2}{AN^8+2\cdot AN^7+6\cdot AN^6+6\cdot AN^5+9\cdot AN^4+6\cdot AN^3+7\cdot AN^2+2\cdot AN+2} = 0$$



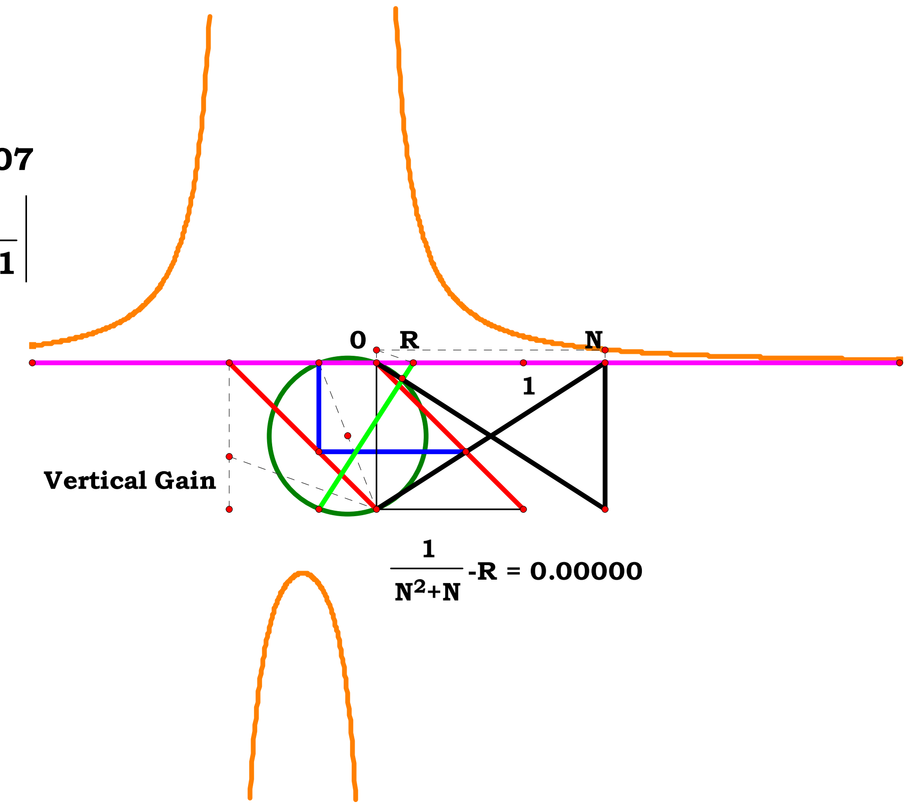
$AB := 1$

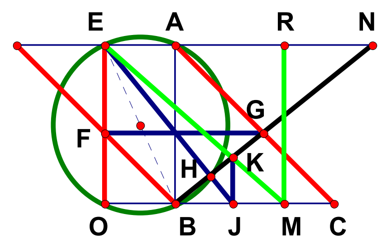
$AN := 1.34507$

$AE := \left| \dfrac{-1}{AN+1} \right|$

$AM := \sqrt{AN^2 + AB^2}$ $\quad JM := AN + AE$ $\quad HJ := \dfrac{AB \cdot JM}{AM}$ $\quad HK := \dfrac{AN \cdot HJ}{AM}$

$JK := \dfrac{AB \cdot HJ}{AM}$ $\quad ER := \dfrac{JK \cdot AB}{HK}$ $\quad AR := ER - AE$ $\quad AR - \dfrac{1}{AN^2 + AN} = 0$

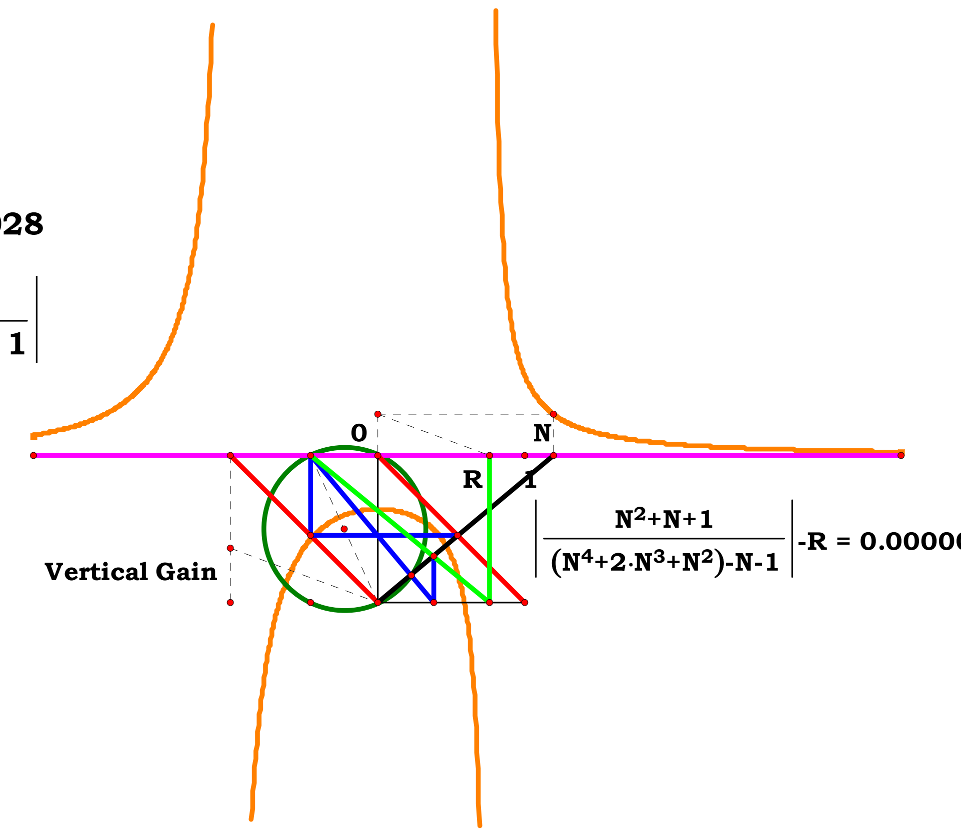




$$AB := 1$$

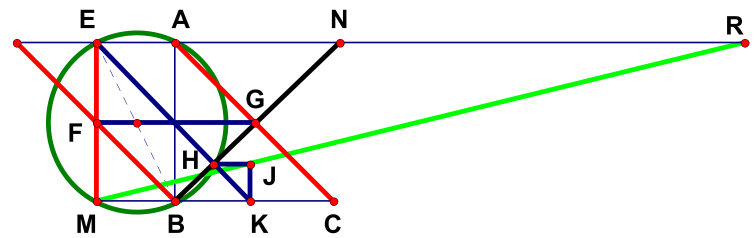
$$AN := 1.38028$$

$$AE := \left|\frac{-1}{AN+1}\right|$$



$$BJ := \frac{1}{AN^2+AN} \qquad KJ := \frac{AB\cdot BJ}{AN} \qquad JO := \frac{AB^2}{AN} \qquad OM := \frac{JO\cdot AB}{AB-KJ}$$

$$AR := OM - AE \qquad AR - \frac{AN^2+AN+1}{AN^4+2\cdot AN^3+AN^2-AN-1} = 0$$



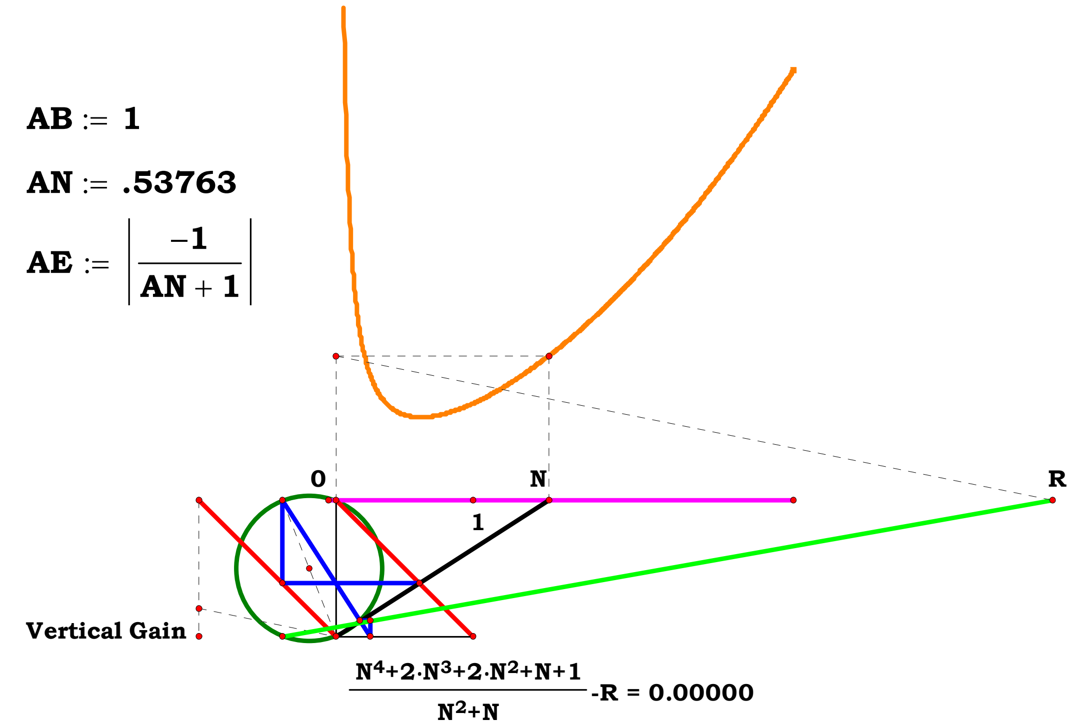
$AB := 1$

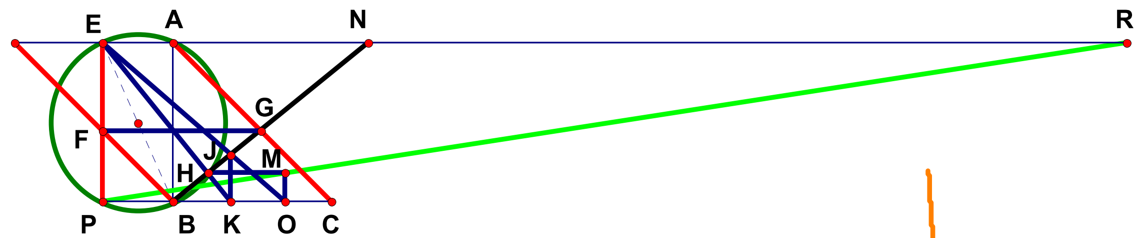
$AN := .53763$

$AE := \left| \dfrac{-1}{AN+1} \right|$

$JK := \dfrac{1}{AN^3 + AN^2 + AN + 1}$ $\quad MK := \dfrac{AB^2}{AN}$ $\quad ER := \dfrac{MK \cdot AB}{JK}$ $\quad AR := ER - AE$

$$AR - \frac{AN^4 + 2 \cdot AN^3 + 2 \cdot AN^2 + AN + 1}{AN^2 + AN} = 0$$



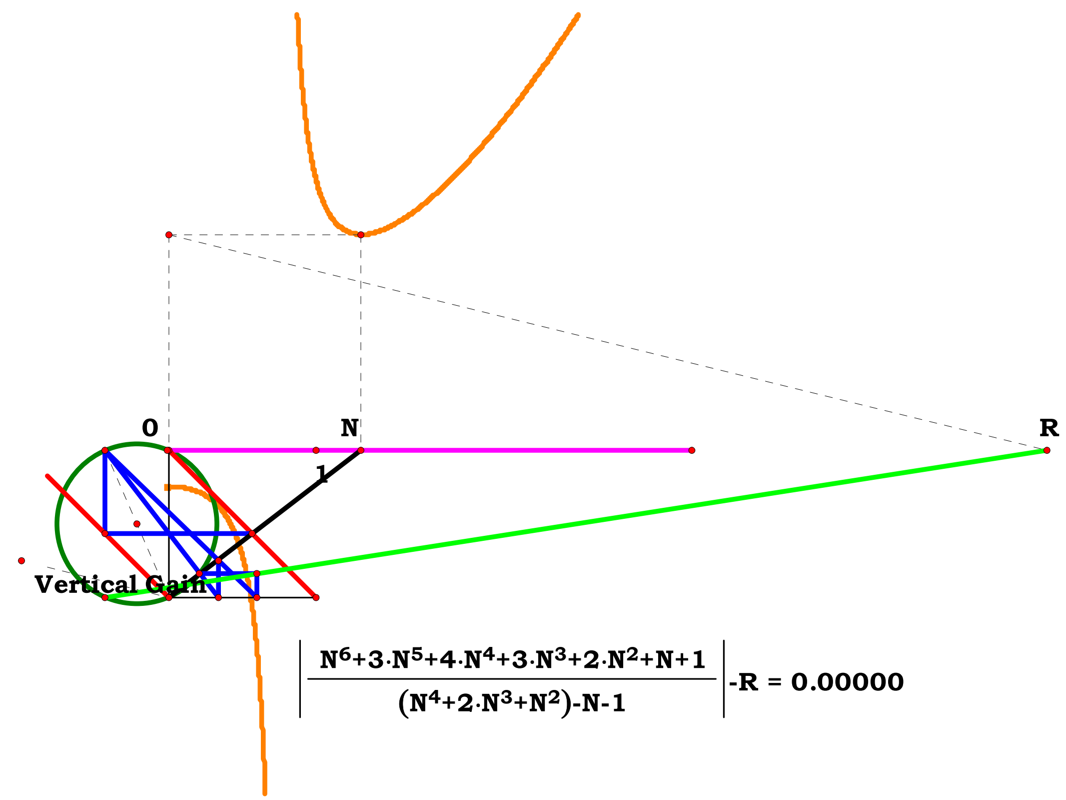


$$AB := 1 \quad AN := 1.25269 \quad AE := \left|\frac{-1}{AN+1}\right| \quad PK := \frac{AB^2}{AN}$$

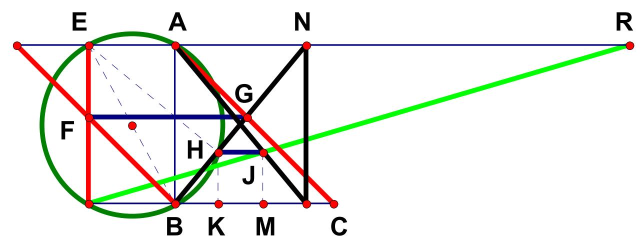
$$MO := \frac{1}{AN^3 + AN^2 + AN + 1} \quad BK := \frac{1}{AN^2 + AN}$$

$$JK := \frac{AB \cdot BK}{AN} \quad PO := \frac{PK \cdot AB}{AB - JK} \quad ER := \frac{PO \cdot AB}{MO} \quad AR := ER - AE$$

$$AR - \frac{AN^6 + 3 \cdot AN^5 + 4 \cdot AN^4 + 3 \cdot AN^3 + 2 \cdot AN^2 + AN + 1}{AN^4 + 2 \cdot AN^3 + AN^2 - AN - 1} = 0$$



$$\left|\frac{N^6+3 \cdot N^5+4 \cdot N^4+3 \cdot N^3+2 \cdot N^2+N+1}{(N^4+2 \cdot N^3+N^2)-N-1}\right| - R = 0.00000$$



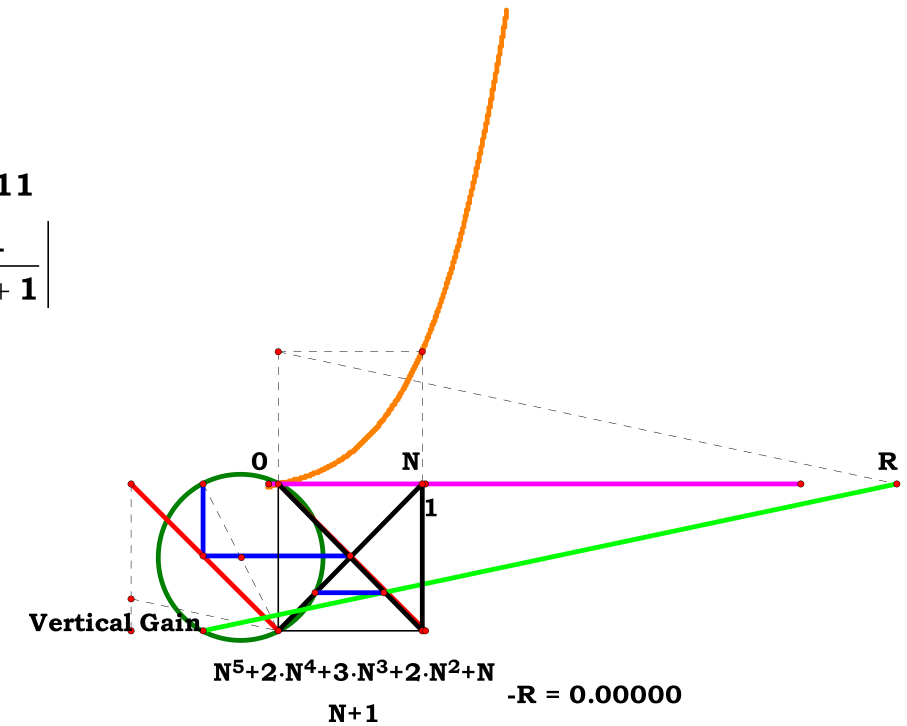
$$AB := 1$$

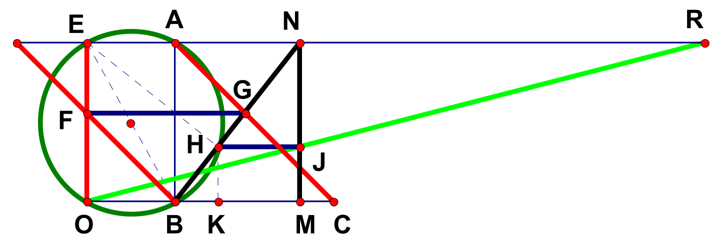
$$AN := .58011$$

$$AE := \left|\frac{-1}{AN+1}\right|$$

$$BK := \frac{AN}{AN^3 + AN^2 + AN + 1} \qquad HK := \frac{1}{AN^3 + AN^2 + AN + 1} \qquad MO := AE + AN - BK$$

$$ER := \frac{MO \cdot AB}{HK} \qquad AR := ER - AE \qquad AR - \frac{AN^5 + 2 \cdot AN^4 + 3 \cdot AN^3 + 2 \cdot AN^2 + AN}{AN + 1} = 0$$

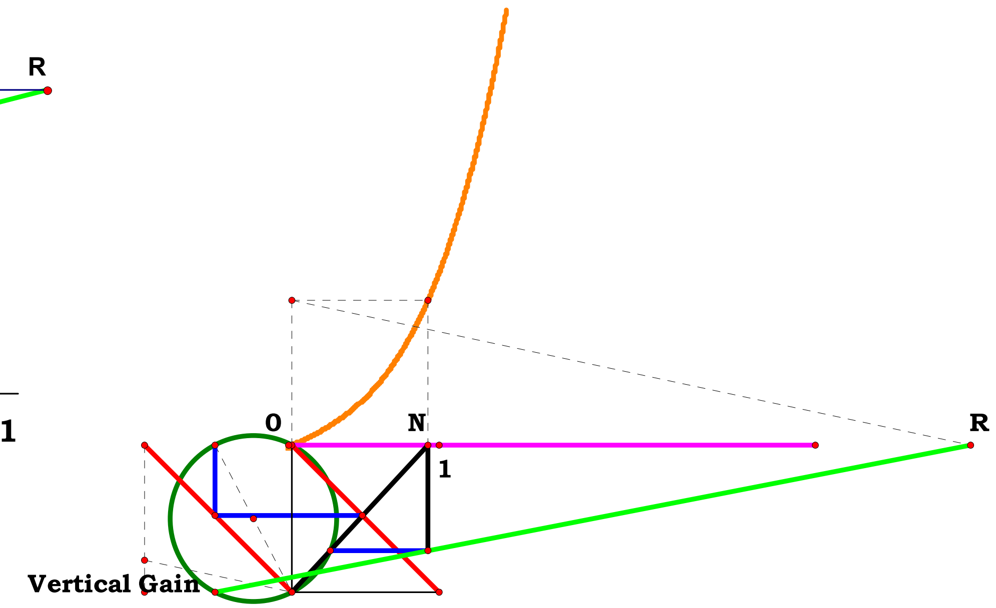




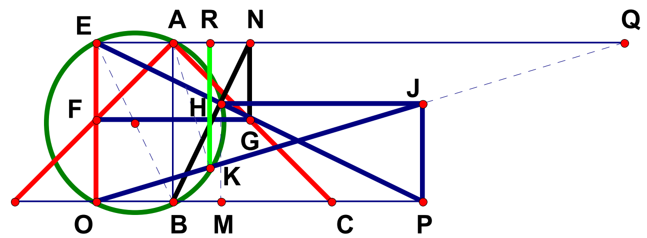
$$AB := 1 \quad AN := .84977 \quad AE := \left|\frac{-1}{AN+1}\right| \quad HK := \frac{1}{AN^3+AN^2+AN+1}$$

$$MO := AN + AE \quad AR := \frac{MO \cdot AB}{HK} - AE$$

$$AR - \frac{AN^5 + 2 \cdot AN^4 + 3 \cdot AN^3 + 3 \cdot AN^2 + 2 \cdot AN}{AN+1} = 0$$



$$\frac{N^5+2\cdot N^4+3\cdot N^3+3\cdot N^2+2\cdot N}{N+1} - R = 0.00000$$



$$\mathbf{AB} := 1$$

$$\mathbf{AN} := .58843$$

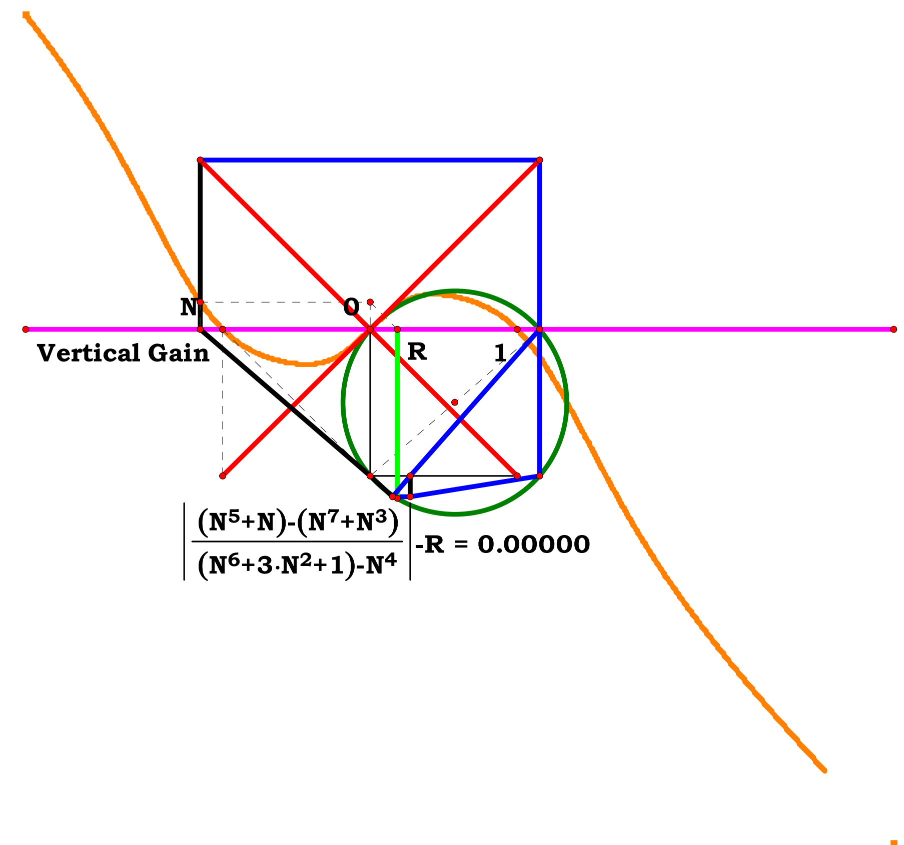
$$\mathbf{AE} := \left|-\mathbf{AN}\right|$$

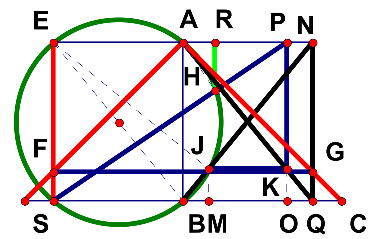
$$\mathbf{EN} := \mathbf{AN} + \mathbf{AE} \qquad \mathbf{BN} := \sqrt{\mathbf{AN}^2 + \mathbf{AB}^2} \qquad \mathbf{HN} := \frac{\mathbf{AN}\cdot\mathbf{EN}}{\mathbf{BN}} \qquad \mathbf{BH} := \mathbf{BN} - \mathbf{HN}$$

$$\mathbf{HM} := \frac{\mathbf{AB}\cdot\mathbf{BH}}{\mathbf{BN}} \qquad \mathbf{JP} := \mathbf{HM} \qquad \mathbf{OP} := \frac{\mathbf{AB}^2}{\mathbf{AN}} \qquad \mathbf{EQ} := \frac{\mathbf{OP}\cdot\mathbf{AB}}{\mathbf{JP}} \qquad \mathbf{AQ} := \mathbf{EQ} - \mathbf{AE}$$

$$\mathbf{OQ} := \sqrt{\mathbf{EQ}^2 + \mathbf{AB}^2} \qquad \mathbf{KQ} := \frac{\mathbf{EQ}\cdot\mathbf{AQ}}{\mathbf{OQ}} \qquad \mathbf{KO} := \mathbf{OQ} - \mathbf{KQ} \qquad \mathbf{ER} := \frac{\mathbf{EQ}\cdot\mathbf{KO}}{\mathbf{OQ}}$$

$$\mathbf{AR} := \mathbf{ER} - \mathbf{AE} \qquad \mathbf{AR} - \frac{\mathbf{AN}^5 - \mathbf{AN}^7 - \mathbf{AN}^3 + \mathbf{AN}}{\mathbf{AN}^6 - \mathbf{AN}^4 + 3\cdot\mathbf{AN}^2 + 1} = 0$$





$\mathbf{AB} := 1$

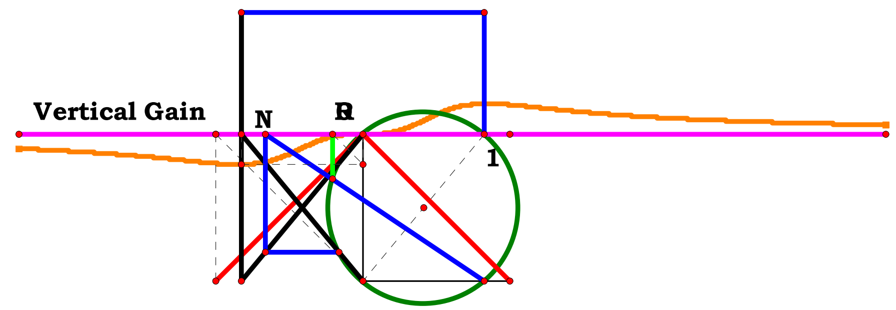
$\mathbf{AN} := .83697$

$\mathbf{AE} := |-\mathbf{AN}|$

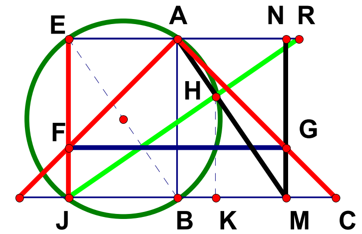
$$\mathbf{BM} := \frac{\mathbf{AN} - \mathbf{AN}^3}{\mathbf{AN}^2 + 1} \qquad \mathbf{PN} := \mathbf{BM} \qquad \mathbf{AP} := \mathbf{AN} - \mathbf{PN} \qquad \mathbf{EP} := \mathbf{AP} + \mathbf{AE}$$

$$\mathbf{PS} := \sqrt{\mathbf{EP}^2 + \mathbf{AB}^2} \qquad \mathbf{HP} := \frac{\mathbf{EP} \cdot \mathbf{AP}}{\mathbf{PS}} \qquad \mathbf{RP} := \frac{\mathbf{EP} \cdot \mathbf{HP}}{\mathbf{PS}} \qquad \mathbf{AR} := \mathbf{AP} - \mathbf{RP}$$

$$\mathbf{AR} - \frac{2 \cdot \mathbf{AN}^5 + 2 \cdot \mathbf{AN}^3}{9 \cdot \mathbf{AN}^6 + 7 \cdot \mathbf{AN}^4 + 3 \cdot \mathbf{AN}^2 + 1} = 0$$



$$\frac{2 \cdot N^5 + 2 \cdot N^3}{9 \cdot N^6 + 7 \cdot N^4 + 3 \cdot N^2 + 1} - R = 0.00000$$



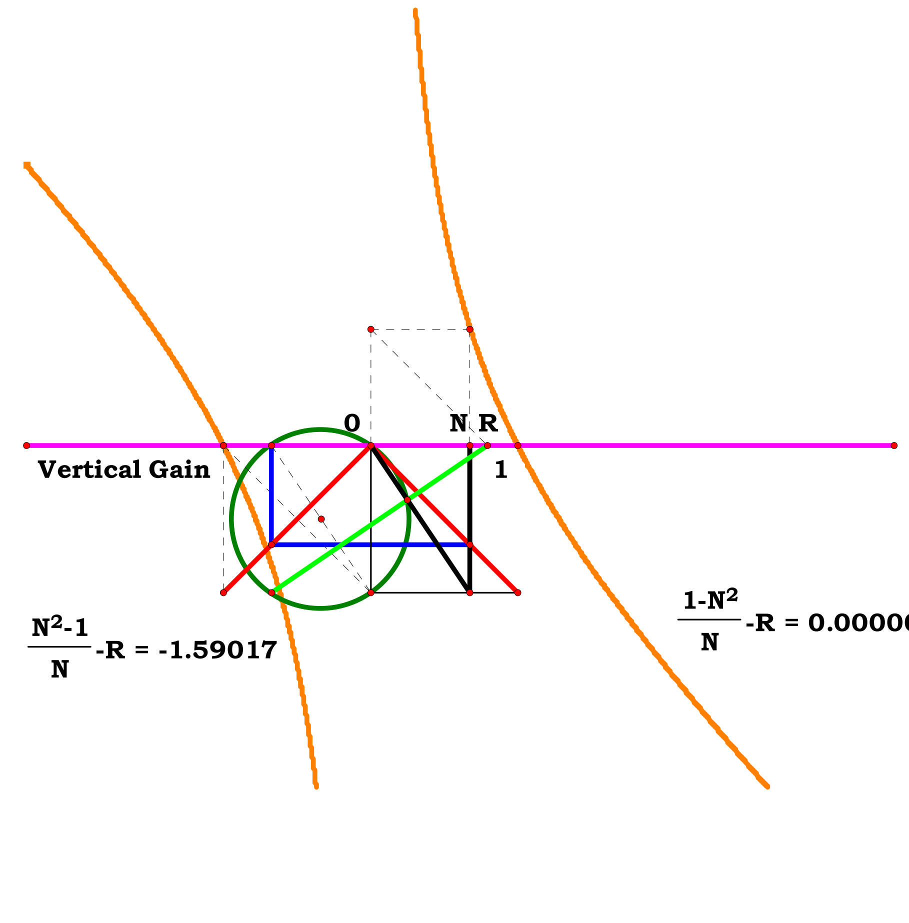
$$AB := 1$$

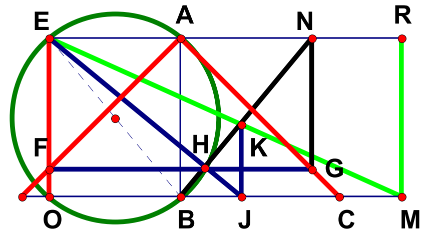
$$AN := 1.34507$$

$$AE := \left|-AN\right|$$

$$AM := \sqrt{AN^2 + AB^2} \quad JM := AN + AE \quad HJ := \frac{AB \cdot JM}{AM} \quad HK := \frac{AN \cdot HJ}{AM}$$

$$JK := \frac{AB \cdot HJ}{AM} \quad ER := \frac{JK \cdot AB}{HK} \quad AR := ER - AE \quad AR - \frac{1 - AN^2}{AN} = 0$$



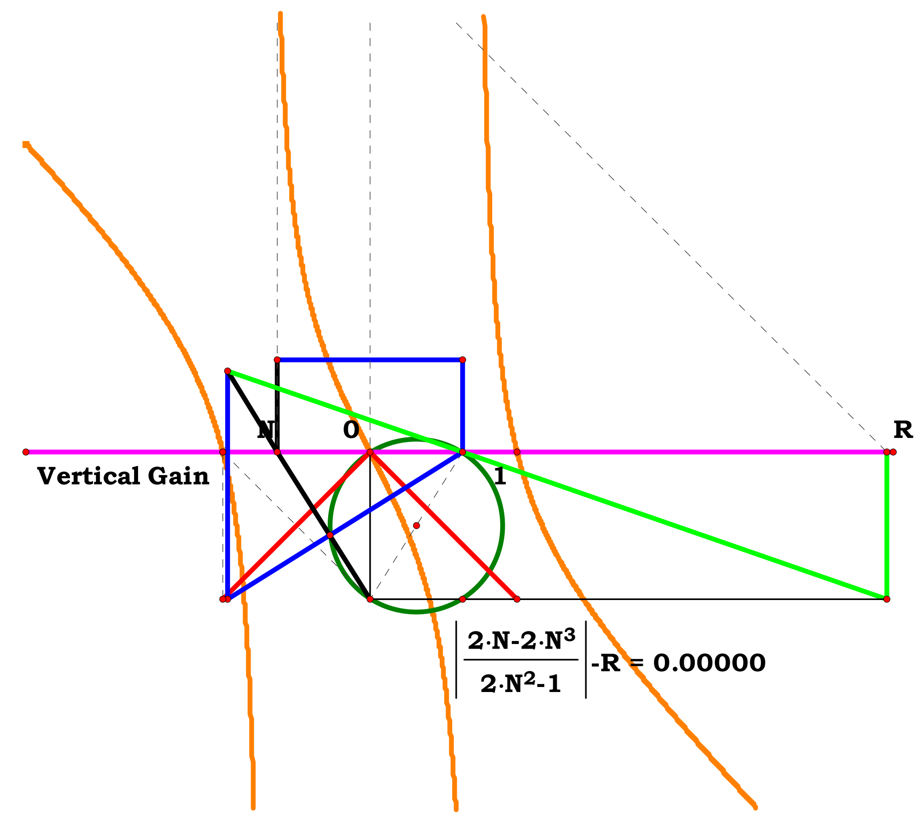


$AB := 1$

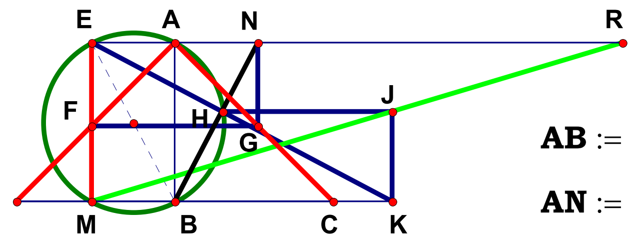
$AN := .8222$

$AE := |-AN|$

$BJ := \dfrac{1 - AN^2}{AN}$ $KJ := \dfrac{AB \cdot BJ}{AN}$ $JO := \dfrac{AB^2}{AN}$ $OM := \dfrac{JO \cdot AB}{AB - KJ}$ $AR := OM - AE$



$$AR - \frac{2 \cdot AN - 2 \cdot AN^3}{2 \cdot AN^2 - 1} = 0$$



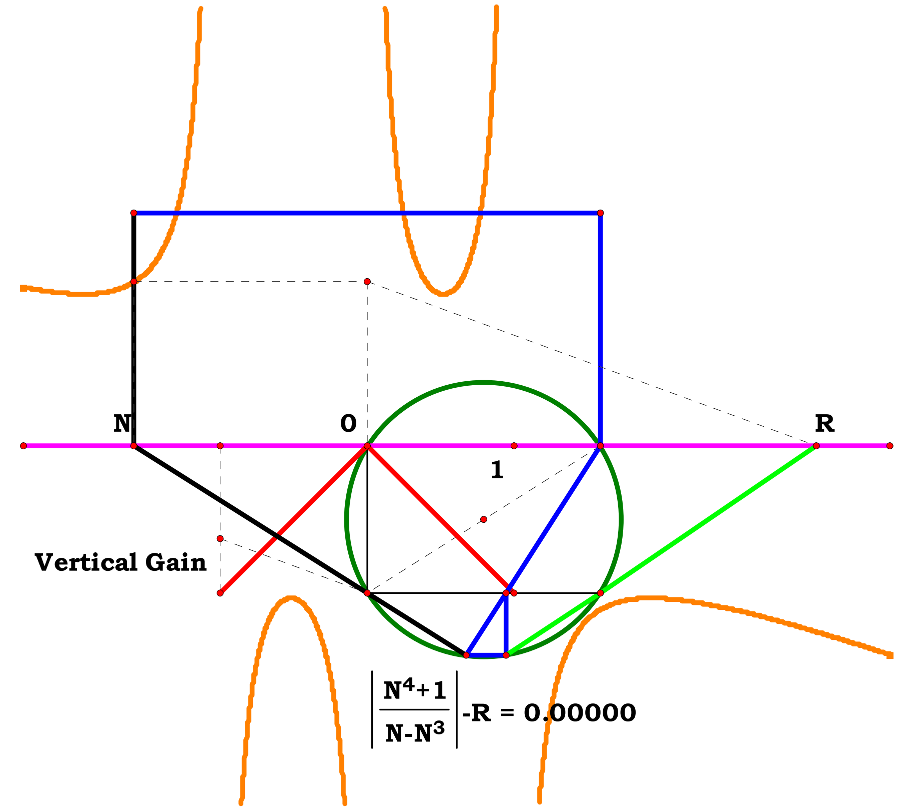
$AB := 1$

$AN := .49762$

$AE := \left| -AN \right|$

$$JK := \frac{1 - AN^2}{AN^2 + 1} \qquad MK := \frac{AB^2}{AN} \qquad ER := \frac{MK \cdot AB}{JK} \qquad AR := ER - AE$$

$$AR - \frac{AN^4 + 1}{AN - AN^3} = 0$$

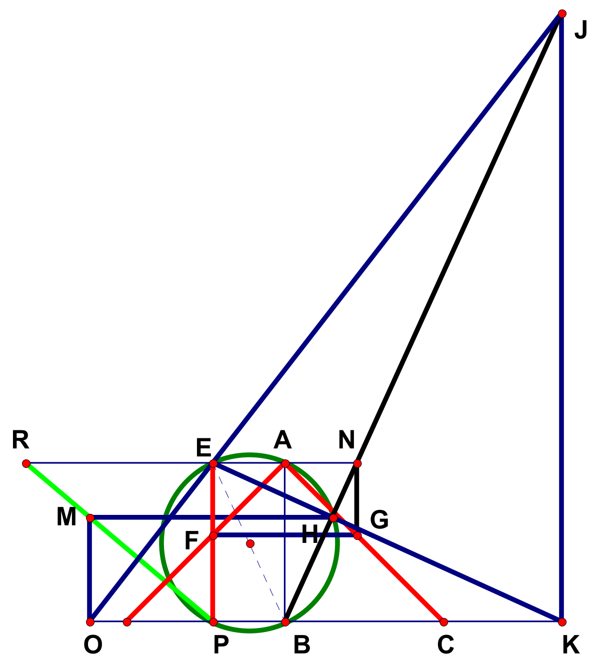


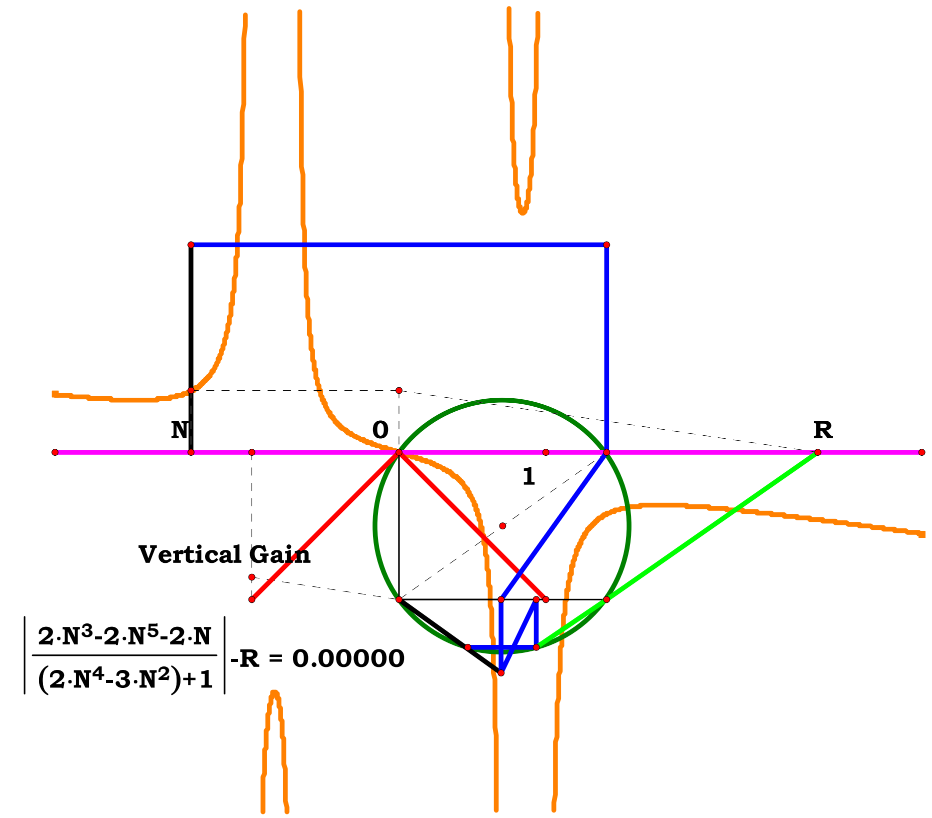
$AB := 1 \quad AN := .75 \quad AE := |-AN|$

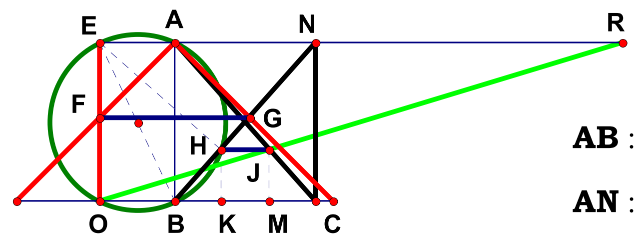
$PK := \dfrac{AB^2}{AN} \quad MO := \dfrac{1 - AN^2}{AN^2 + 1}$

$BK := \dfrac{1 - AN^2}{AN} \quad JK := \dfrac{AB \cdot BK}{AN}$

$PO := \dfrac{PK \cdot AB}{AB - JK} \quad ER := \dfrac{PO \cdot AB}{MO} \quad AR := ER - AE \quad AR - \dfrac{2 \cdot AN^3 - 2 \cdot AN^5 - 2 \cdot AN}{2 \cdot AN^4 - 3AN^2 + 1} = 0$







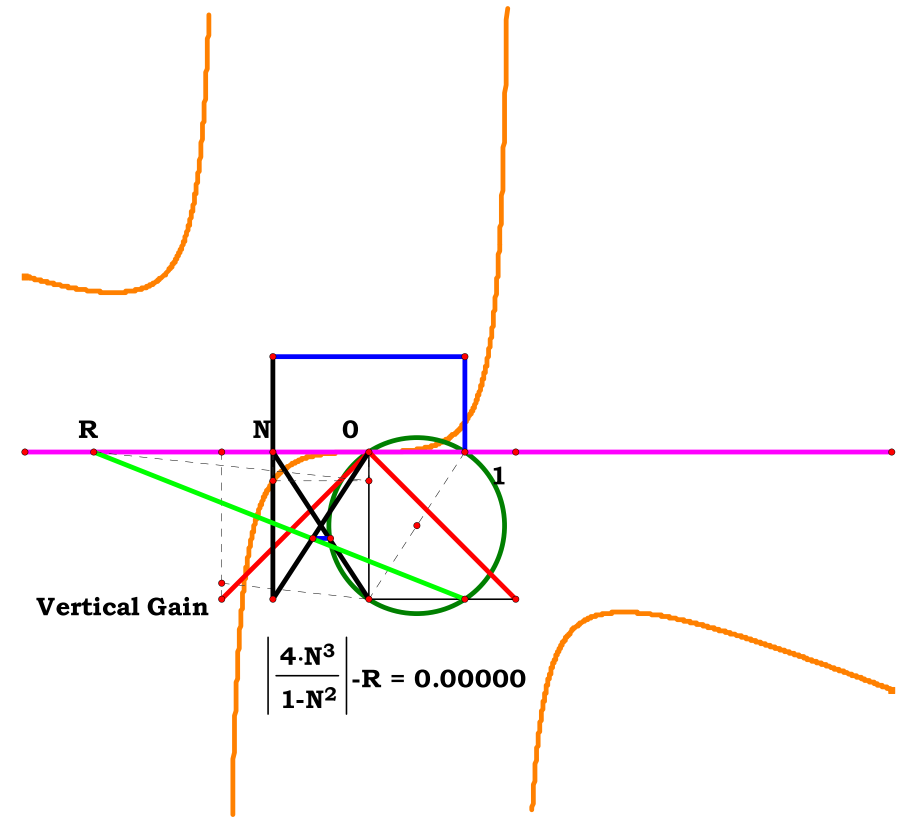
$$AB := 1$$

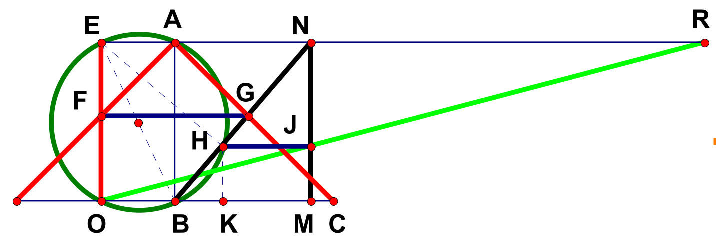
$$AN := .66635$$

$$AE := \left|-AN\right|$$

$$BK := \frac{AN - AN^3}{AN^2 + 1} \qquad HK := \frac{1 - AN^2}{AN^2 + 1} \qquad MO := AE + AN - BK$$

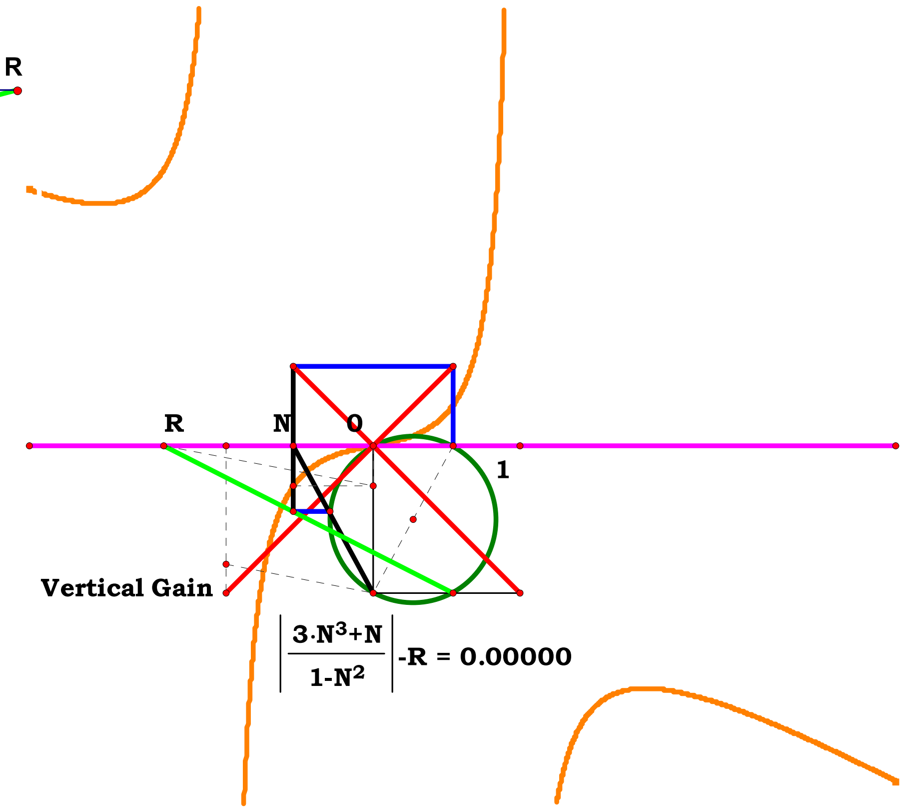
$$ER := \frac{MO \cdot AB}{HK} \qquad AR := ER - AE \qquad AR - \frac{4AN^3}{1 - AN^2} = 0$$



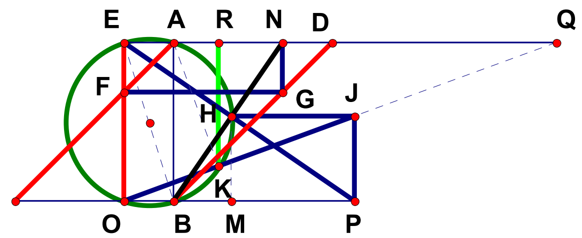


$AB := 1 \quad AN := .56214 \quad AE := |-AN| \quad HK := \dfrac{1 - AN^2}{AN^2 + 1}$

$MO := AN + AE \quad AR := \dfrac{MO \cdot AB}{HK} - AE \quad AR - \dfrac{3 \cdot AN^3 + AN}{1 - AN^2} = 0$



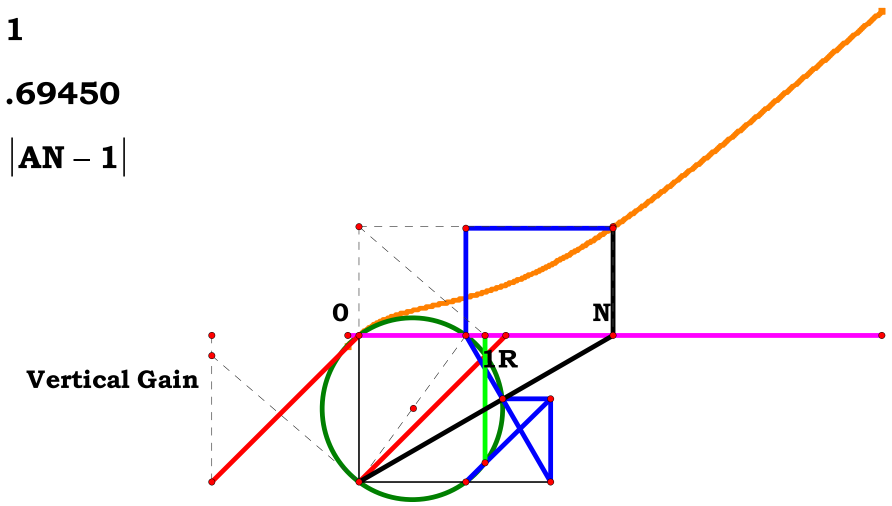
$\left|\dfrac{3 \cdot N^3 + N}{1 - N^2}\right|$-R = 0.00000



$$AB := 1$$

$$AN := .69450$$

$$AE := |AN - 1|$$



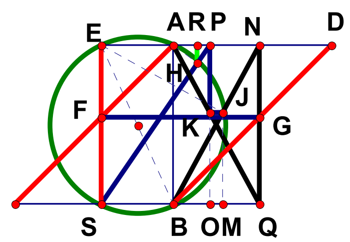
$$\left|\frac{(((((N^7-3\cdot N^6)+6\cdot N^5)-6\cdot N^4)+5\cdot N^3)-2\cdot N^2)+N}{(((N^6-2\cdot N^5)+4\cdot N^4)-2\cdot N^3)+3\cdot N^2+1}\right|-R = 0.00000$$

$$EN := AN + AE \quad BN := \sqrt{AN^2 + AB^2} \quad HN := \frac{AN \cdot EN}{BN} \quad BH := BN - HN$$

$$HM := \frac{AB \cdot BH}{BN} \quad JP := HM \quad OP := \frac{AB^2}{AN} \quad EQ := \frac{OP \cdot AB}{JP} \quad AQ := EQ - AE$$

$$OQ := \sqrt{EQ^2 + AB^2} \quad KQ := \frac{EQ \cdot AQ}{OQ} \quad KO := OQ - KQ \quad ER := \frac{EQ \cdot KO}{OQ}$$

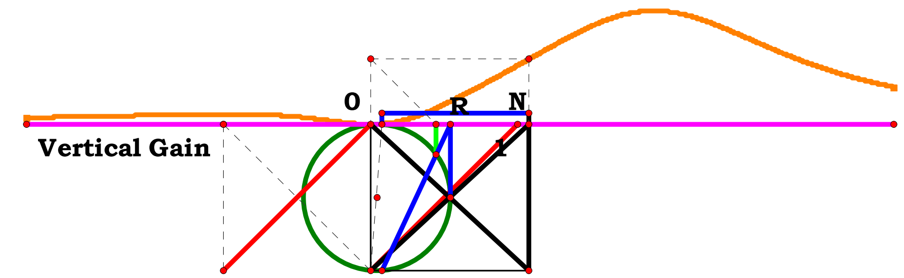
$$AR := ER - AE \quad AR - \frac{AN^7 - 3 \cdot AN^6 + 6 \cdot AN^5 - 6 \cdot AN^4 + 5 \cdot AN^3 - 2 \cdot AN^2 + AN}{AN^6 - 2 \cdot AN^5 + 4 \cdot AN^4 - 2 \cdot AN^3 + 3 \cdot AN^2 + 1} = 0$$



$$\mathbf{AB} := 1$$

$$\mathbf{AN} := .52386$$

$$\mathbf{AE} := \left|\mathbf{AN} - 1\right|$$

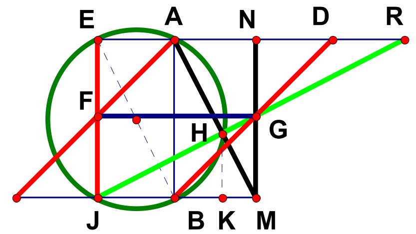


$$\frac{N^4+N^2}{(((((N^6-4\cdot N^5)+7\cdot N^4)-6\cdot N^3)+7\cdot N^2)-2\cdot N)+2}-R = 0.00000$$

$$\mathbf{BM} := \frac{\mathbf{AN}^3 - \mathbf{AN}^2 + \mathbf{AN}}{\mathbf{AN}^2 + 1} \quad \mathbf{PN} := \mathbf{BM} \quad \mathbf{AP} := \mathbf{AN} - \mathbf{PN} \quad \mathbf{EP} := \mathbf{AP} + \mathbf{AE}$$

$$\mathbf{PS} := \sqrt{\mathbf{EP}^2 + \mathbf{AB}^2} \quad \mathbf{HP} := \frac{\mathbf{EP}\cdot\mathbf{AP}}{\mathbf{PS}} \quad \mathbf{RP} := \frac{\mathbf{EP}\cdot\mathbf{HP}}{\mathbf{PS}} \quad \mathbf{AR} := \mathbf{AP} - \mathbf{RP}$$

$$\mathbf{AR} - \frac{\mathbf{AN}^4 + \mathbf{AN}^2}{\mathbf{AN}^6 - 4\cdot\mathbf{AN}^5 + 7\cdot\mathbf{AN}^4 - 6\cdot\mathbf{AN}^3 + 7\cdot\mathbf{AN}^2 - 2\cdot\mathbf{AN} + 2} = 0$$



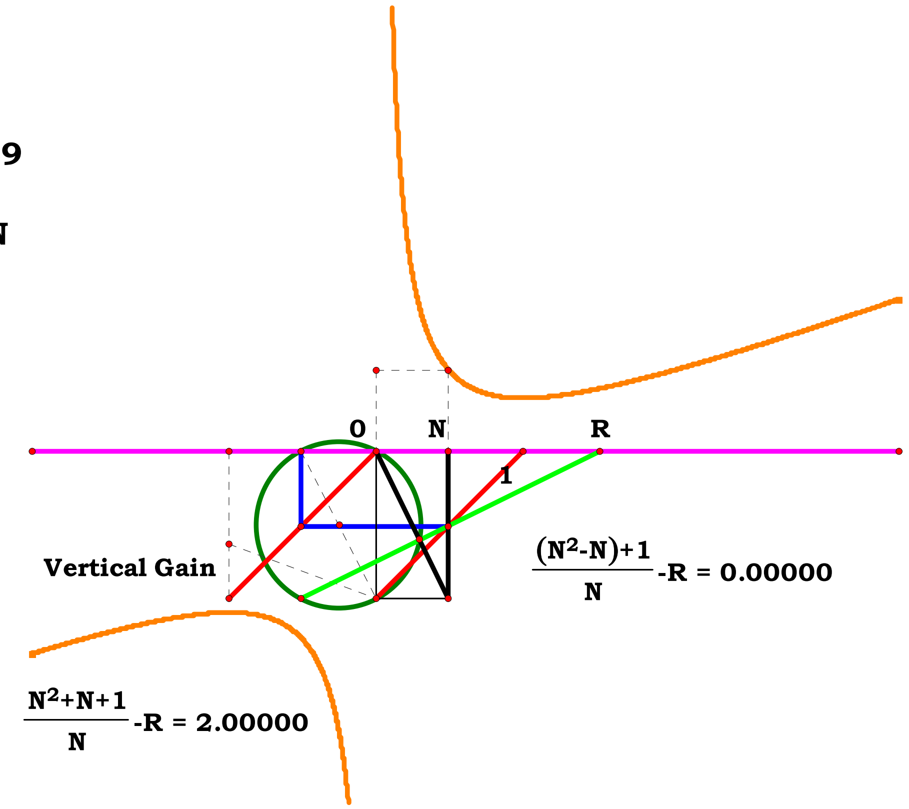
$$AB := 1$$

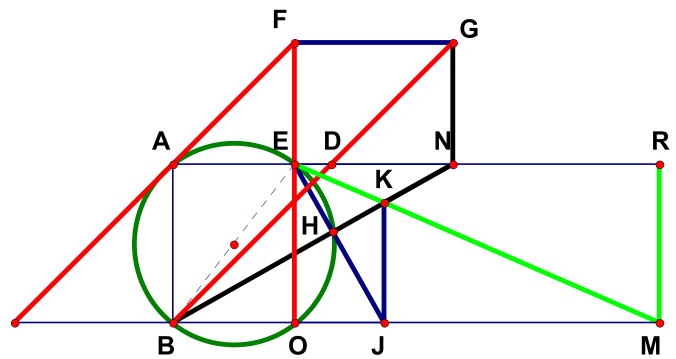
$$AN := 2.3059$$

$$AE := 1 - AN$$

$$AM := \sqrt{AN^2 + AB^2} \quad JM := AN + AE \quad HJ := \frac{AB \cdot JM}{AM} \quad HK := \frac{AN \cdot HJ}{AM}$$

$$JK := \frac{AB \cdot HJ}{AM} \quad ER := \frac{JK \cdot AB}{HK} \quad AR := ER - AE \quad AR - \frac{AN^2 - AN + 1}{AN} = 0$$



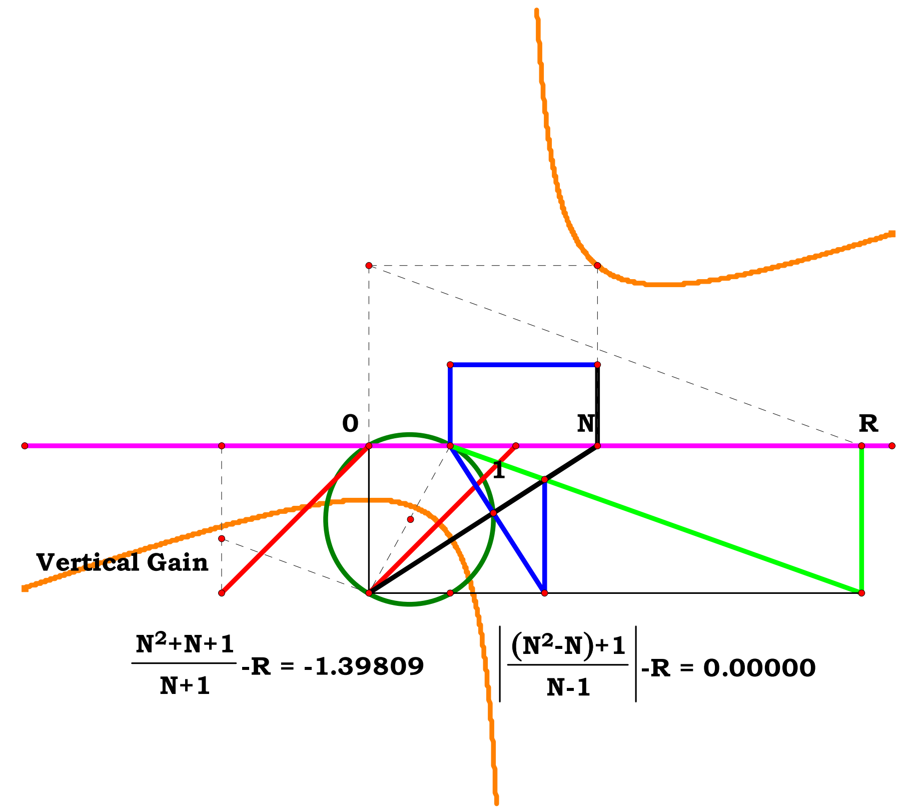


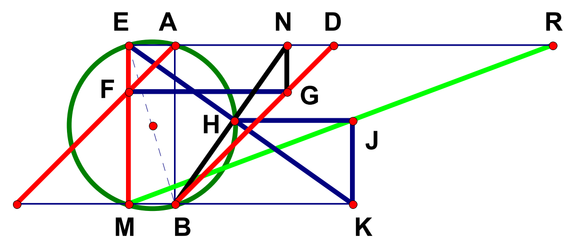
$$AB := 1 \qquad AN := 3.5398$$

$$AE := AN - 1$$

$$BJ := \frac{AN^2 - AN + 1}{AN} \qquad KJ := \frac{AB \cdot BJ}{AN} \qquad JO := \frac{AB^2}{AN} \qquad OM := \frac{JO \cdot AB}{AB - KJ}$$

$$AR := OM + AE \qquad AR - \frac{AN^2 - AN + 1}{AN - 1} = 0$$





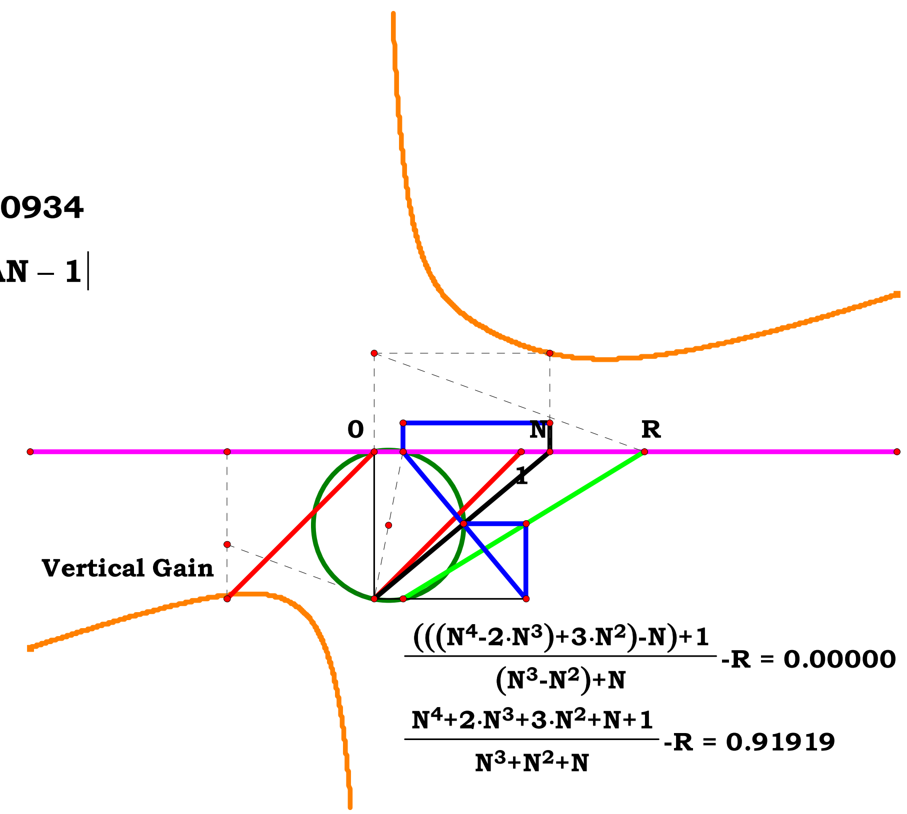
$AB := 1$

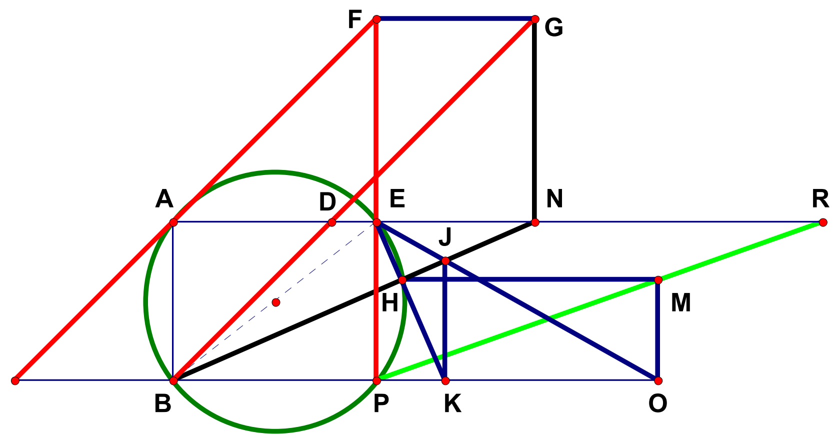
$AN := .70934$

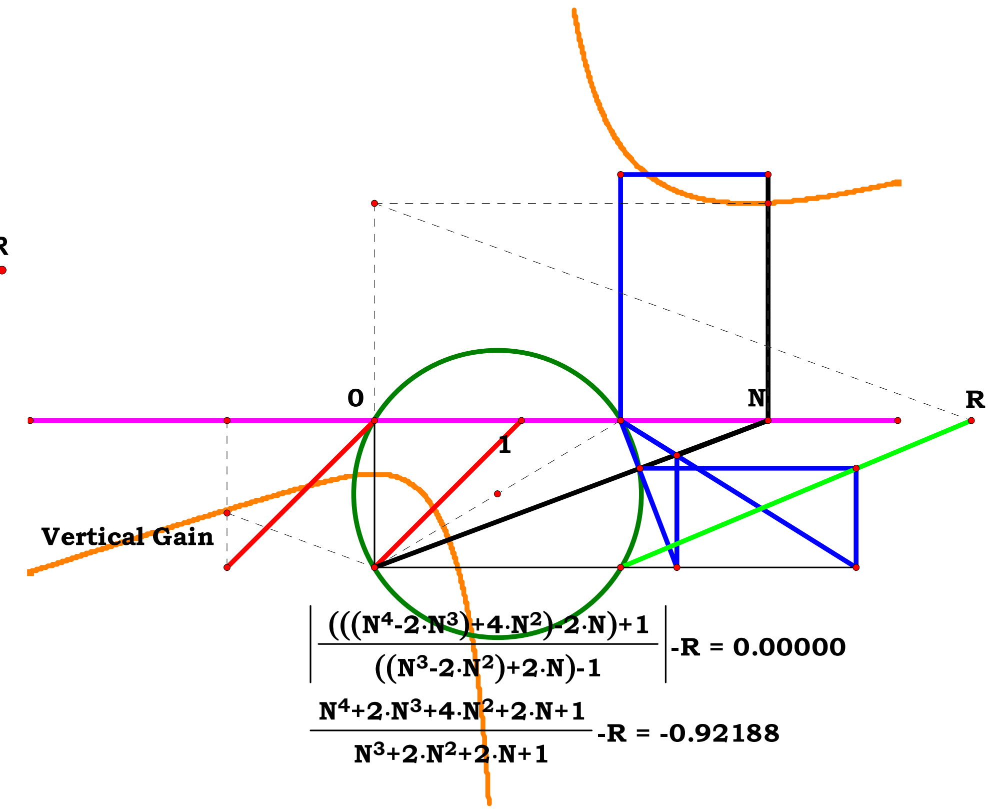
$AE := |AN - 1|$

$$JK := \frac{AN^2 - AN + 1}{AN^2 + 1} \qquad MK := \frac{AB^2}{AN} \qquad ER := \frac{MK \cdot AB}{JK} \qquad AR := ER - AE$$

$$AR - \frac{AN^4 - 2 \cdot AN^3 + 3 \cdot AN^2 - AN + 1}{AN^3 - AN^2 + AN} = 0$$





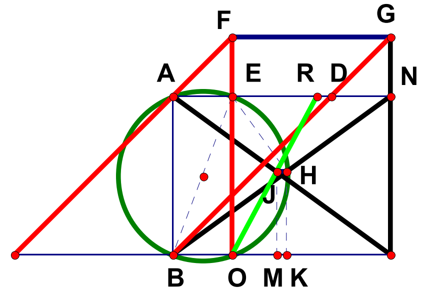


$$AB := 1 \qquad AN := 1.79237$$

$$AE := AN - 1 \qquad PK := \frac{AB^2}{AN}$$

$$MO := \frac{AN^2 - AN + 1}{AN^2 + 1} \qquad BK := \frac{AN^2 - AN + 1}{AN} \qquad JK := \frac{AB \cdot BK}{AN} \qquad PO := \frac{PK \cdot AB}{AB - JK}$$

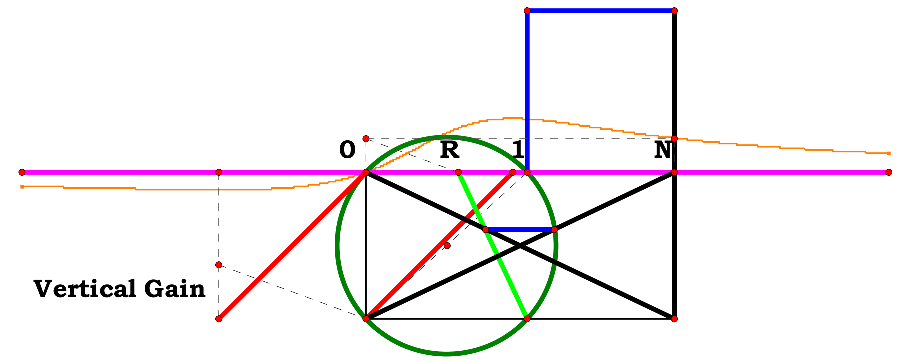
$$ER := \frac{PO \cdot AB}{MO} \qquad AR := ER + AE \qquad AR - \frac{AN^4 - 2AN^3 + 4AN^2 - 2AN + 1}{AN^3 - 2AN^2 + 2AN - 1} = 0$$



$AB := 1$

$AN := 1.44321$

$AE := AN - 1$

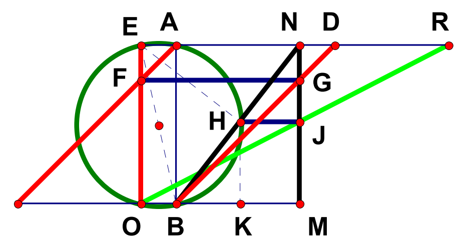


$$\frac{N}{(N^2-N)+1}-R = 0.00000$$

$$\frac{N}{N^2+N+1}-R = -0.35419$$

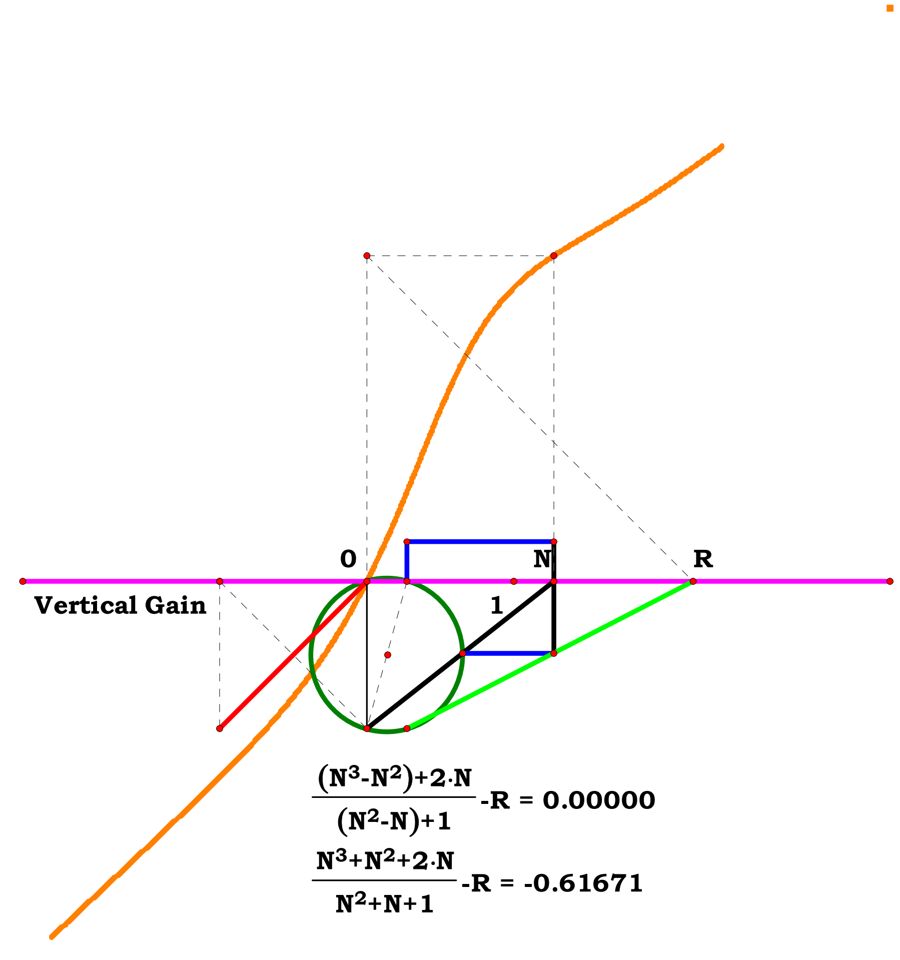
$$BK := \frac{AN^3 - AN^2 + AN}{AN^2 + 1} \qquad HK := \frac{AN^2 - AN + 1}{AN^2 + 1} \qquad MO := AN - BK - AE$$

$$ER := \frac{MO \cdot AB}{HK} \qquad AR := ER + AE \qquad AR - \frac{AN}{AN^2 - AN + 1} = 0$$



$AB := 1 \quad AN := 1.58883 \quad AE := AN - 1 \quad HK := \dfrac{AN^2 - AN + 1}{AN^2 + 1}$

$MO := AN - AE \quad AR := \dfrac{MO \cdot AB}{HK} + AE \quad AR - \dfrac{AN^3 - AN^2 + 2 \cdot AN}{AN^2 - AN + 1} = 0$



5CST1

$N_1 = 1.25275$

$N_2 = 0.52747$

$$\frac{N_1 \cdot N_2}{N_1 - N_2} - R_0 = 0$$

$$N_1 - N_2 - R_1 = 0$$

$$\frac{N_1^2 - N_1 \cdot N_2}{N_2} - R_2 = 0$$

$$\frac{N_1 \cdot N_2}{N_1 - 2 \cdot N_2} - R_3 = 0$$

$$\frac{N_1^2 - N_1 \cdot N_2}{N_1 - 2 \cdot N_2} - R_4 = 0$$

$$\frac{N_1^2 - 2 \cdot N_1 \cdot N_2}{N_1 - N_2} - R_5 = 0$$

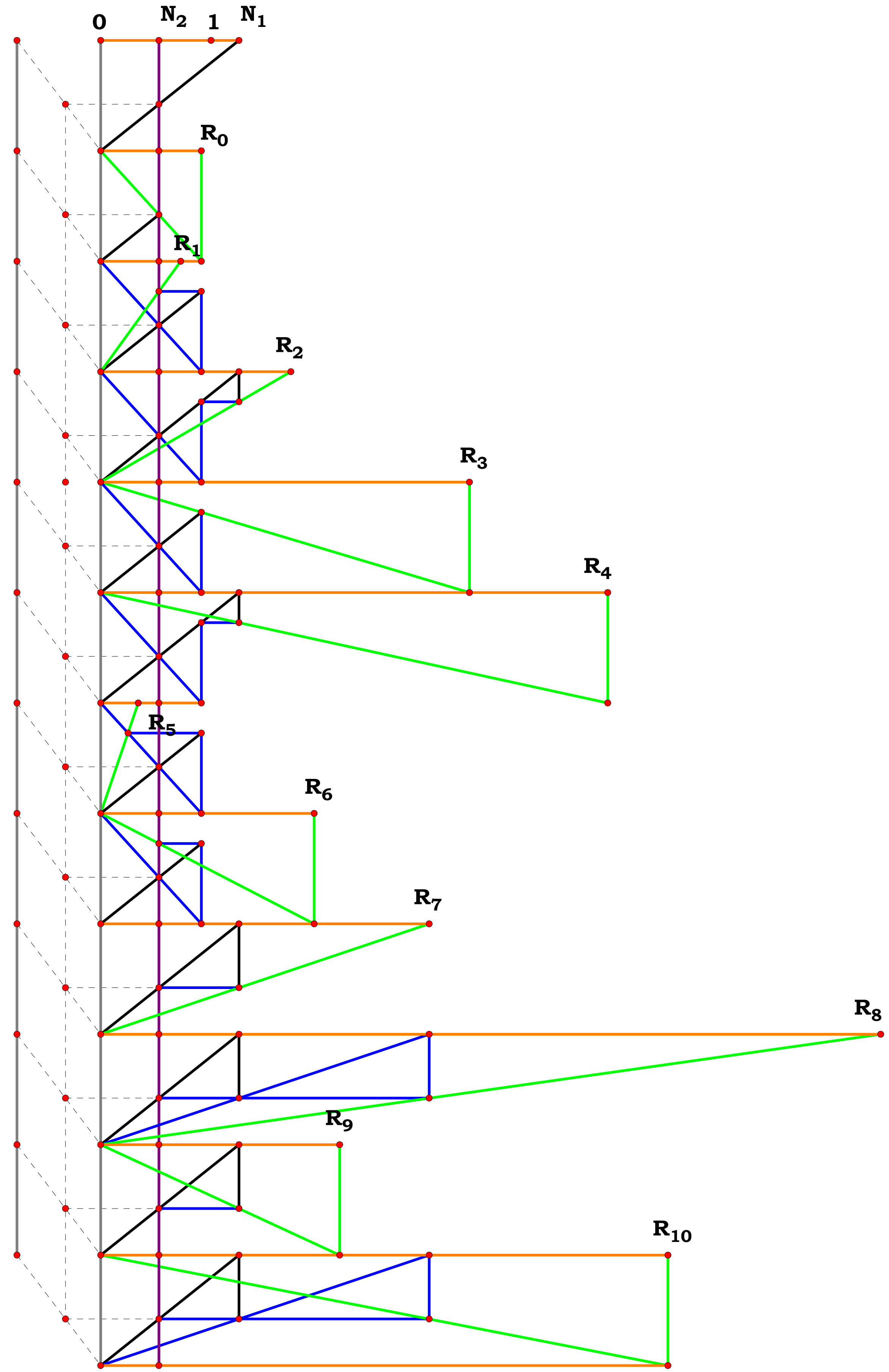
$$\frac{N_1 \cdot N_2 - N_2^2}{N_1 - 2 \cdot N_2} - R_6 = 0$$

$$\frac{N_1^2}{N_2} - R_7 = 0$$

$$\frac{N_1^3}{N_2^2} - R_8 = 0$$

$$\frac{N_1^2}{N_1 - N_2} - R_9 = 0$$

$$\frac{N_1^3}{N_1 \cdot N_2 - N_2^2} - R_{10} = 0$$



5CST2

$N_1 = 2.00000$

$N_2 = 0.90110$

$$\frac{N_1 \cdot N_2}{N_1+N_2}-R_0 = 0$$

$$\frac{N_1 \cdot N_2}{2 \cdot N_1+N_2}-R_1 = 0$$

$$\frac{2 \cdot N_1^2 \cdot N_2+N_1 \cdot N_2^2}{N_1^2+2 \cdot N_1 \cdot N_2+N_2^2}-R_2 = 0$$

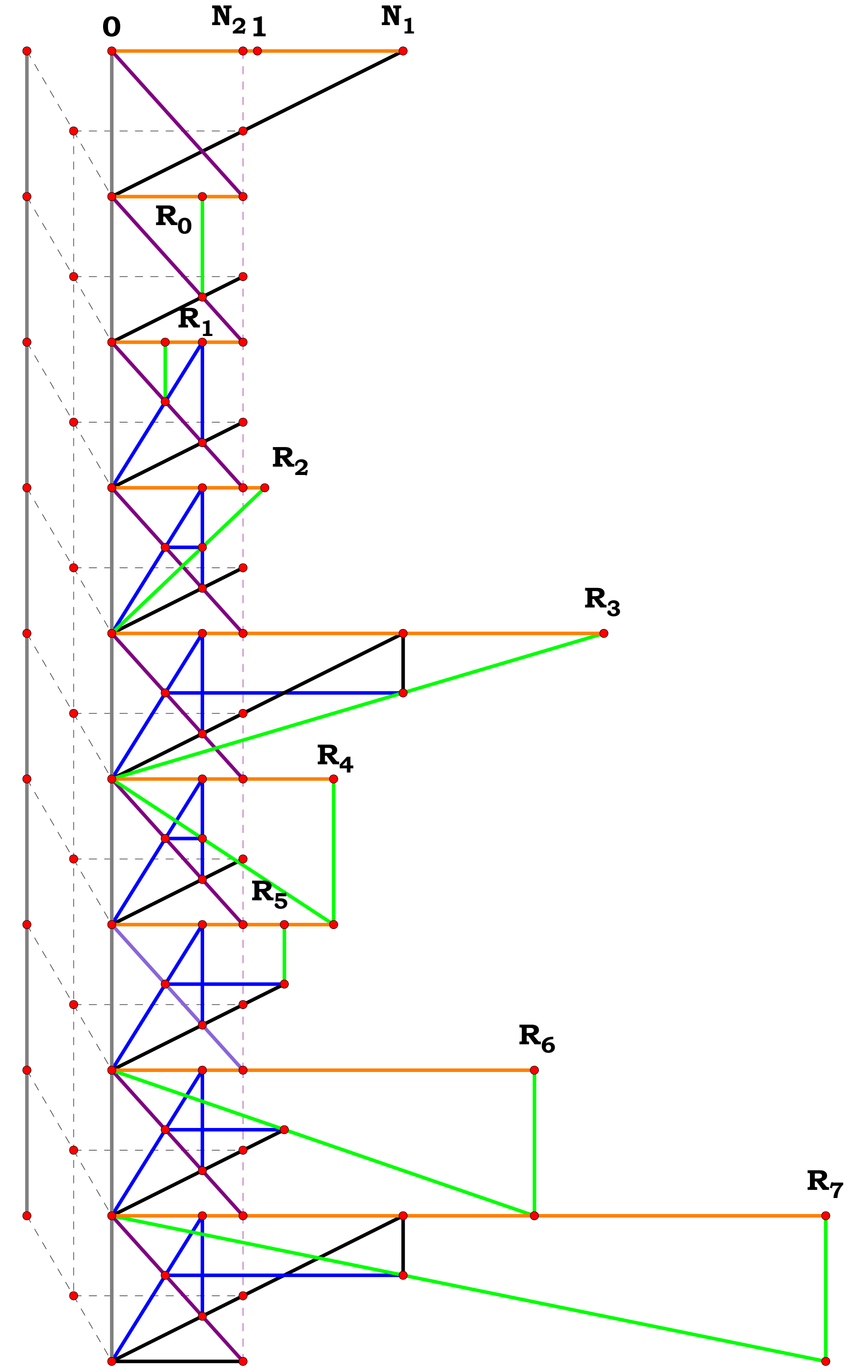
$$\frac{2 \cdot N_1^2+N_1 \cdot N_2}{N_1+N_2}-R_3 = 0$$

$$\frac{2 \cdot N_1 \cdot N_2+N_2^2}{N_1+N_2}-R_4 = 0$$

$$\frac{N_1^2 \cdot N_2+N_1 \cdot N_2^2}{2 \cdot N_1 \cdot N_2+N_2^2}-R_5 = 0$$

$$(N_1+N_2)-R_6 = 0$$

$$(2 \cdot N_1+N_2)-R_7 = 0$$



5CST3A

$N_1 = 2.32967$

$N_2 = 1.73626$

$$\frac{N_2}{N_1} - R_0 = 0$$

$$\frac{N_2}{N_1^2} - R_1 = 0$$

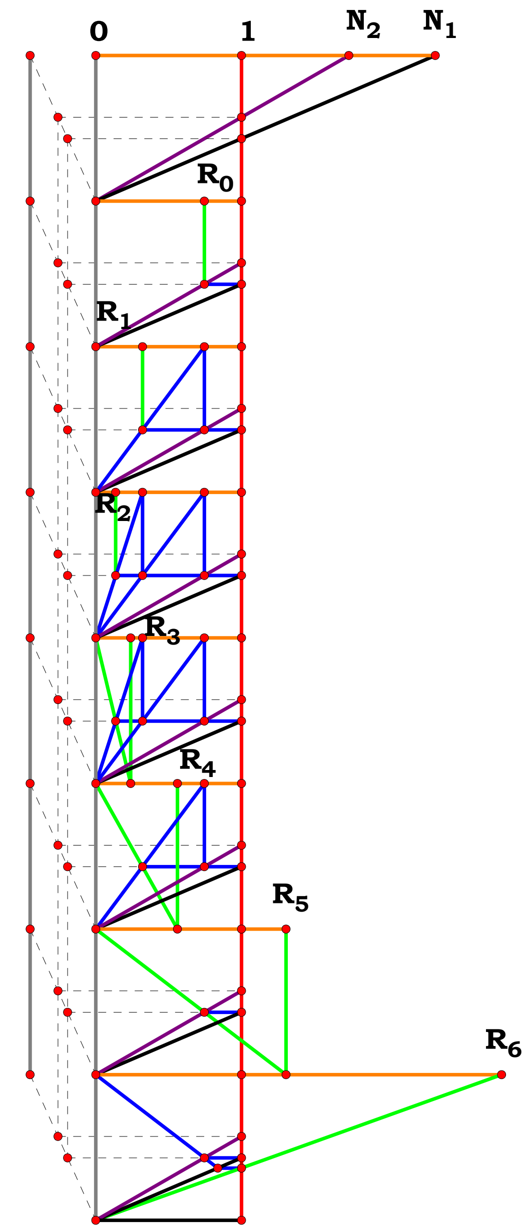
$$\frac{N_2}{N_1^3} - R_2 = 0$$

$$\frac{N_2}{N_1^3 - N_1^2} - R_3 = 0$$

$$\frac{N_2}{N_1^2 - N_1} - R_4 = 0$$

$$\frac{N_2}{N_1 - 1} - R_5 = 0$$

$$\frac{(N_1^2 - N_1) + N_2}{N_2} - R_6 = 0$$



5CST3B

$N_1 = 2.32967$

$N_2 = 1.53846$

$$\frac{N_2}{N_1} - R_0 = 0$$

$$\frac{N_2^2}{N_1^2} - R_1 = 0$$

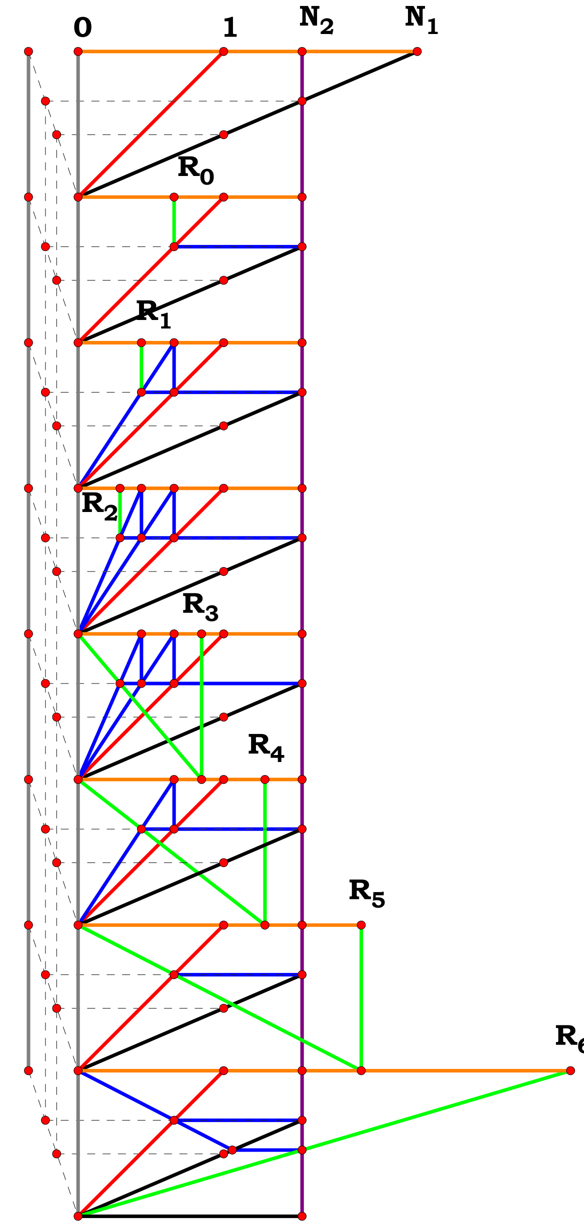
$$\frac{N_2^3}{N_1^3} - R_2 = 0$$

$$\frac{N_2^3}{N_1^3 - N_1^2 \cdot N_2} - R_3 = 0$$

$$\frac{N_2^2}{N_1^2 - N_1 \cdot N_2} - R_4 = 0$$

$$\frac{N_2}{N_1 - N_2} - R_5 = 0$$

$$((N_1^2 - N_1 \cdot N_2) + N_2) - R_6 = 0$$



5CST3C

$N_1 = 2.61538$

$N_2 = 1.48352$

$$\frac{N_2^2}{N_1} - R_0 = 0$$

$$\frac{N_2^3}{N_1^2} - R_1 = 0$$

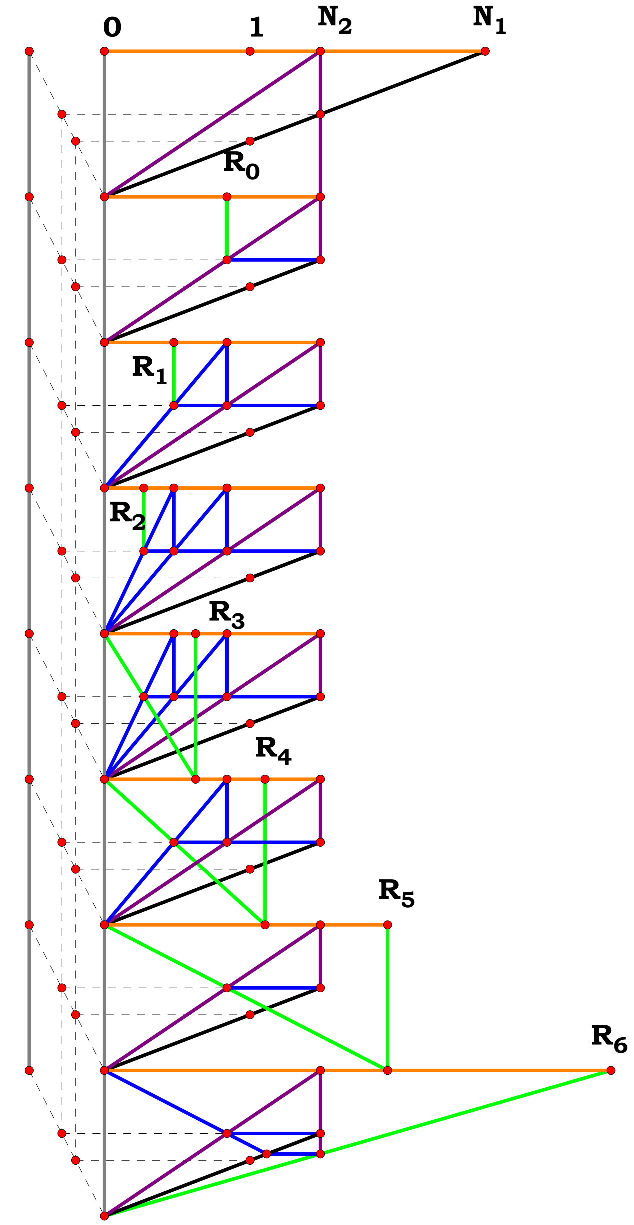
$$\frac{N_2^4}{N_1^3} - R_2 = 0$$

$$\frac{N_2^4}{N_1^3 - N_1^2 \cdot N_2} - R_3 = 0$$

$$\frac{N_2^3}{N_1^2 - N_1 \cdot N_2} - R_4 = 0$$

$$\frac{N_2^2}{N_1 - N_2} - R_5 = 0$$

$$\frac{(N_1^2 - N_1 \cdot N_2) + N_2^2}{N_2} - R_6 = 0$$



5CST4A

$N_1 = 1.97802$

$N_2 = 1.14286$

$$\frac{N_1 - N_2}{N_1} - R_0 = 0$$

$$\frac{N_1 - N_2}{N_2} - R_1 = 0$$

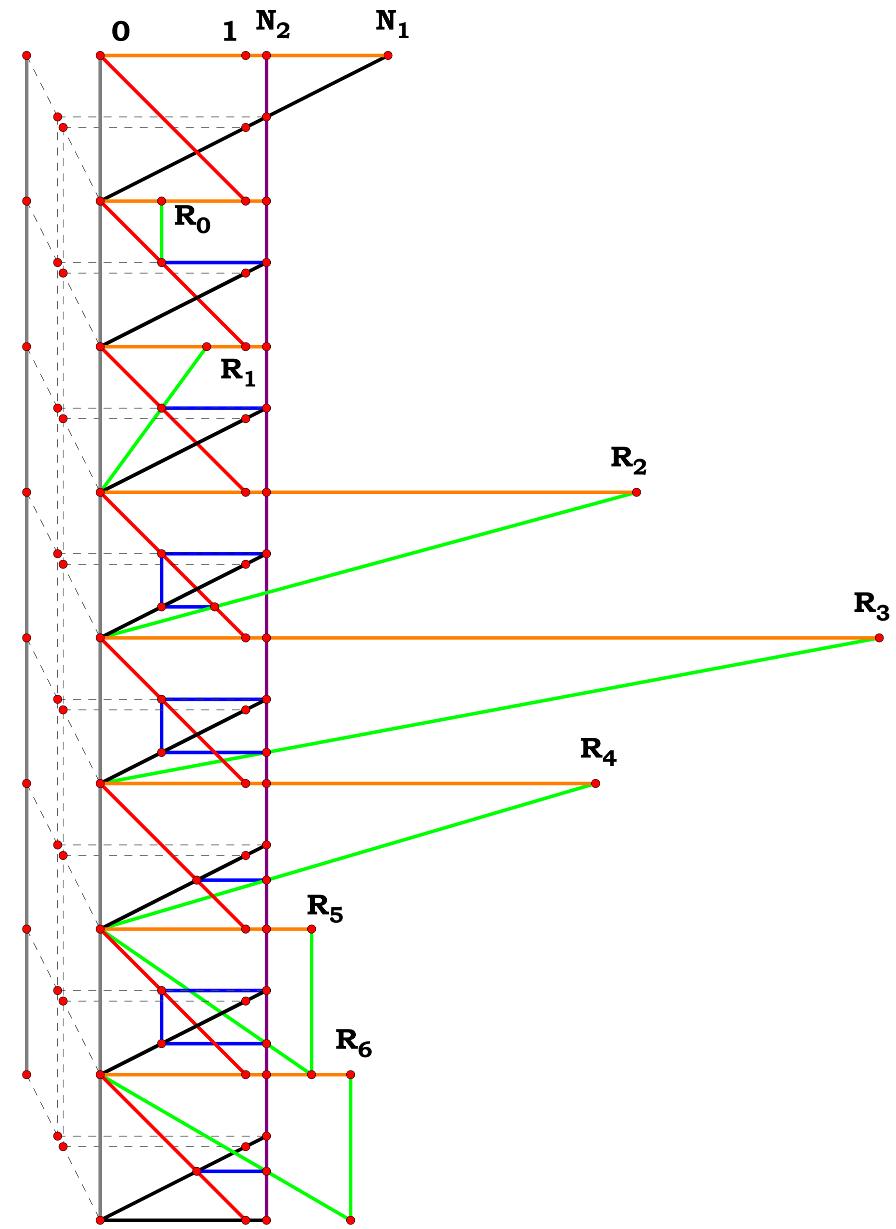
$$\frac{(N_1^2 - N_1) + N_2}{N_1 - N_2} - R_2 = 0$$

$$\frac{N_1^2 \cdot N_2}{N_1 - N_2} - R_3 = 0$$

$$(N_1 \cdot N_2 + N_2) - R_4 = 0$$

$$\frac{N_1^2 \cdot N_2}{(N_1^2 - N_1) + N_2} - R_5 = 0$$

$$\frac{N_1 \cdot N_2 + N_2}{N_1} - R_6 = 0$$



5CST4B

$N_1 = 1.83516$

$N_2 = 1.30769$

$$\frac{N_1 \cdot N_2 - N_2}{N_1} - R_0 = 0$$

$$N_1 \cdot N_2 - N_2 - R_1 = 0$$

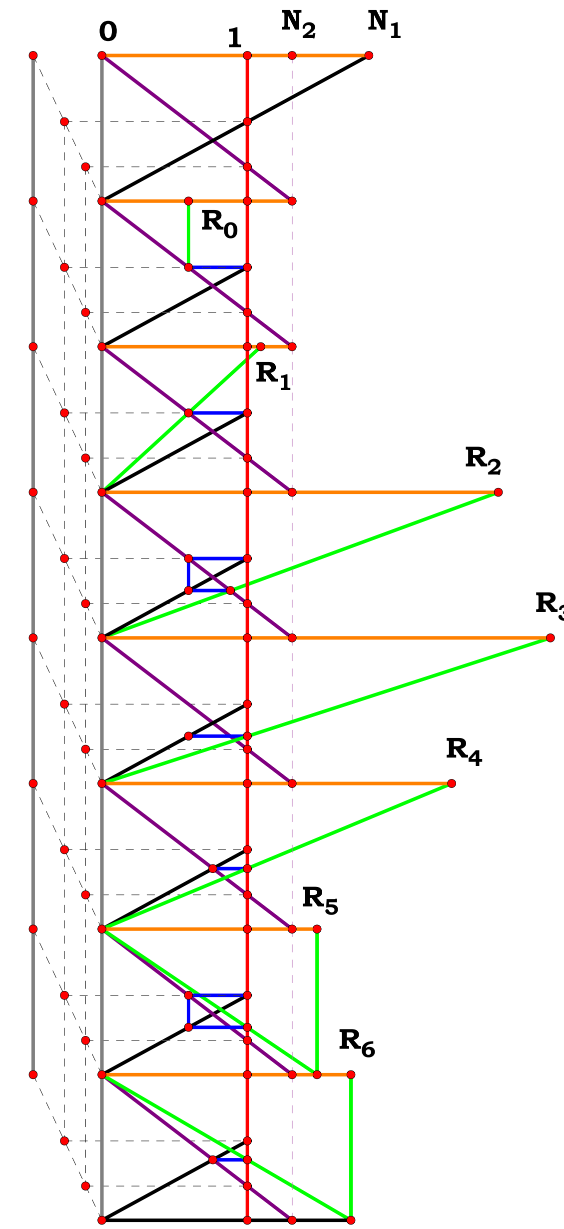
$$\frac{(N_1^2 - N_1 \cdot N_2) + N_2}{N_1 - 1} - R_2 = 0$$

$$\frac{N_1^2}{N_1 \cdot N_2 - N_2} - R_3 = 0$$

$$\frac{N_1 + N_2}{N_2} - R_4 = 0$$

$$\frac{N_1^2}{(N_1^2 - N_1 \cdot N_2) + N_2} - R_5 = 0$$

$$\frac{N_1 + N_2}{N_1} - R_6 = 0$$



5CST4C

$N_1 = 2.08791$

$N_2 = 1.21978$

$$\frac{N_1 \cdot N_2 - N_2^2}{N_1} - R_0 = 0$$

$$N_1 - N_2 - R_1 = 0$$

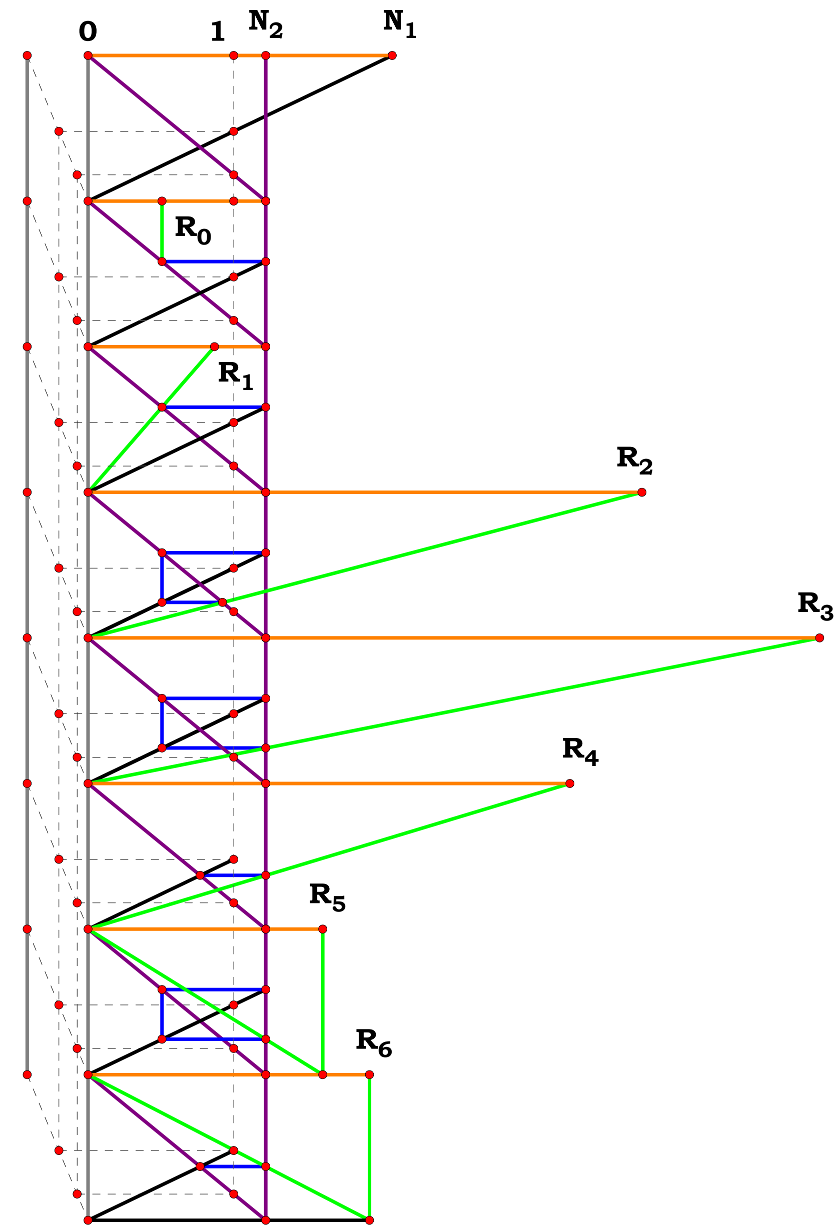
$$\frac{(N_1^2 - N_1 \cdot N_2) + N_2^2}{N_1 - N_2} - R_2 = 0$$

$$\frac{N_1^2}{N_1 - N_2} - R_3 = 0$$

$$(N_1 + N_2) - R_4 = 0$$

$$\frac{N_1^2 \cdot N_2}{(N_1^2 - N_1 \cdot N_2) + N_2^2} - R_5 = 0$$

$$\frac{N_1 \cdot N_2 + N_2^2}{N_1} - R_6 = 0$$



5CST5A

$N_1 = 2.00000$

$N_2 = 1.27473$

$$\frac{N_2^2}{(N_1^2+N_1)-N_2}-R_0 = 0$$

$$\frac{N_2}{N_1+1}-R_1 = 0$$

$$\frac{N_2}{(N_1-N_2)+1}-R_2 = 0$$

$$\frac{N_1^2}{N_1 \cdot N_2+N_2}-R_3 = 0$$

$$\frac{((N_1^3+N_1^2)-N_1 \cdot N_2)+N_2^2}{N_1 \cdot N_2+N_2}-R_4 = 0$$

$$\frac{N_1^2}{N_2}-R_5 = 0$$

$$(N_1 \cdot N_2+N_2)-R_6 = 0$$

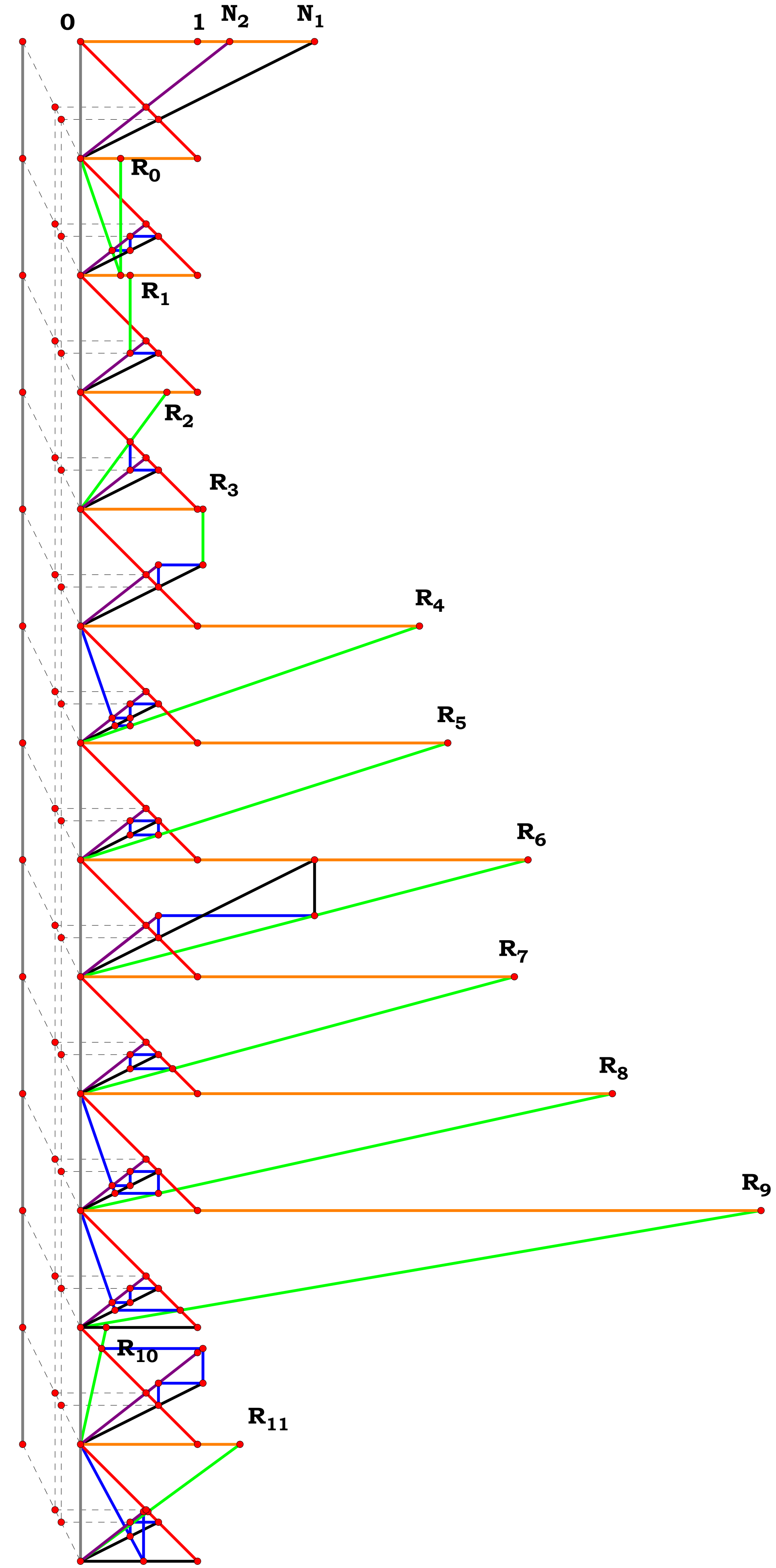
$$\frac{(N_1^2+N_1)-N_2}{N_2}-R_7 = 0$$

$$\frac{((N_1^4+N_1^3)-N_1^2 \cdot N_2)+N_1 \cdot N_2^2}{N_1 \cdot N_2^2+N_2^2}-R_8 = 0$$

$$\frac{(N_1^3+N_1^2)-N_1 \cdot N_2}{N_2^2}-R_9 = 0$$

$$\frac{(N_1 \cdot N_2^2+N_2^2)-N_1^2}{N_1^2}-R_{10} = 0$$

$$\frac{N_1^2-N_2}{N_1}-R_{11} = 0$$



5CST5B

$N_1 = 1.26374$

$N_2 = 1.14286$

$$\frac{N_2}{(N_1^2+N_1 \cdot N_2)-N_2}-R_0 = 0$$

$$\frac{N_2}{N_1+N_2}-R_1 = 0$$

$$\frac{N_2}{(N_1+N_2)-1}-R_2 = 0$$

$$\frac{N_1^2 \cdot N_2}{N_1+N_2}-R_3 = 0$$

$$\frac{((N_1^3+N_1^2 \cdot N_2)-N_1 \cdot N_2)+N_2}{N_1+N_2}-R_4 = 0$$

$$N_1^2-R_5 = 0$$

$$\frac{N_1+N_2}{N_2}-R_6 = 0$$

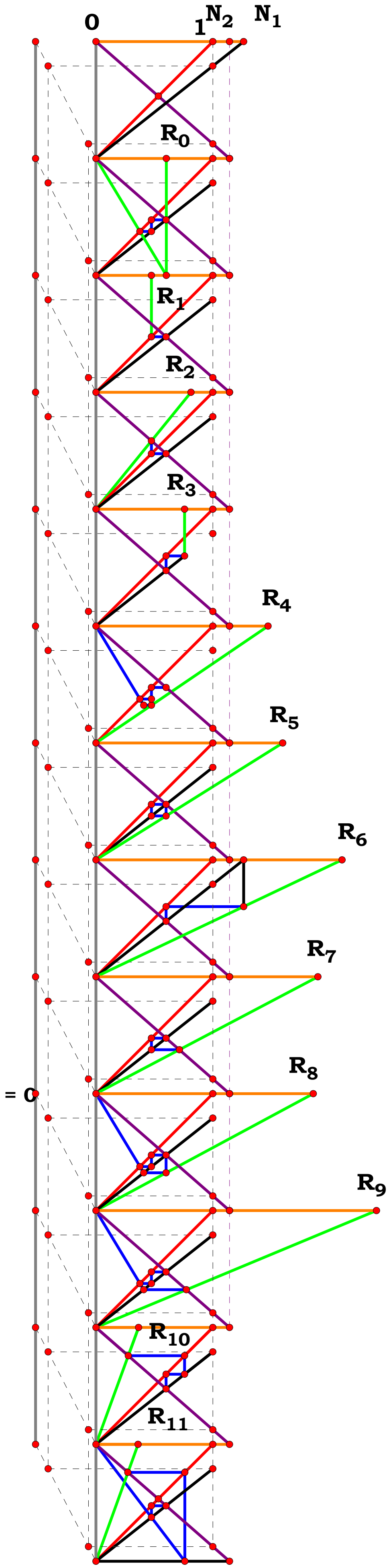
$$(N_1^2+N_1 \cdot N_2)-N_2-R_7 = 0$$

$$\frac{((N_1^4+N_1^3 \cdot N_2)-N_1^2 \cdot N_2)+N_1 \cdot N_2}{N_1+N_2}-R_8 = 0$$

$$(N_1^3+N_1^2 \cdot N_2)-N_1 \cdot N_2-R_9 = 0$$

$$\frac{(N_1+N_2)-N_1^2 \cdot N_2}{N_1^2}-R_{10} = 0$$

$$\frac{N_1^2-N_2}{N_1}-R_{11} = 0$$



5CST5C

$N_1 = 1.87912$

$N_2 = 1.40659$

$$\frac{N_2^3}{(N_1^2+N_1 \cdot N_2)-N_2^2}-R_0 = 0$$

$$\frac{N_2^2}{N_1+N_2}-R_1 = 0$$

$$\frac{N_2^2}{N_1}-R_2 = 0$$

$$\frac{N_1^2}{N_1+N_2}-R_3 = 0$$

$$\frac{((N_1^3+N_1^2 \cdot N_2)-N_1 \cdot N_2^2)+N_2^3}{N_1 \cdot N_2+N_2^2}-R_4 = 0$$

$$\frac{N_1^2}{N_2}-R_5 = 0$$

$$(N_1+N_2)-R_6 = 0$$

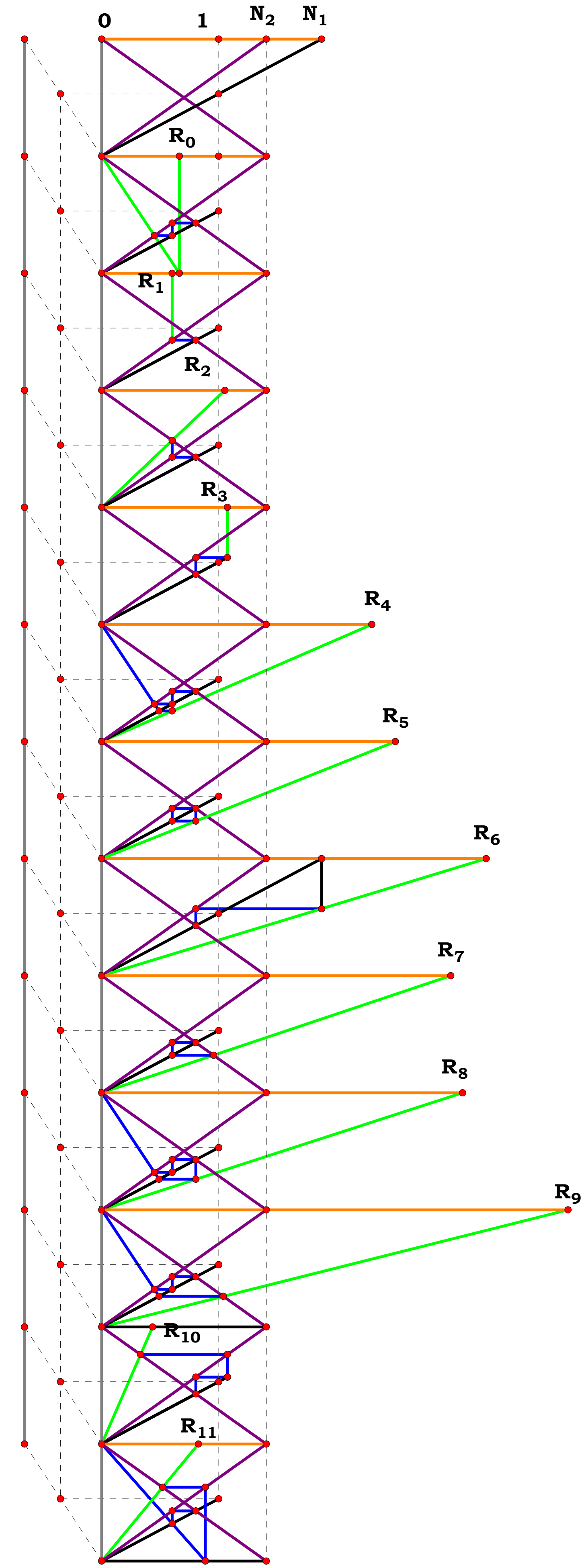
$$\frac{(N_1^2+N_1 \cdot N_2)-N_2^2}{N_2}-R_7 = 0$$

$$\frac{((N_1^4+N_1^3 \cdot N_2)-N_1^2 \cdot N_2^2)+N_1 \cdot N_2^3}{N_1 \cdot N_2^2+N_2^3}-R_8 = 0$$

$$\frac{(N_1^3+N_1^2 \cdot N_2)-N_1 \cdot N_2^2}{N_2^2}-R_9 = 0$$

$$\frac{(N_1 \cdot N_2^2+N_2^3)-N_1^2 \cdot N_2}{N_1^2}-R_{10} = 0$$

$$\frac{N_1^2-N_2^2}{N_1}-R_{11} = 0$$



5CST6A

$N_1 = 1.69231$

$N_2 = 1.26374$

$$\frac{N_1^2 - N_1 \cdot N_2}{N_2} - R_0 = 0$$

$$\frac{N_1 \cdot N_2 + N_2}{N_1^2} - R_1 = 0$$

$$\frac{(N_1^2 \cdot N_2 + N_1 \cdot N_2) - N_2}{N_1} - R_2 = 0$$

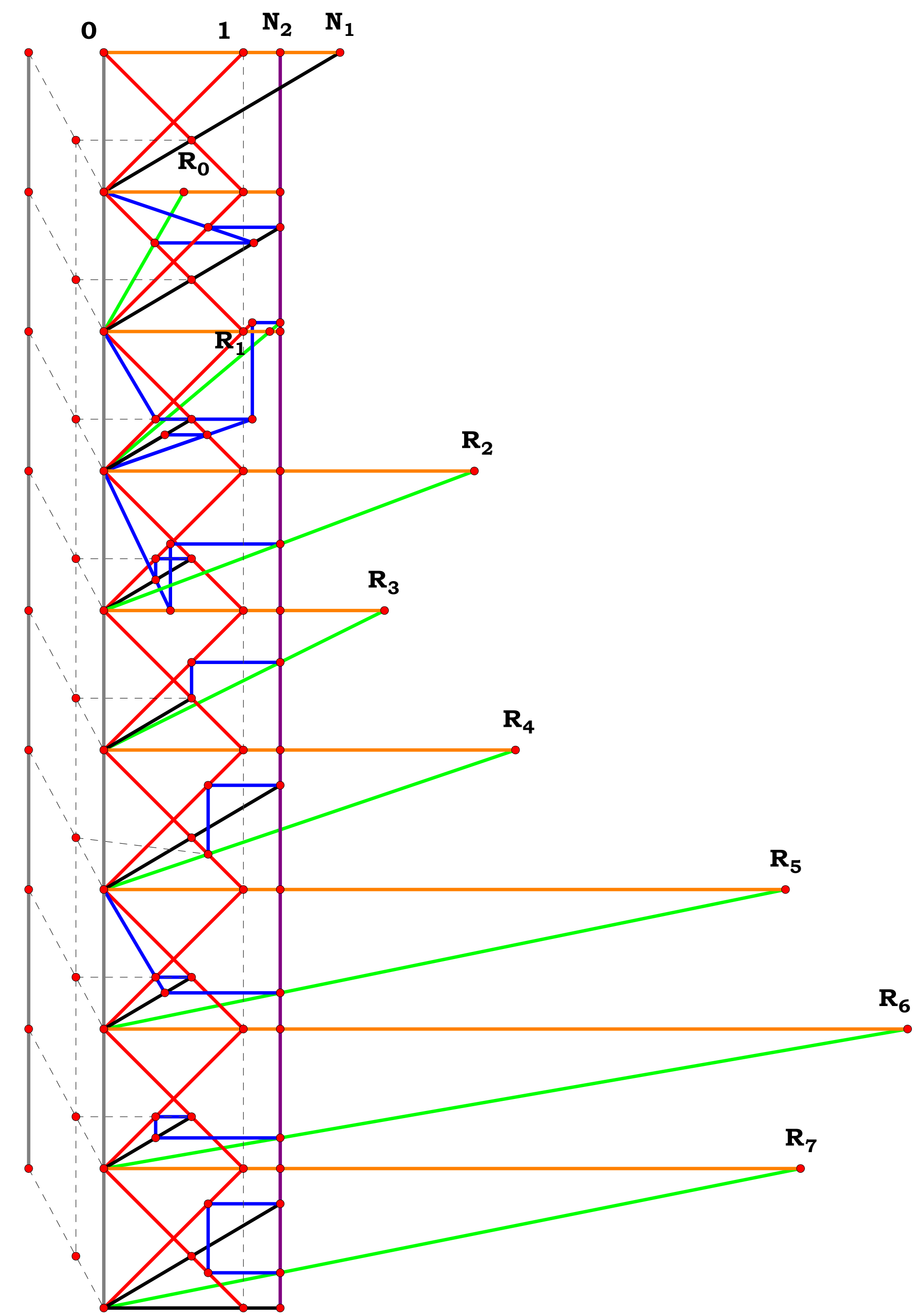
$$\frac{N_1 \cdot N_2 + N_2}{N_1} - R_3 = 0$$

$$\frac{N_2}{N_1 - N_2} - R_4 = 0$$

$$(N_1^2 \cdot N_2 + N_2) - R_5 = 0$$

$$(N_1^2 \cdot N_2 + N_1 \cdot N_2) - R_6 = 0$$

$$\frac{N_1 \cdot N_2}{N_1 - N_2} - R_7 = 0$$



5CST6B

$N_1 = 1.84615$

$N_2 = 1.53846$

$N_1{}^2 \cdot N_2 - N_1 \cdot N_2 - R_0 = 0$

$\frac{N_1+N_2}{N_1{}^2 \cdot N_2} - R_1 = 0$

$\frac{(N_1{}^2+N_1 \cdot N_2)-N_2}{N_1 \cdot N_2} - R_2 = 0$

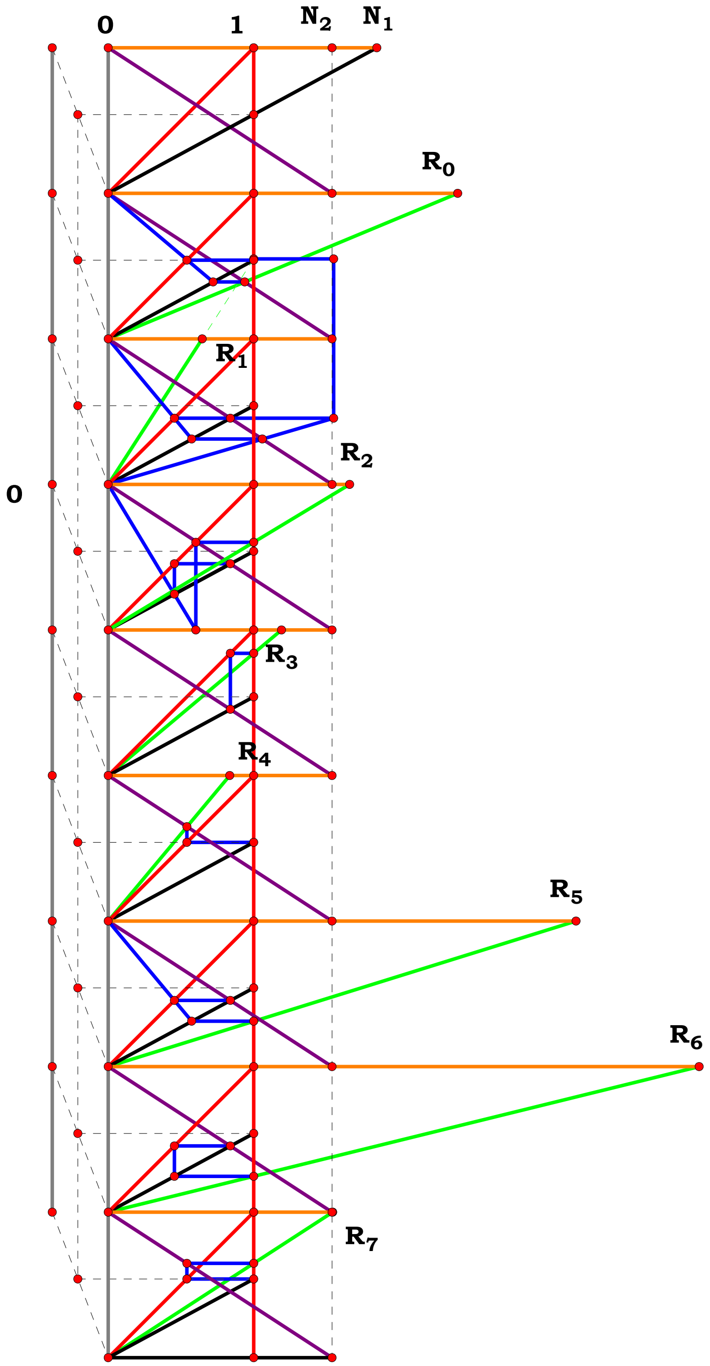
$\frac{N_1+N_2}{N_1 \cdot N_2} - R_3 = 0$

$\frac{N_2}{N_1 \cdot N_2 - 1} - R_4 = 0$

$\frac{N_1{}^2+N_2}{N_2} - R_5 = 0$

$\frac{N_1{}^2+N_1 \cdot N_2}{N_2} - R_6 = 0$

$\frac{N_1 \cdot N_2}{N_1 \cdot N_2 - 1} - R_7 = 0$



5CST6C

$N_1 = 1.59341$

$N_2 = 1.20879$

$$\frac{N_1{}^2 - N_1}{N_2} - R_0 = 0$$

$$\frac{N_1 \cdot N_2{}^2 + N_2{}^2}{N_1{}^2} - R_1 = 0$$

$$\frac{(N_1{}^2 + N_1) - N_2}{N_1} - R_2 = 0$$

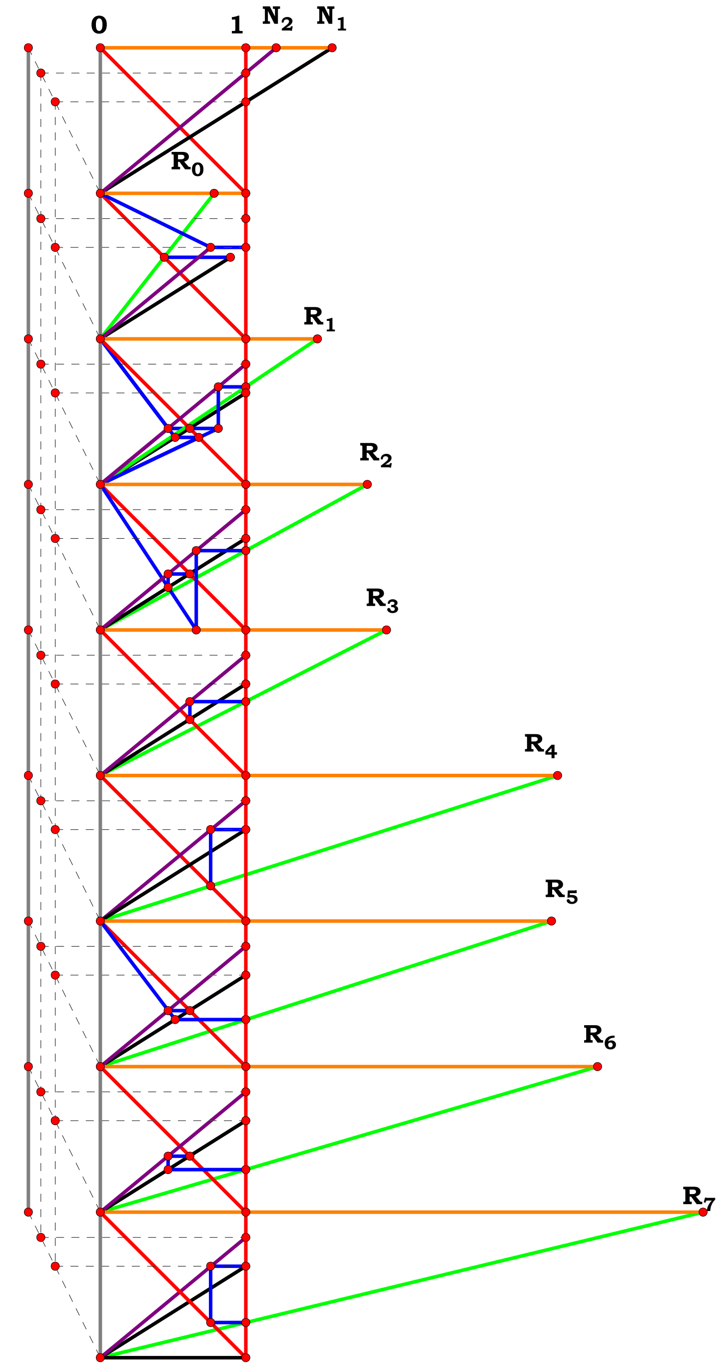
$$\frac{N_1 \cdot N_2 + N_2}{N_1} - R_3 = 0$$

$$\frac{N_2}{N_1 - N_2} - R_4 = 0$$

$$\frac{N_1{}^2 + N_2}{N_2} - R_5 = 0$$

$$\frac{N_1{}^2 + N_1}{N_2} - R_6 = 0$$

$$\frac{N_1}{N_1 - N_2} - R_7 = 0$$



5CST6D

$N_1 = 1.63736$

$N_2 = 1.29670$

$N_1^2 - N_1 \cdot N_2 - R_0 = 0$

$\frac{N_1+N_2}{N_1^2} - R_1 = 0$

$\frac{(N_1^2+N_1 \cdot N_2)-N_2}{N_1} - R_2 = 0$

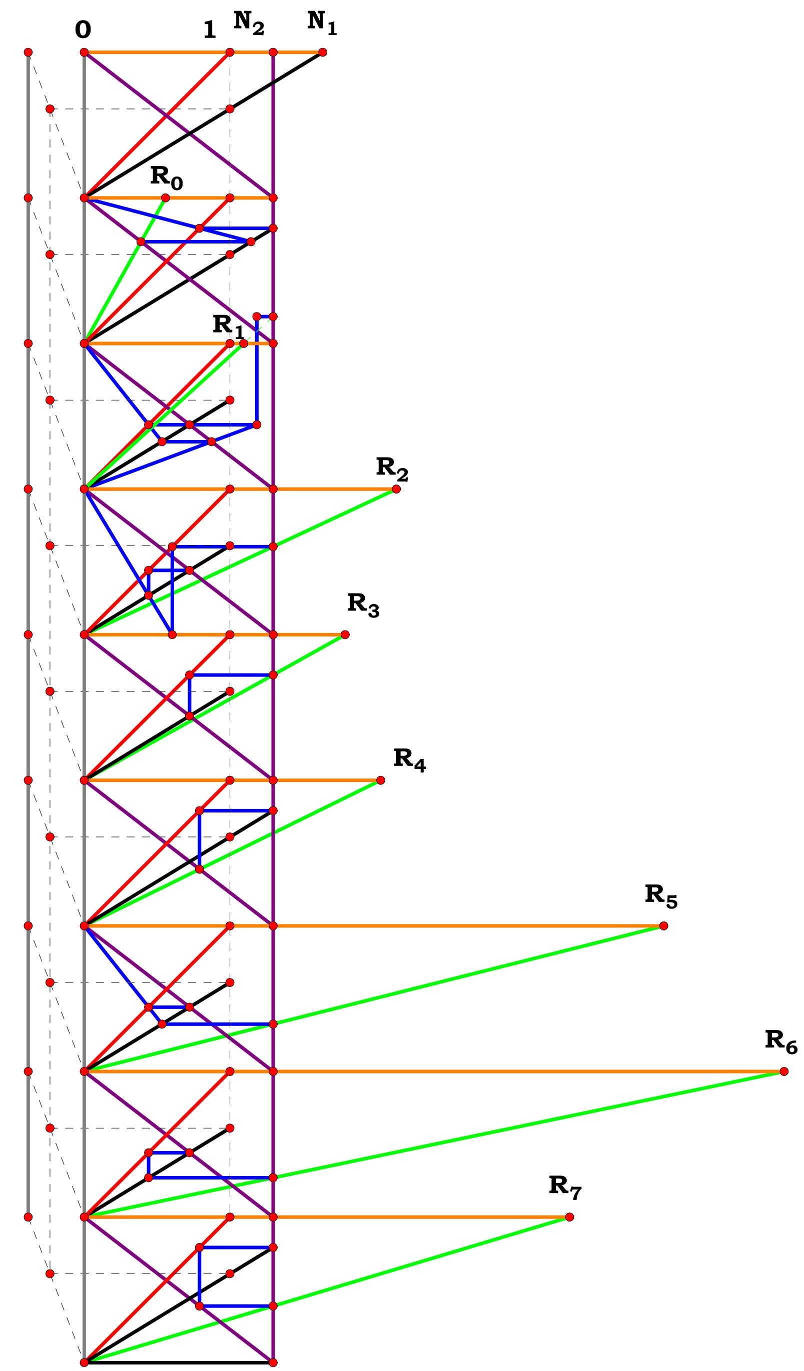
$\frac{N_1+N_2}{N_1} - R_3 = 0$

$\frac{N_2}{N_1-1} - R_4 = 0$

$(N_1^2+N_2) - R_5 = 0$

$(N_1^2+N_1 \cdot N_2) - R_6 = 0$

$\frac{N_1 \cdot N_2}{N_1-1} - R_7 = 0$



5CST6E

$N_1 = 2.58242$

$N_2 = 1.47253$

$$\frac{N_1^2 - N_1 \cdot N_2}{N_2^2} - R_0 = 0$$

$$\frac{N_1 \cdot N_2^3 + N_2^3}{N_1^2} - R_1 = 0$$

$$\frac{\left(N_1^2 \cdot N_2 + N_1 \cdot N_2\right) - N_2^2}{N_1} - R_2 = 0$$

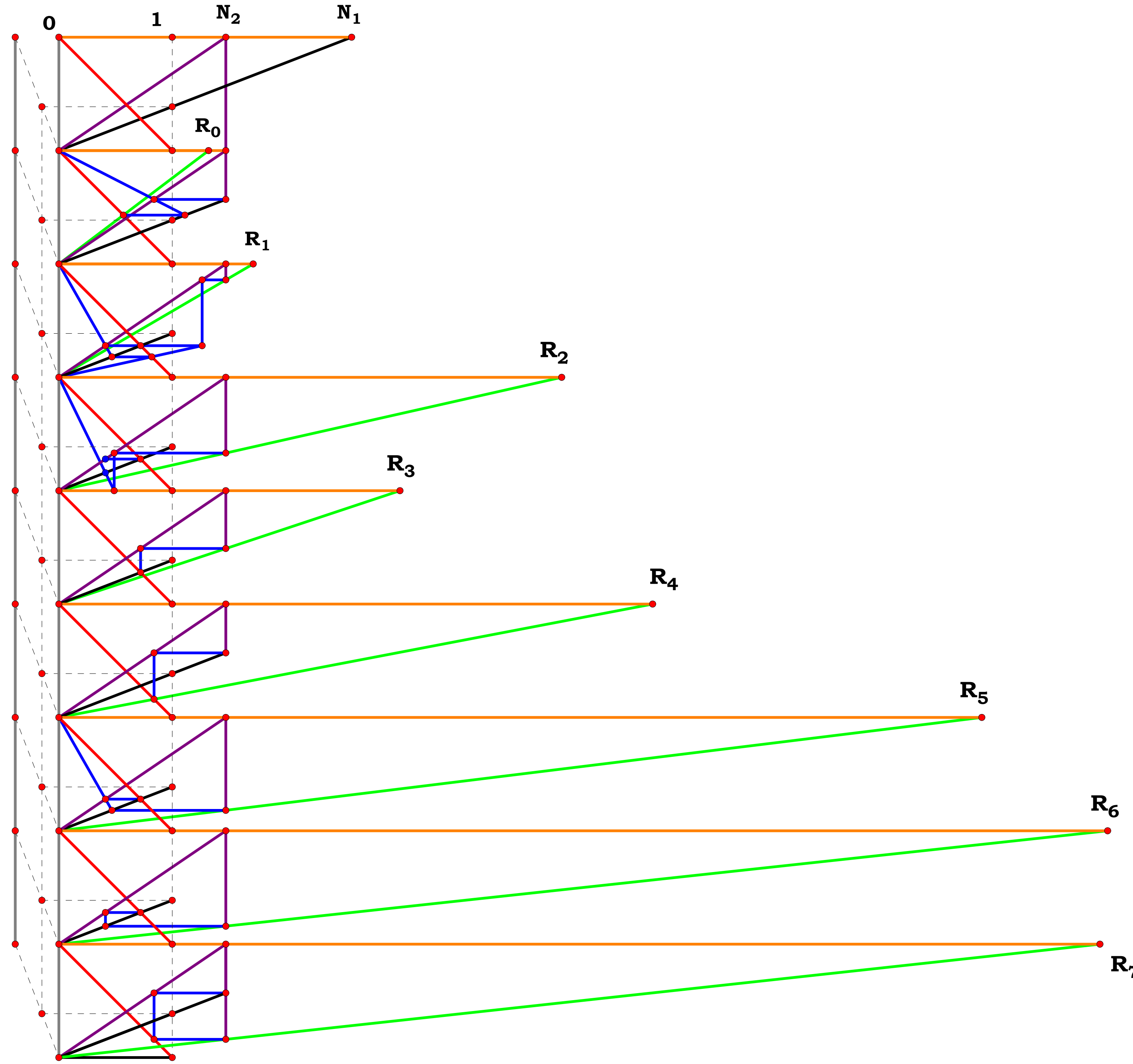
$$\frac{N_1 \cdot N_2^2 + N_2^2}{N_1} - R_3 = 0$$

$$\frac{N_2^2}{N_1 - N_2^2} - R_4 = 0$$

$$\left(N_1^2 + N_2\right) - R_5 = 0$$

$$\left(N_1^2 + N_1\right) - R_6 = 0$$

$$\frac{N_1 \cdot N_2}{N_1 - N_2^2} - R_7 = 0$$



5CST6F

$N_1 = 1.37363$

$N_2 = 0.89011$

$N_1^2 - N_1 - R_0 = 0$

$$\frac{N_1 \cdot N_2 + N_2^2}{N_1^2} - R_1 = 0$$

$$\frac{(N_1^2 + N_1 \cdot N_2) - N_2^2}{N_1 \cdot N_2} - R_2 = 0$$

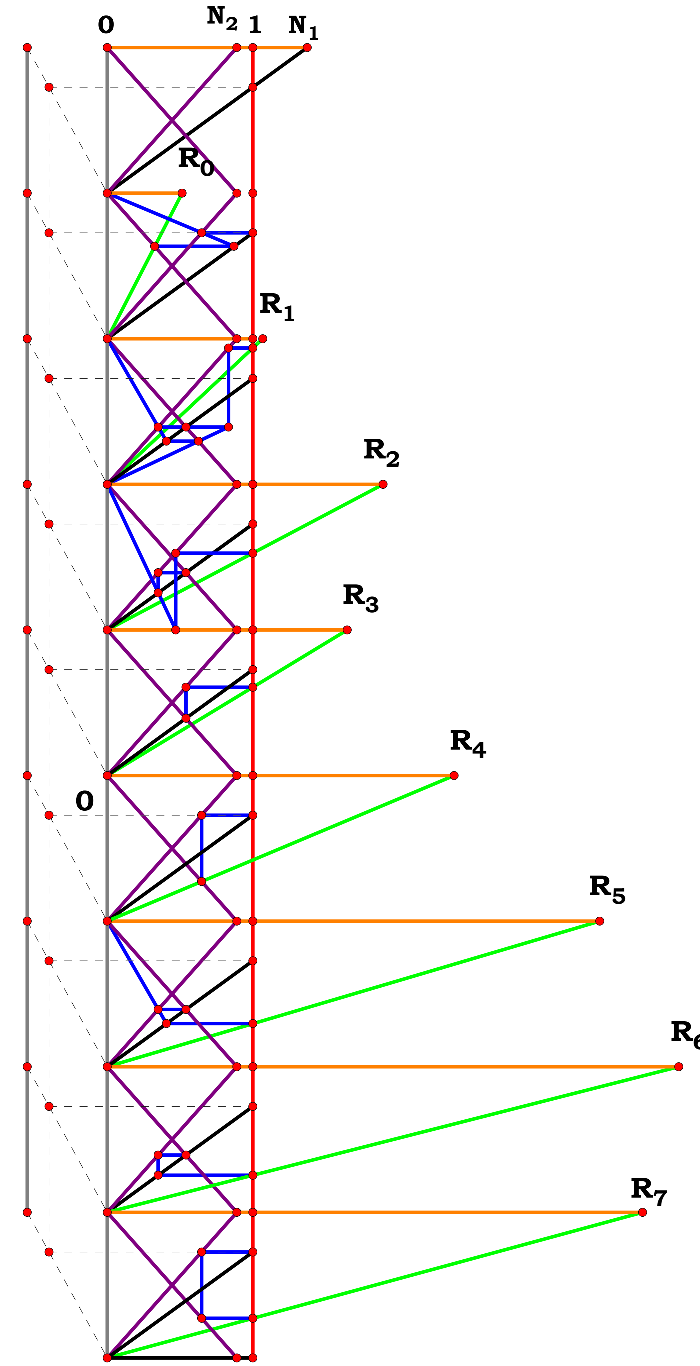
$$\frac{N_1 + N_2}{N_1} - R_3 = 0$$

$$\frac{N_2}{N_1 - 1} - R_4 = 0$$

$$\frac{N_1^2 + N_2^2}{N_2^2} - R_5 = 0$$

$$\frac{N_1^2 + N_1 \cdot N_2}{N_2^2} - R_6 = 0$$

$$\frac{N_1}{N_1 - 1} - R_7 = 0$$



5CST6G

$N_1 = 1.56044$

$N_2 = 1.07692$

$$\frac{N_1^2 - N_1 \cdot N_2}{N_2} - R_0 = 0$$

$$\frac{N_1 \cdot N_2^2 + N_2^3}{N_1^2} - R_1 = 0$$

$$\frac{(N_1^2 + N_1 \cdot N_2) - N_2^2}{N_1} - R_2 = 0$$

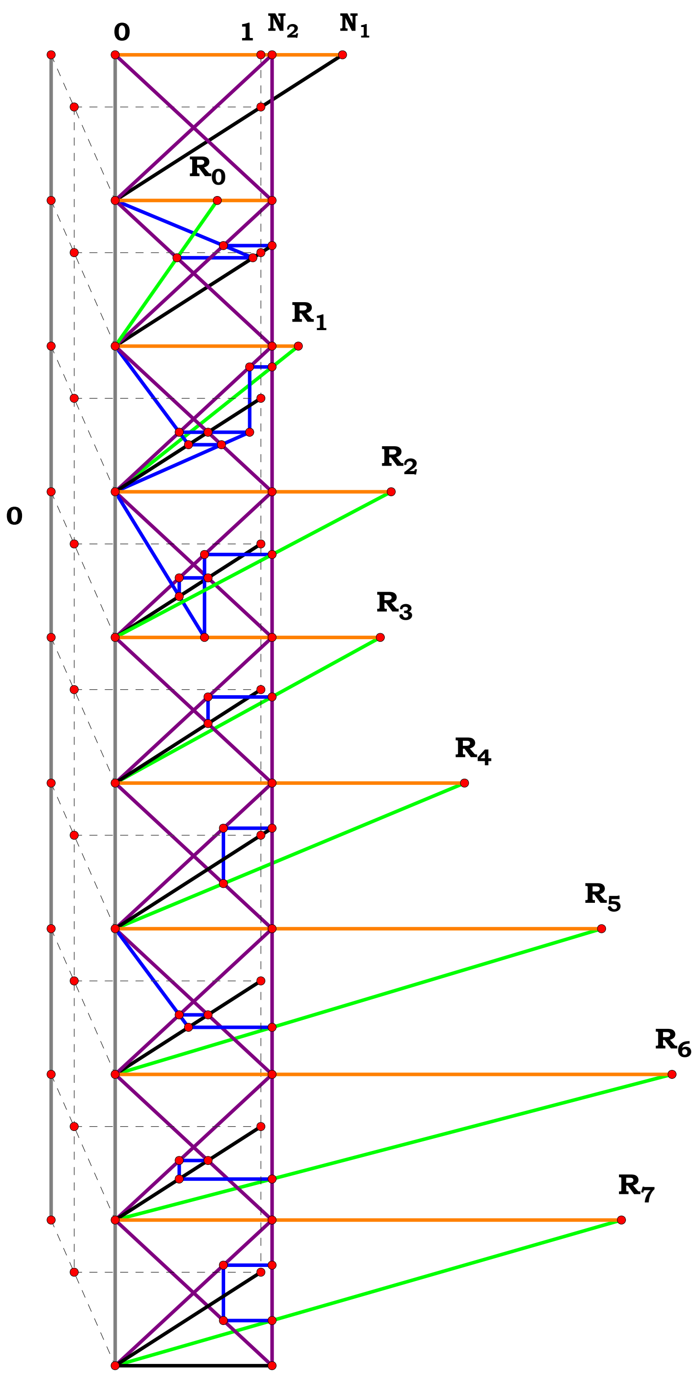
$$\frac{N_1 \cdot N_2 + N_2^2}{N_1} - R_3 = 0$$

$$\frac{N_2^2}{N_1 - N_2} - R_4 = 0$$

$$\frac{N_1^2 + N_2^2}{N_2} - R_5 = 0$$

$$\frac{N_1^2 + N_1 \cdot N_2}{N_2} - R_6 = 0$$

$$\frac{N_1 \cdot N_2}{N_1 - N_2} - R_7 = 0$$



5CST7A

$N_1 = 3.24176$

$N_2 = 1.21978$

$$\frac{N_1 \cdot N_2 - N_2^2}{N_1^2} - R_0 = 0$$

$$\frac{N_1 - N_2}{N_1} - R_1 = 0$$

$$\frac{N_1 \cdot N_2}{(N_1 \cdot N_2 + N_1) - N_2} - R_2 = 0$$

$$\frac{(N_1 \cdot N_2 + N_1 \cdot N_2^2) - N_2^2}{(N_1 \cdot N_2^2 + N_1 \cdot N_2 + N_1) - N_2 - N_2^2} - R_3 = 0$$

$$\frac{N_1^2}{(N_1^2 + N_1) - N_2} - R_4 = 0$$

$$\frac{(N_1^2 \cdot N_2 + N_1 \cdot N_2) - N_2^2}{(N_1^2 \cdot N_2 + N_1 \cdot N_2 + N_1) - N_2 - N_2^2} - R_5 = 0$$

$$\frac{(N_1^2 \cdot N_2 + N_1 \cdot N_2) - N_2^2}{N_1^2} - R_6 = 0$$

$$\frac{N_1^2 \cdot N_2}{(N_1^2 - N_1 \cdot N_2) + N_2^2} - R_7 = 0$$

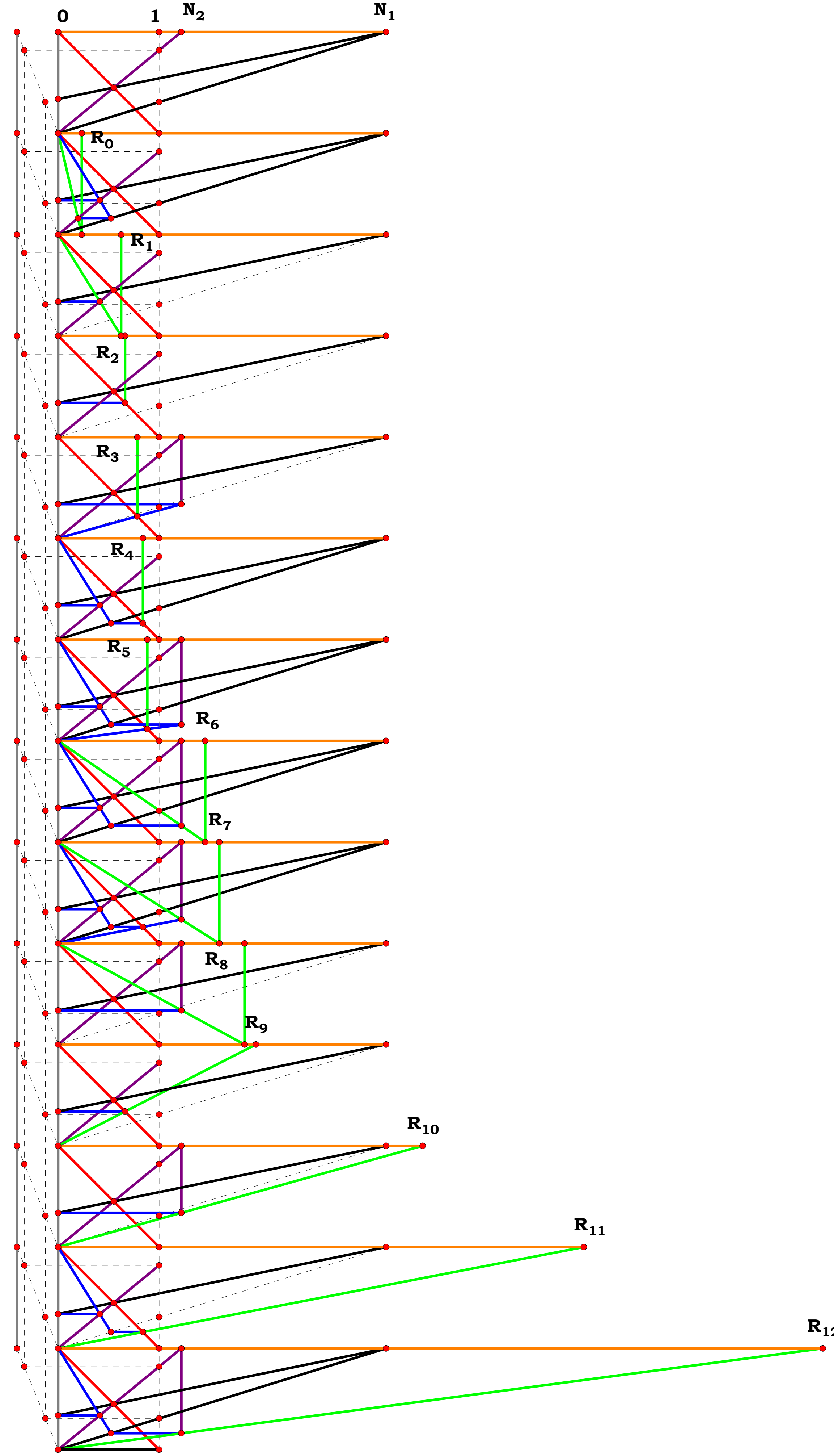
$$\frac{(N_1 + N_1 \cdot N_2) - N_2}{N_1} - R_8 = 0$$

$$\frac{N_1 \cdot N_2}{N_1 - N_2} - R_9 = 0$$

$$\frac{(N_1 \cdot N_2^2 + N_1 \cdot N_2) - N_2^2}{N_1 - N_2} - R_{10} = 0$$

$$\frac{N_1^2}{N_1 - N_2} - R_{11} = 0$$

$$\frac{(N_1^2 \cdot N_2 + N_1 \cdot N_2) - N_2^2}{N_1 - N_2} - R_{12} = 0$$



5CST7B

$N_1 = 2.54945$

$N_2 = 1.97802$

$$\frac{N_1 \cdot N_2 - N_2}{N_1^2} - R_0 = 0$$

$$\frac{N_1 \cdot N_2 - N_2}{N_1} - R_1 = 0$$

$$\frac{N_1 \cdot N_2}{(N_1 \cdot N_2 + N_1) - N_2} - R_2 = 0$$

$$\frac{(N_1 \cdot N_2^2 + N_1 \cdot N_2) - N_2^2}{(2 \cdot N_1 \cdot N_2 + N_1) - 2 \cdot N_2} - R_3 = 0$$

$$\frac{N_1^2 \cdot N_2}{(N_1^2 + N_1 \cdot N_2) - N_2} - R_4 = 0$$

$$\frac{(N_1^2 \cdot N_2 + N_1 \cdot N_2^2) - N_2^2}{(N_1^2 + 2 \cdot N_1 \cdot N_2) - 2 \cdot N_2} - R_5 = 0$$

$$\frac{(N_1^2 \cdot N_2 + N_1 \cdot N_2^2) - N_2^2}{N_1^2} - R_6 = 0$$

$$\frac{N_1^2 \cdot N_2}{(N_1^2 - N_1 \cdot N_2) + N_2} - R_7 = 0$$

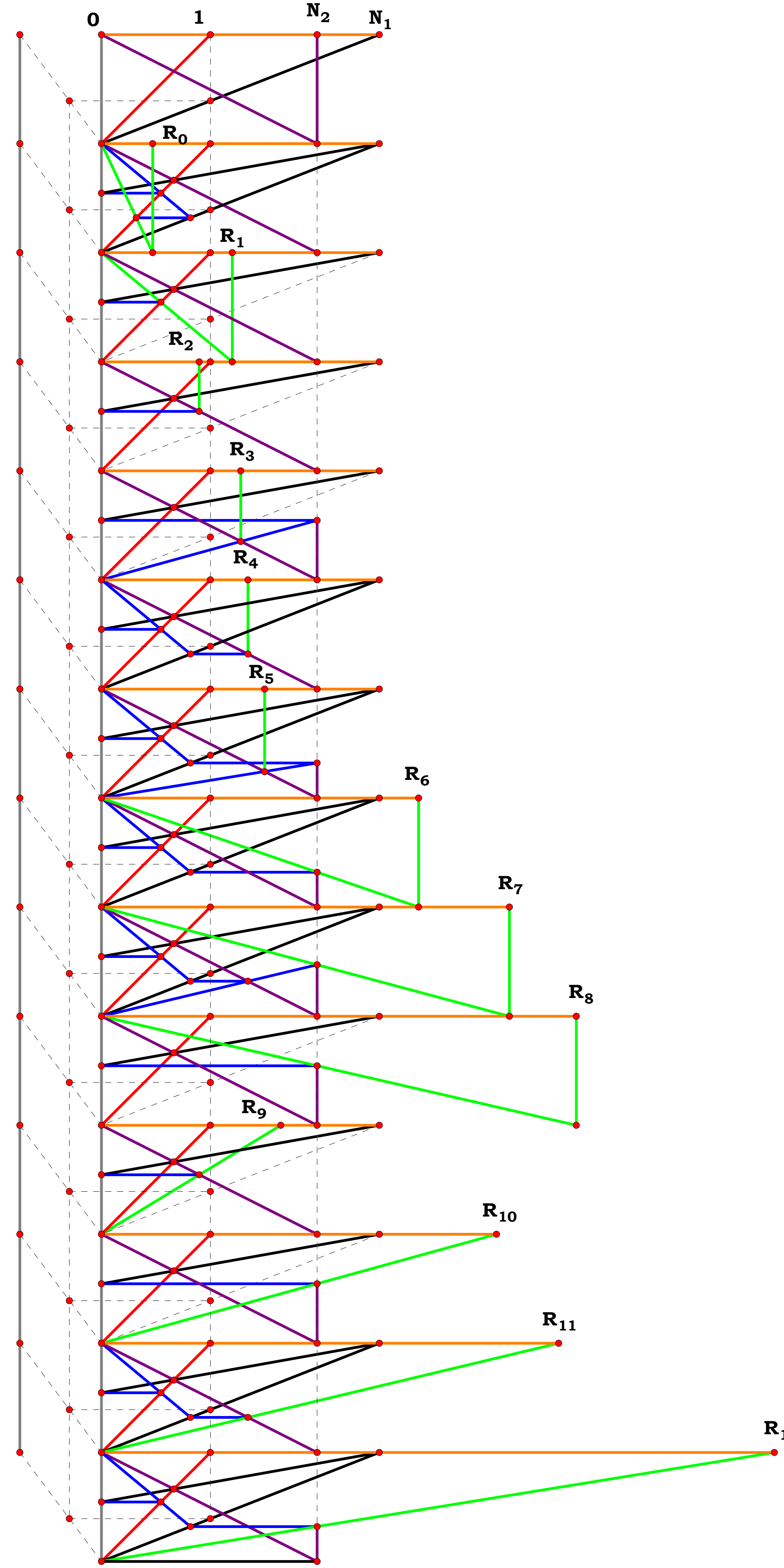
$$\frac{(N_1 \cdot N_2^2 + N_1 \cdot N_2) - N_2^2}{N_1} - R_8 = 0$$

$$\frac{N_1}{N_1 - 1} - R_9 = 0$$

$$\frac{(N_1 + N_1 \cdot N_2) - N_2}{N_1 - 1} - R_{10} = 0$$

$$\frac{N_1^2}{N_1 - 1} - R_{11} = 0$$

$$\frac{(N_1^2 + N_1 \cdot N_2) - N_2}{N_1 - 1} - R_{12} = 0$$



5CST7C

$N_1 = 3.36264$

$N_2 = 1.47253$

$$\frac{N_1 \cdot N_2^2 - N_2^3}{N_1^2} - R_0 = 0$$

$$\frac{N_1 \cdot N_2 - N_2^2}{N_1} - R_1 = 0$$

$$\frac{N_1 \cdot N_2}{2 \cdot N_1 - N_2} - R_2 = 0$$

$$\frac{2 \cdot N_1 \cdot N_2 - N_2^2}{3 \cdot N_1 - 2 \cdot N_2} - R_3 = 0$$

$$\frac{N_1^2 \cdot N_2}{(N_1^2 + N_1 \cdot N_2) - N_2^2} - R_4 = 0$$

$$\frac{(N_1^2 \cdot N_2 + N_1 \cdot N_2^2) - N_2^3}{(N_1^2 + 2 \cdot N_1 \cdot N_2) - 2 \cdot N_2^2} - R_5 = 0$$

$$\frac{(N_1^2 \cdot N_2 + N_1 \cdot N_2^2) - N_2^3}{N_1^2} - R_6 = 0$$

$$\frac{N_1^2 \cdot N_2}{(N_1^2 - N_1 \cdot N_2) + N_2^2} - R_7 = 0$$

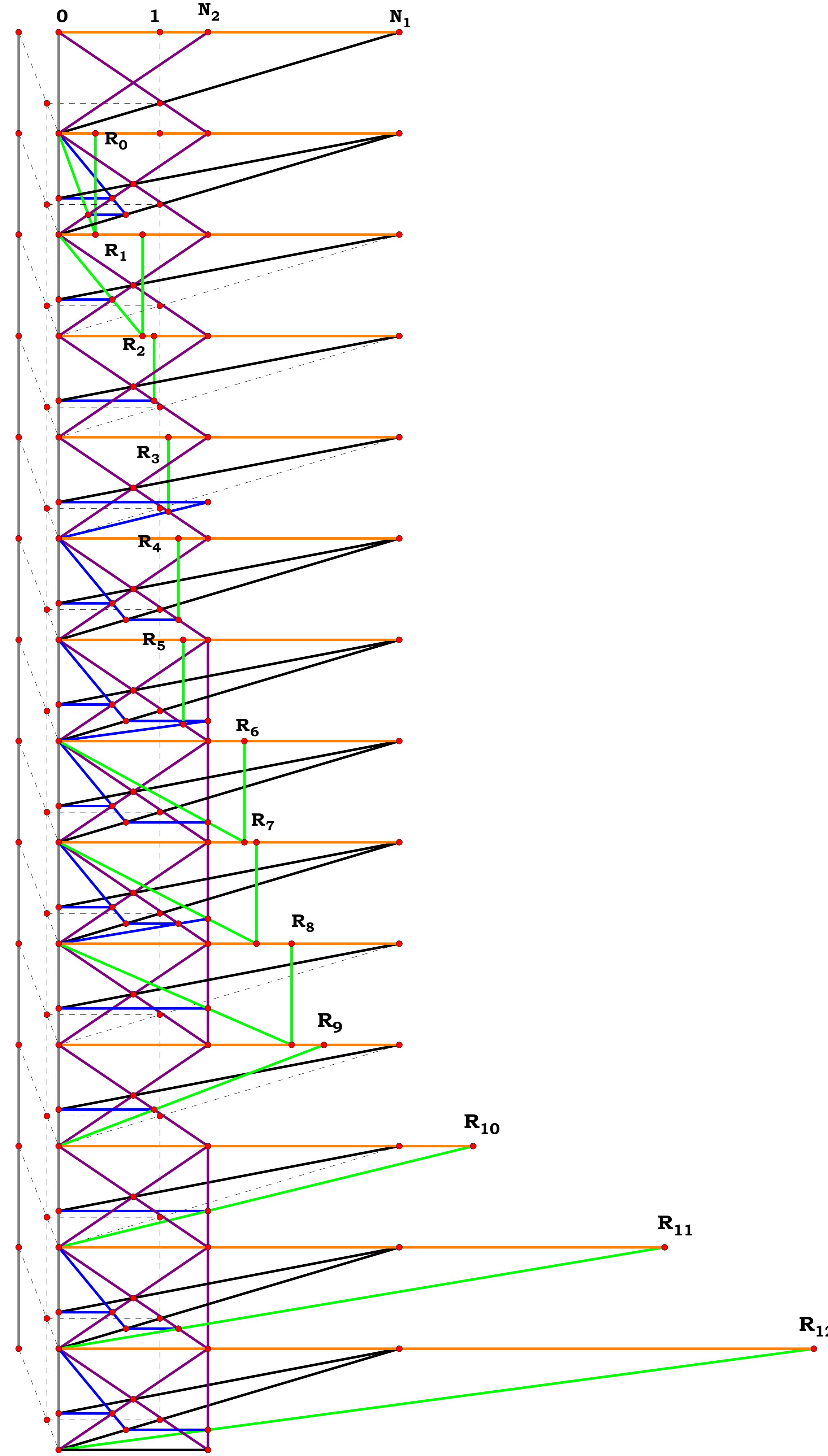
$$\frac{2 \cdot N_1 \cdot N_2 - N_2^2}{N_1} - R_8 = 0$$

$$\frac{N_1 \cdot N_2}{N_1 - N_2} - R_9 = 0$$

$$\frac{2 \cdot N_1 \cdot N_2 - N_2^2}{N_1 - N_2} - R_{10} = 0$$

$$\frac{N_1^2}{N_1 - N_2} - R_{11} = 0$$

$$\frac{(N_1^2 + N_1 \cdot N_2) - N_2^2}{N_1 - N_2} - R_{12} = 0$$



BOOK I.

OF

EUCLID'S ELEMENTS

TRANSLATED FROM THE TEXT OF HEIBERG

BY

SIR THOMAS L. HEATH,

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HONORARY FELLOW (SOMETIME FELLOW) OF TRINITY COLLEGE CAMBRIDGE

2013 *EDITION*

REVISED WITH SUBTRACTIONS

AND *SUBSTITUTIONS*.

REFORMATTED AND ABRIDGED FOR STUDY OF *THE ELEMENTS*.

BY JOHN CLARK.

BOOK I.

DEFINITIONS.

1. A **POINT** IS THAT WHICH HAS NO PART.

2. A **LINE** IS BREADTHLESS LENGTH.

3. THE EXTREMITIES OF A LINE ARE **POINTS**.

4. A **STRAIGHT LINE** IS A LINE WHICH LIES EVENLY WITH THE POINTS ON ITSELF.

5. A **SURFACE** IS THAT WHICH HAS LENGTH AND BREADTH ONLY.

6. THE EXTREMITIES OF A SURFACE ARE **LINES**.

7. A **PLANE SURFACE** IS A SURFACE WHICH LIES EVENLY WITH THE STRAIGHT LINES ON ITSELF.

8. A **PLANE ANGLE** IS THE INCLINATION TO ONE ANOTHER OF TWO LINES IN A PLANE WHICH MEET ONE ANOTHER AND DO NOT LIE IN A STRAIGHT LINE.

9. AND WHEN THE LINES CONTAINING THE ANGLE ARE STRAIGHT, THE ANGLE IS CALLED **RECTILINEAL**.

10. WHEN A STRAIGHT LINE SET UP ON A STRAIGHT LINE MAKES THE ADJACENT ANGLES EQUAL, TO ONE ANOTHER, EACH, OF THE EQUAL ANGLES IS **RIGHT**, AND THE STRAIGHT LINE STANDING ON THE OTHER IS CALLED A **PERPENDICULAR** TO THAT ON WHICH IT STANDS.

11. AN **OBTUSE ANGLE** IS AN ANGLE GREATER THAN A RIGHT ANGLE.

12. AN **ACUTE ANGLE** IS AN ANGLE LESS THAN A RIGHT ANGLE.

13. A **BOUNDARY** IS THAT WHICH IS AN EXTREMITY OF ANYTHING.

14. A **FIGURE** IS THAT WHICH IS CONTAINED BY ANY BOUNDARY OR BOUNDARIES.

15. A **CIRCLE** IS A PLANE FIGURE CONTAINED BY ONE LINE SUCH THAT ALL THE STRAIGHT LINES FALLING UPON IT FROM ONE POINT AMONG THOSE LYING WITHIN THE FIGURE ARE EQUAL, TO ONE ANOTHER;

16. AND THE POINT IS CALLED THE **CENTRE** OF THE CIRCLE.

17. A **DIAMETER** OF THE CIRCLE IS ANY STRAIGHT LINE DRAWN THROUGH THE CENTRE AND TERMINATED IN BOTH DIRECTIONS BY THE CIRCUMFERENCE OF THE CIRCLE, AND SUCH A STRAIGHT LINE, ALSO, BISECTS THE CIRCLE.

18. A **SEMICIRCLE** IS THE FIGURE CONTAINED BY THE DIAMETER AND THE CIRCUMFERENCE CUT OFF BY IT. AND THE CENTRE OF THE SEMICIRCLE IS THE SAME AS THAT OF THE CIRCLE.

19. **RECTILINEAL FIGURES** ARE THOSE WHICH ARE CONTAINED BY STRAIGHT LINES, **TRILATERAL** FIGURES BEING THOSE CONTAINED BY THREE, **QUADRILATERAL** THOSE CONTAINED BY

FOUR, AND **MULTILATERAL** THOSE CONTAINED BY MORE THAN FOUR STRAIGHT LINES.

20. OF TRILATERAL FIGURES, AN **EQUILATERAL TRIANGLE** IS THAT WHICH HAS ITS THREE SIDES EQUAL, AN **ISOSCELES TRIANGLE** THAT WHICH HAS TWO OF ITS SIDES ALONE EQUAL, AND A **SCALENE TRIANGLE** THAT WHICH HAS ITS THREE SIDES UNEQUAL.

21. FURTHER, OF TRILATERAL FIGURES, A **RIGHT-ANGLED TRIANGLE** IS THAT WHICH HAS A RIGHT ANGLE, AN **OBTUSE-ANGLED TRIANGLE** THAT WHICH HAS AN OBTUSE ANGLE, AND AN **ACUTE-ANGLED TRIANGLE** THAT WHICH HAS ITS THREE ANGLES ACUTE .

22. OF QUADRILATERAL FIGURES, A **SQUARE** IS THAT WHICH IS BOTH EQUILATERAL AND RIGHT-ANGLED; AN **OBLONG** THAT WHICH IS RIGHT-ANGLED BUT NOT EQUILATERAL; A **RHOMBUS** THAT WHICH IS EQUILATERAL BUT NOT RIGHT-ANGLED; AND A **RHOMBOID** THAT WHICH HAS ITS OPPOSITE SIDES AND ANGLES EQUAL, TO ONE ANOTHER BUT IS NEITHER EQUILATERAL NOR RIGHT-ANGLED. AND LET QUADRILATERALS OTHER THAN THESE BE CALLED **TRAPEZIA**.

23. **PARALLEL** STRAIGHT LINES ARE STRAIGHT LINES WHICH, BEING IN THE SAME PLANE AND BEING PRODUCED INDEFINITELY IN BOTH DIRECTIONS, DO NOT MEET ONE ANOTHER IN EITHER DIRECTION.

Postulates.

Let the following be postulated:

1. To draw a straight line from any point to any point.

2. To produce a finite straight line continuously in a straight line.

3. To describe a circle with any centre and distance.

4. That all right angles are equal, to one another.

5. That, if a straight line intersecting two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Common Notions.

1. Things which are equal, to the same thing are, also, equal, to one another.

2. If equals be added to equals, the wholes are equal.

3. If equals be subtracted from equals, the remainders are equal.

[7] 4. Things which coincide with one another are equal, to one another.

[8] 5. The whole is greater than the part.

NOTES.

DEFINITION 1.

A POINT *IS THAT WHICH HAS NO PART*.

NOTES.

DEFINITION 2.

A LINE *IS BREADTHLESS LENGTH*.

Notes.

Definition 3.

The extremities of a line are points.

NOTES.

DEFINITION 4.

A STRAIGHT LINE *IS A LINE WHICH LIES EVENLY WITH THE POINTS ON ITSELF*.

Notes.

Definition 5.

A SURFACE *is that which has length and breadth only*.

NOTES.

DEFINITION 6.

THE EXTREMITIES OF A SURFACE ARE LINES.

NOTES.

DEFINITION 7.

A PLANE *SURFACE IS A SURFACE WHICH LIES EVENLY WITH THE STRAIGHT LINES ON ITSELF*.

NOTES.

DEFINITION 8.

A PLANE ANGLE *IS THE INCLINATION TO ONE ANOTHER OF TWO LINES IN A PLANE WHICH MEET ONE ANOTHER AND DO NOT LIE IN A STRAIGHT LINE.*

NOTES.

DEFINITION 9.

AND WHEN THE LINES CONTAINING THE ANGLE ARE STRAIGHT, THE ANGLE IS CALLED RECTILINEAL.

Notes.

Definition 10.

When a straight line set up on a straight line makes the adjacent angles equal, to one another, each, of the equal angles is RIGHT, *and the straight line standing on the other is called a* PERPENDICULAR *to that on which it stands*.

NOTES.

DEFINITION 11.

An OBTUSE ANGLE *is an angle greater than a right angle.*

NOTES.

DEFINITION 12.

12. *An* ACUTE ANGLE *is an angle less than a right angle*.

Notes.

Definition 13.

A BOUNDARY *is that which is an extremity of anything*.

NOTES.

DEFINITION 14.

A FIGURE *IS THAT WHICH IS CONTAINED BY ANY BOUNDARY OR BOUNDARIES*.

NOTES.

DEFINITION 15.

15. A CIRCLE *IS A PLANE FIGURE CONTAINED BY ONE LINE SUCH THAT ALL THE STRAIGHT LINES FALLING UPON IT FROM ONE POINT AMONG THOSE LYING WITHIN THE FIGURE ARE EQUAL, TO ONE ANOTHER*;

Notes.

Definition 16.

16. *And the point is called the* centre *of the circle.*

Notes.

Definition 17.

A DIAMETER *of the circle is any straight line drawn through the centre and terminated in both directions by the circumference of ⊙and such a straight line, also, bisects the circle.*

NOTES.

DEFINITION 18.

A SEMICIRCLE *IS THE FIGURE CONTAINED BY THE DIAMETER AND THE CIRCUMFERENCE CUT OFF BY IT. AND THE CENTRE OF THE SEMICIRCLE IS THE SAME AS THAT OF THE CIRCLE.*

Notes.

Definition 19.

19. Rectilineal figures *are those which are contained by straight lines*, trilateral figures *being those contained by three*, quadrilateral *those contained by four*, and multilateral *those contained by more than four straight lines*.

Notes.

Definition 20.

20. *Of trilateral figures, an* equilateral triangle *is that which has its three sides equal, an* isosceles triangle *that which has two of its sides alone equal, and a* scalene triangle *that which has its three sides unequal*.

Notes.

Definition 21.

21. *Further, of trilateral figures, a* RIGHT-ANGLED TRIANGLE *is that which has a right angle, an* OBTUSE-ANGLED TRIANGLE *that which has an obtuse angle, and an* ACUTE-ANGLED TRIANGLE *that which has its three angles acute.*

NOTES.

DEFINITION 22.

OF QUADRILATERAL FIGURES, A SQUARE *IS THAT WHICH IS BOTH EQUILATERAL AND RIGHT-ANGLED; AN* OBLONG *THAT WHICH IS RIGHT-ANGLED BUT NOT EQUILATERAL; A* RHOMBUS *THAT WHICH IS EQUILATERAL BUT NOT RIGHT-ANGLED; AND A* RHOMBOID *THAT WHICH HAS ITS OPPOSITE SIDES AND ANGLES EQUAL, TO ONE ANOTHER BUT IS NEITHER EQUILATERAL NOR RIGHT-ANGLED. AND LET QUADRILATERALS OTHER THAN THESE BE CALLED* TRAPEZIA.

Notes.

Definition 23.

Parallel *straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.*

NOTES.

POSTULATE 1.

LET THE FOLLOWING BE POSTULATED: TO DRAW A STRAIGHT LINE FROM ANY POINT TO ANY POINT.

NOTES.

POSTULATE 2.

TO PRODUCE A FINITE STRAIGHT LINE CONTINUOUSLY IN A STRAIGHT LINE.

NOTES.

POSTULATE 3.

TO DESCRIBE A CIRCLE WITH ANY CENTRE AND DISTANCE.

NOTES.

POSTULATE 4.

THAT ALL RIGHT ANGLES ARE EQUAL, TO ONE ANOTHER.

Notes.

Postulate 5.

That, if a straight line intersecting two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Notes.

Common Notion 1.

Things which are equal, to the same thing are, also, equal, to one another.

Notes.

Common Notions 2.

2. *If equals be added to equals, the wholes are equal.*

Notes.

Common Notions 3.

3. *If equals be subtracted from equals, the remainders are equal.*

Notes.

Common Notion 4.

Things which coincide with one another are equal, to one another.

NOTES.

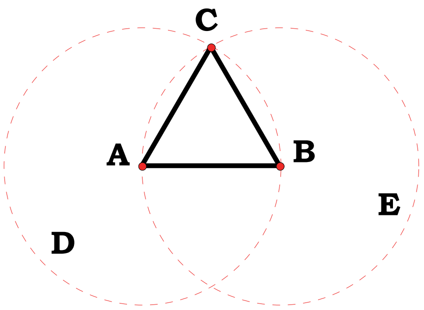
COMMON NOTION 5.

THE WHOLE IS GREATER THAN THE PART.

BOOK I.

PROPOSITIONS.

PROPOSITION 1.



ON A GIVEN FINITE STRAIGHT LINE TO CONSTRUCT AN EQUILATERAL TRIANGLE.

LET,

AB BE GIVEN.

THUS IT IS REQUIRED,

TO CONSTRUCT AN EQUILATERAL TRIANGLE ON AB.

[POST. 3]

WITH,

CENTRE, A, AND DISTANCE, AB, LET,

$\odot AB$, BE DESCRIBED;

[POST. 3] AGAIN WITH,

CENTRE, B, AND DISTANCE, BA, LET,

$\odot BA$, BE DESCRIBED;

[POST. 1] AND FROM,

THE POINT, C, IN WHICH

THE CIRCLES INTERSECT ONE ANOTHER, TO A, B, LET,

DESCRIBE CA, AND CB.

[DEF. 15] NOW, SINCE,

A, IS THE CENTRE OF $\odot AB$, $AC = AB$.

[DEF. 15] AGAIN SINCE ,

B, IS THE CENTRE OF $\odot BA$, $BC = BA$.

BUT,

$AC = AB$;

[*C. N.* 1] THEREFORE,

$AC = AB$, $BC = AB$. AND,

THINGS WHICH ARE EQUAL, TO THE SAME THING, ARE, ALSO, EQUAL, TO ONE ANOTHER;

THEREFORE,

$CA = CB$.

THEREFORE,

CA, AB, BC, ARE EQUAL, TO ONE ANOTHER.

THEREFORE,

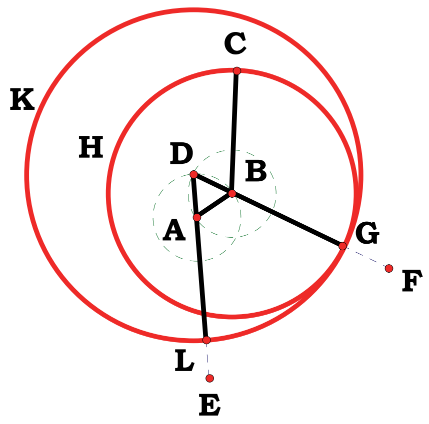
ΔABC, IS EQUILATERAL; AND

IT HAS BEEN CONSTRUCTED ON

THE GIVEN FINITE STRAIGHT LINE, AB.

(BEING) WHAT IT WAS REQUIRED *TO* DO.

Proposition 2.



To place, at a given point (as an extremity), a straight line equal, to a given straight line.

Let,

A and *BC*, be given,,

Thus it is required,

to place, at *A*,

a line equal, to *BC*.

[Post. 1]

From,

A, to *B*,

describe *AB*; and

[I. 1] on it let,

the equilateral Δ*DAB*, be constructed.

[Post. 2] Let,

AE, *BF*, be described collinear with *DA*, *DB*;

[Post. 3] with,

centre, *B*, and distance, *BC*, let, ⊙*BC*, be described;

[Post. 3] and again, with,

centre, *D*, and distance, *DG*, let,

⊙*DG*, be described.

Then, since,

B, is the centre of ⊙*BC*, *BC* = *BG*.

Again, since,

D, is the centre of ⊙*DG*, *DL* = *DG*,

And, in these,

DA = *DB*;

[*C. N.* 3]

therefore,

the remainders, *AL* = *BG*. But, also,

BC = *BG*;

therefore,

AL = *BG*, *BC* = *BG*.

[*C. N.* 1] And,

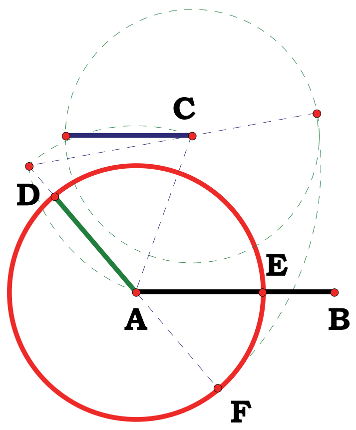
THINGS WHICH ARE EQUAL,
TO THE SAME THING ARE, ALSO, EQUAL, TO ONE ANOTHER;

THEREFORE,
$AL = BC$.

THEREFORE,
AT A, AL, IS PLACED EQUAL, TO BC.

(BEING) WHAT IT WAS REQUIRED TO DO.

PROPOSITION 3.



GIVEN TWO UNEQUAL STRAIGHT LINES, TO SUBTRACT FROM THE GREATER, A STRAIGHT LINE EQUAL, TO THE LESS.

LET,

AB, *C*, BE GIVEN,

AND LET,

AB BE THE GREATER OF THEM.

THUS IT IS REQUIRED,

TO SUBTRACT FROM *AB*, *C*, THE LESS.

[I. 2] [POST. 3] LET,

AT *A*, $AD = C$; AND WITH CENTRE, *A*, AND DISTANCE, *AD*,

LET,

$\odot AD$, BE DESCRIBED.

[DEF. 15] NOW, SINCE,

A, IS THE CENTRE OF $\odot AD$, $AE = AD$. BUT, $C = AD$.

[*C. N.* 1]

THEREFORE,

$AE = AD$, $C = AD$;

SO THAT,

$AE = C$.

THEREFORE,

GIVEN *AB*, *C*, FROM,

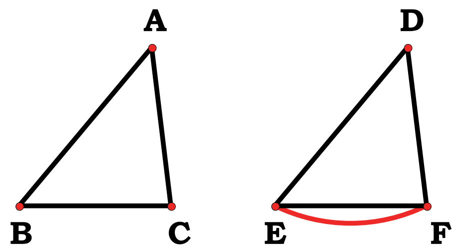
AB, THE GREATER,

AE HAS BEEN CUT OFF EQUAL, TO *C*, THE LESS.

(BEING) WHAT IT WAS REQUIRED TO DO.

PROPOSITION 4.

IF TWO TRIANGLES HAVE THE TWO SIDES EQUAL, TO TWO SIDES RESPECTIVELY, AND HAVE THE ANGLES CONTAINED BY THE EQUAL STRAIGHT LINES EQUAL, THEY WILL, ALSO, HAVE THE BASE EQUAL, TO THE BASE, THE TRIANGLE WILL BE EQUAL, TO THE TRIANGLE, AND THE REMAINING ANGLES WILL BE EQUAL, TO THE REMAINING ANGLES RESPECTIVELY, NAMELY THOSE WHICH THE EQUAL SIDES SUBTEND.



LET,

ΔABC, ΔDEF, HAVE

$AB = DE$, AND $AC = DF$, AND $\angle BAC = \angle EDF$.

I SAY THAT;

$BC = EF$, $\Delta ABC = \Delta DEF$, AND

THE REMAINING ANGLES WILL BE EQUAL, TO THE REMAINING ANGLES RESPECTIVELY,

NAMELY,

THOSE WHICH THE EQUAL SIDES SUBTEND,

THAT IS,

$\angle ABC = \angle DEF$, $\angle ACB = \angle DFE$.

FOR, IF,

ΔABC, BE APPLIED TO ΔDEF,

AND IF,

A, BE PLACED ON THE POINT, D, AND AB, ON DE,

THEN,

B, WILL, ALSO, COINCIDE WITH E, BECAUSE,

$AB = DE$.

AGAIN,

AB COINCIDING WITH DE,

AC, WILL, ALSO, COINCIDE WITH DF, BECAUSE,

$\angle BAC = \angle EDF$;

HENCE, ALSO,

C, WILL COINCIDE WITH F, BECAUSE,

$AC = DF$.

BUT, ALSO,

B COINCIDED WITH E;

HENCE,

BC, WILL COINCIDE WITH EF.

[FOR IF,

WHEN B COINCIDES WITH E AND

C WITH F,

BC, DOES NOT COINCIDE WITH EF.

THEN,

TWO STRAIGHT LINES WILL ENCLOSE A SPACE: WHICH, IS IMPOSSIBLE.

[*C. N.* 4]

THEREFORE,

BC, WILL COINCIDE WITH, EF] AND WILL BE EQUAL, TO IT.

THUS,

THE WHOLE ΔABC, WILL COINCIDE WITH

THE WHOLE ΔDEF, AND WILL BE EQUAL, TO IT.

AND,

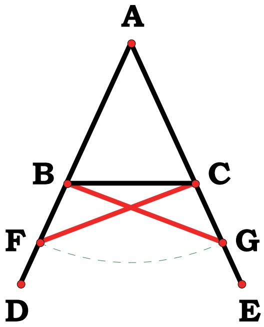
THE REMAINING ANGLES WILL, ALSO, COINCIDE WITH THE REMAINING ANGLES, AND WILL BE EQUAL, TO THEM,

$\angle ABC = \angle DEF$, $\angle ACB = \angle DFE$.

THEREFORE ETC.

(BEING) WHAT IT WAS REQUIRED TO PROVE.

PROPOSITION 5.



IN ISOSCELES TRIANGLES THE ANGLES AT THE BASE ARE EQUAL, TO ONE ANOTHER, AND, IF THE EQUAL STRAIGHT LINES BE PRODUCED FURTHER, THE ANGLES UNDER THE BASE WILL BE EQUAL, TO ONE ANOTHER.

[POST. 2]

LET,

ABC BE AN ISOSCELES TRIANGLE HAVING
$AB = AC$;

AND LET,

BD, CE, BE PRODUCED FURTHER,
COLLINEAR WITH AB, AC.

I SAY THAT;

$\angle ABC = \angle ACB$, $\angle CBD = \angle BCE$.

LET,

F, BE ASSERTED AT RANDOM, ON BD;

[I. 3]

LET FROM,

AE, THE GREATER, $AG = AF$, THE LESS;

[POST. 1]

AND LET,

FC, GB.

THEN, SINCE,

$AF = AG$, AND $AB = AC$,
$FA = GA$, $AC = AB$; AND
$\angle FAG$ IS COMMON.

THEREFORE,

$FC = GB$, $\Delta AFC = \Delta AGB$, AND
THE REMAINING ANGLES
WILL BE EQUAL, TO THE REMAINING ANGLES, RESPECTIVELY,

NAMELY,

THOSE WHICH THE EQUAL SIDES SUBTEND,

[I. 4]

THAT IS,

$\angle ACF = \angle ABG$, $\angle AFC = \angle AGB$.

AND, SINCE,

$AF = AG$, AND IN THESE,

$AB = AC$, THE REMAINDERS,

$BF = CG$. BUT,

$FC = GB$;

THEREFORE,

$BF = CG$, $FC = GB$; AND

$\angle BFC = \angle CGB$, WHILE

BC IS COMMON TO THEM;

THEREFORE,

$\Delta BFC = \Delta CGB$, AND

THE REMAINING ANGLES WILL BE EQUAL, TO THE REMAINING ANGLES, RESPECTIVELY,

NAMELY,

THOSE WHICH THE EQUAL SIDES SUBTEND;

THEREFORE,

$\angle FBC = \angle GCB$, AND $\angle BCF = \angle CBG$.

ACCORDINGLY, SINCE, THE WHOLE

$\angle ABG = \angle ACF$, AND IN THESE

$\angle CBG = \angle BCF$, THE REMAINING,

$\angle ABC = \angle ACB$; AND THEY ARE AT THE BASE OF

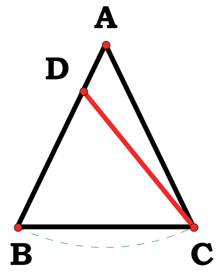
ΔABC. BUT,

$\angle FBC = \angle GCB$; AND THEY ARE UNDER THE BASE.

THEREFORE ETC.

Q. E. D.

PROPOSITION 6.



IF IN A TRIANGLE TWO ANGLES BE EQUAL, TO ONE ANOTHER, THE SIDES WHICH SUBTEND THE EQUAL ANGLES WILL, ALSO, *BE EQUAL, TO ONE ANOTHER.*

LET,

ΔABC, HAVE $\angle ABC = \angle ACB$;

I SAY THAT;

$AB = AC$. FOR, IF,

$AB \neq AC$,

THEN,

ONE OF THEM IS GREATER.

LET,

AB BE GREATER;

AND LET FROM,

AB, THE GREATER,

$DB = AC$, THE LESS;

LET,

DC BE DESCRIBED.

THEN, SINCE,

$DB = AC$, AND

BC IS COMMON,

$DB = AC$, $BC = CB$; AND $\angle DBC = \angle ACB$;

THEREFORE,

$DC = AB$, AND $\Delta DBC = \Delta ACB$,

THE LESS TO THE GREATER: WHICH,

IS ABSURD.

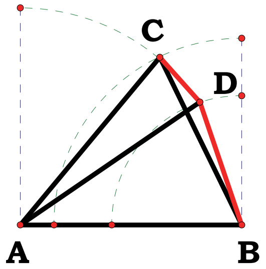
THEREFORE,

$AB = AC$;

THEREFORE ETC.

Q. E. D.

PROPOSITION 7.



GIVEN TWO STRAIGHT LINES CONSTRUCTED ON A STRAIGHT LINE (FROM ITS EXTREMITIES) AND MEETING IN A POINT, THERE CANNOT BE CONSTRUCTED ON THE SAME STRAIGHT LINE (FROM ITS EXTREMITIES), AND ON THE SAME SIDE OF IT, TWO OTHER STRAIGHT LINES MEETING IN ANOTHER POINT AND EQUAL, TO THE FORMER TWO RESPECTIVELY, NAMELY EACH TO THAT WHICH HAS THE SAME EXTREMITY WITH IT.

FOR, IF POSSIBLE, GIVEN,

AC, CB,

CONSTRUCTED ON AB, AND INTERSECTING AT C,

LET,

AD, DB,

BE CONSTRUCTED, ON AB, ON THE SAME SIDE OF IT,

INTERSECTING AT D, AND

EQUAL, TO THE FORMER TWO, RESPECTIVELY,

NAMELY,

EACH TO THAT WHICH HAS THE SAME EXTREMITY WITH IT,

SO THAT,

$CA = DA$,

WHICH,

HAS THE SAME EXTREMITY, A, WITH IT, AND

$CB = DB$, WHICH HAS THE SAME EXTREMITY, B, WITH IT;

AND LET,

CD BE DESCRIBED.

[I. 5]

THEN, SINCE,

$AC = AD$, $\angle ACD = \angle ADC$; THEREFORE,

$\angle ADC > \angle DCB$; THEREFORE,

$\angle CDB$, IS MUCH GREATER THAN $\angle DCB$. AGAIN, SINCE,

$CB = DB$, $\angle CDB = \angle DCB$. BUT,

IT WAS, ALSO, PROVED MUCH GREATER THAN IT:

WHICH,

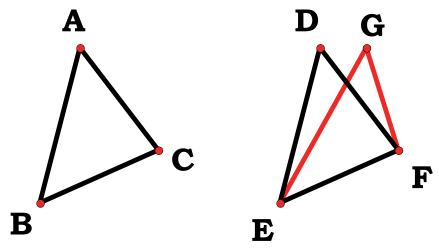
IS IMPOSSIBLE.

THEREFORE ETC.

Q. E. D.

PROPOSITION 8.

IF TWO TRIANGLES HAVE THE TWO SIDES EQUAL, TO TWO SIDES RESPECTIVELY, AND HAVE, ALSO, *THE BASE EQUAL, TO THE BASE, THEY WILL,* ALSO, *HAVE THE ANGLES EQUAL WHICH ARE CONTAINED BY THE EQUAL STRAIGHT LINES.*



LET,

ΔABC, ΔDEF HAVE
$AB = DE$, $AC = DF$, AND
$BC = EF$;

I SAY THAT;

$\angle BAC = \angle EDF$.

FOR, IF,

ΔABC, BE APPLIED TO ΔDEF,

AND IF,
B, BE PLACED ON E, AND
BC, ON EF,
C, WILL, ALSO, COINCIDE WITH F,

BECAUSE,
$BC = EF$.

THEN,
BC COINCIDING WITH EF,
BA, AC WILL, ALSO, COINCIDE WITH ED, DF;

FOR, IF,
BC, COINCIDES WITH EF, AND
BA, AC, DO NOT COINCIDE WITH, ED, DF,

BUT,
FALL BESIDE THEM AS EG, GF,

THEN,
GIVEN TWO STRAIGHT LINES CONSTRUCTED ON
A STRAIGHT LINE (FROM ITS EXTREMITIES), AND
MEETING IN A POINT,
THERE WILL HAVE BEEN CONSTRUCTED ON
THE SAME STRAIGHT LINE (FROM ITS EXTREMITIES), AND
ON THE SAME SIDE OF IT,
TWO OTHER STRAIGHT LINES MEETING IN ANOTHER POINT, AND
EQUAL, TO THE FORMER TWO, RESPECTIVELY,

NAMELY,

EACH TO THAT WHICH HAS THE SAME EXTREMITY WITH IT.

[I. 7]

BUT,

THEY CANNOT BE SO CONSTRUCTED.

THEREFORE,

IT IS NOT POSSIBLE THAT,

IF,

BC, BE APPLIED TO *EF*,
BA, *AC*, SHOULD NOT COINCIDE WITH *ED*, *DF*;

THEREFORE,

THEY WILL COINCIDE,

SO THAT,

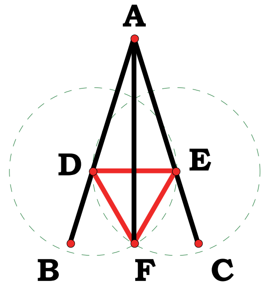
$\angle BAC$, WILL, ALSO, COINCIDE WITH

$\angle EDF$, AND WILL BE EQUAL, TO IT.

IF THEREFORE ETC.

Q. E. D.

Proposition 9.



To bisect a given rectilineal angle.

Let,

$\angle BAC$, be

the given rectilineal angle.

Thus it is required,

to bisect it.

Let,

D, be asserted at random, to AB;

[I. 3] let, from AC,

AE, = AD;

let,

describe DE, and on DE,

let,

the equilateral ΔDEF, be constructed;

let,

AF be described.

I say that;

$\angle BAC$, has been bisected with AF.

For, since,

$AD = AE$, and AF is common,

the two sides, $DA = EA$, $AF = AF$. And,

$DF = EF$;

therefore,

$\angle DAF = \angle EAF$.

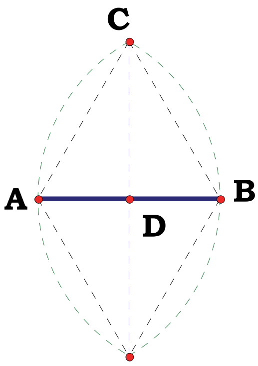
Therefore,

the given rectilineal $\angle BAC$,

has been bisected with the straight line, AF.

Q. E. F.

PROPOSITION 10.



TO BISECT A GIVEN FINITE STRAIGHT LINE.

LET,
AB BE GIVEN.

THUS IT IS REQUIRED,
TO BISECT AB.

[I. 1]

LET,
THE EQUILATERAL ΔABC, BE DESCRIBED,

[I. 9]

AND LET,
$\angle ACB$, BE BISECTED WITH CD;

I SAY THAT;
AB, HAS BEEN BISECTED AT D.

FOR, SINCE,
$AC = CB$, AND CD IS COMMON,
$AC = BC$, $CD = CD$; AND $\angle ACD = \angle BCD$;

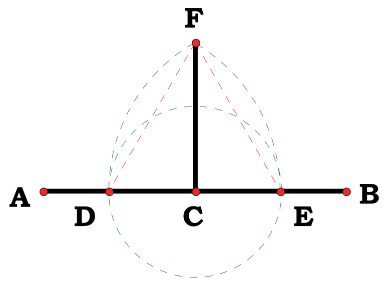
[I. 4]

THEREFORE,
$AD = BD$.

THEREFORE,
AB HAS BEEN BISECTED AT D.

Q. E. F.

PROPOSITION 11.



TO DRAW A STRAIGHT LINE AT RIGHT ANGLES TO A GIVEN STRAIGHT LINE FROM A GIVEN POINT ON IT.

LET,

AB BE GIVEN,

AND,

C GIVEN ON IT.

THUS IT IS REQUIRED,

TO DRAW FROM C, A LINE AT RIGHT ANGLES TO AB.

LET,

D, BE ASSERTED AT RANDOM, OF AC;

[I. 3] LET,

$CE = CD$;

[I. 1] LET,

ON DE, THE EQUILATERAL ΔFDE, BE DESCRIBED,

AND LET,

FC BE DESCRIBED;

I SAY THAT;

FC, HAS BEEN DRAWN AT
RIGHT ANGLES TO AB, FROM C, ON IT.

FOR, SINCE,

$DC = CE$,

AND,

CF IS COMMON,

$DC = EC$, $CF = CF$; AND $DF = FE$;

[I. 8]

THEREFORE,

$\angle DCF = \angle ECF$; AND THEY ARE ADJACENT ANGLES.

[DEF. 10]

BUT,

WHEN A STRAIGHT LINE SET UP ON A STRAIGHT LINE MAKES
THE ADJACENT ANGLES EQUAL, TO ONE ANOTHER,
EACH, OF THE EQUAL ANGLES IS RIGHT;

THEREFORE,

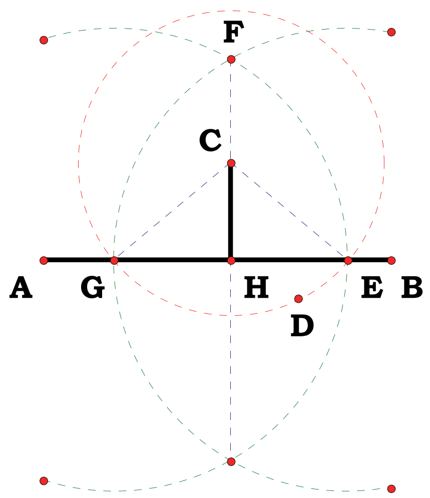
EACH, OF ∟DCF, ∟FCE, IS RIGHT.

THEREFORE,

CF, HAS BEEN DRAWN AT RIGHT ANGLES TO *AB*,
FROM *C*, ON IT.

Q. E. F.

PROPOSITION 12.



TO A GIVEN INFINITE STRAIGHT LINE, FROM A GIVEN POINT WHICH IS NOT ON IT, TO DRAW A PERPENDICULAR STRAIGHT LINE.

LET,

AB BE THE GIVEN INFINITE STRAIGHT LINE, AND

C, THE GIVEN POINT WHICH IS NOT ON IT;

THUS IT IS REQUIRED,

TO DRAW, TO THE GIVEN INFINITE STRAIGHT LINE, AB,
FROM THE GIVEN POINT, C, WHICH IS NOT ON IT,
A PERPENDICULAR STRAIGHT LINE.

FOR LET, AT RANDOM,

D, BE TAKEN ON
THE OTHER SIDE OF AB, AND
WITH CENTRE, C, AND DISTANCE, CD,

[POST. 3] LET,

$\odot CD$, BE DESCRIBED;

[I. 10] LET,

THE EG, BE BISECTED, AT H,

[POST 1] AND LET,

CG, CH, CE, BE DESCRIBED.

I SAY THAT;

CH HAS BEEN DRAWN PERPENDICULAR TO AB, FROM
C, WHICH IS NOT ON IT.

FOR, SINCE,

$GH = HE$, AND HC IS COMMON,
$GH = EH$, $HC = HC$; AND $CG = CE$;

[I. 8] THEREFORE,

$\angle CHG = \angle EHC$.

AND THEY ARE ADJACENT ANGLES.

[DEF. 10] BUT,

WHEN A STRAIGHT LINE SET UP ON A STRAIGHT LINE MAKES
THE ADJACENT ANGLES EQUAL, TO ONE ANOTHER,
EACH, OF THE EQUAL ANGLES IS RIGHT, AND

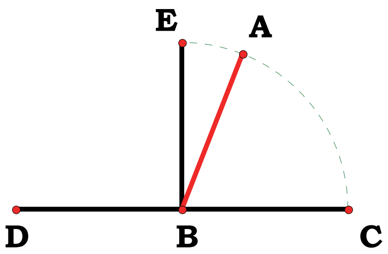
THE STRAIGHT LINE STANDING ON
THE OTHER IS CALLED A PERPENDICULAR TO THAT
ON WHICH IT STANDS.

THEREFORE,

CH HAS BEEN DRAWN PERPENDICULAR TO AB, FROM
C, WHICH IS NOT ON IT.

Q. E. F.

PROPOSITION 13.



IF A STRAIGHT LINE SET UP ON A STRAIGHT LINE MAKE ANGLES, IT WILL MAKE EITHER TWO RIGHT ANGLES OR ANGLES EQUAL, TO TWO RIGHT ANGLES.

FOR LET,

AB, SET UP ON *CD*, MAKE

∠*CBA*, ∠*ABD*;

I SAY THAT;

∠*CBA*, ∠*ABD*, ARE EITHER TWO RIGHT ANGLES, OR EQUAL, TO TWO RIGHT ANGLES.

NOW, IF,

∠*CBA* = ∠*ABD*,

[DEF. 10] THEN,

THEY ARE TWO RIGHT ANGLES.

[I. 11] BUT, IF NOT, LET,

BE, BE DRAWN FROM *B*, AT RIGHT ANGLES, TO *CD*;

THEREFORE,

∟*CBE*, ∟*EBD*, ARE TWO RIGHT ANGLES.

THEN, SINCE,

∠*CBE* = ∠*CBA* + ∠*ABE*,

LET,

∠*EBD*, BE ADDED TO EACH;

[*C. N.* 2] THEREFORE,

∠*CBE* + ∠*EBD* = *CBA* + *ABE* + *EBD*.

AGAIN, SINCE,

∠*DBA* = ∠*DBE* + ∠*EBA*,

LET,

∠*ABC*, BE ADDED TO EACH;

[*C. N.* 2] THEREFORE,

∠*DBA* + ∠*ABC*, = ∠*DBE* + ∠*EBA* + ∠*ABC*.

[*C. N.* 1] BUT,

∠*CBE* + ∠*EBD*, WERE, ALSO, PROVED EQUAL, TO

THE SAME THREE ANGLES; AND
THINGS WHICH ARE EQUAL, TO THE SAME THING ARE ALSO EQUAL, TO ONE ANOTHER;

THEREFORE,

$\angle CBE + \angle EBD = \angle DBA + \angle ABC$.

BUT,

$\angle CBE + \angle EBD$, ARE TWO RIGHT ANGLES;

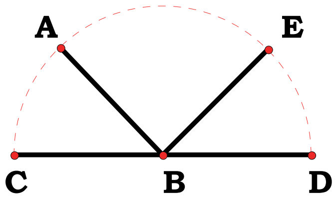
THEREFORE,

$\angle DBA + ABC$, ARE, ALSO, EQUAL, TO TWO RIGHT ANGLES.

THEREFORE ETC.

Q. E. D.

Proposition 14.



If with any straight line, and at a point on it, two straight lines not lying on the same side make the adjacent angles equal, to two right angles, the two straight lines will be in a straight line with one another.

For, let

with any AB, and at B, on it, BC, BD,
not lying on the same side, make
the adjacent angles, ABC, ABD, equal, to
two right angles;

I say that;

BD is collinear with CB. For, if,
BD is not collinear with BC, let,
BE, be collinear with CB.

[I. 13] Then, since,

AB, intersects CBE,

$\angle ABC + \angle ABE$, are equal, to two right angles. But,

$\angle ABC + \angle ABD$, are, also, equal, to two right angles;

[Post. 4 and *C. N.* 1] therefore,

$\angle CBA + ABE = \angle CBA + \angle ABD$.

Let,

$\angle CBA$, be subtracted from each;

[*C. N.* 3] therefore,

the remaining, $\angle ABE = ABD$, the less to the greater:

which,

is impossible.

Therefore,

BE is not collinear with CB.

Similarly, we can prove that,

neither is any other straight line except, BD.

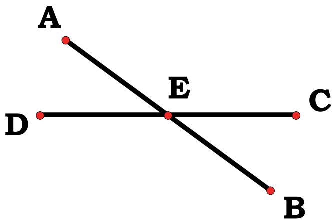
Therefore,

CB is collinear with BD.

Therefore etc.

Q. E. D.

PROPOSITION 15.



IF TWO STRAIGHT LINES INTERSECT ONE ANOTHER, THEY MAKE THE VERTICAL ANGLES EQUAL, TO ONE ANOTHER.

FOR LET,

AB, CD, INTERSECT ONE ANOTHER AT E;

I SAY THAT;

$\angle AEC = \angle DEB$, AND $\angle CEB = \angle AED$.

FOR, SINCE,

AE, INTERSECTS CD, MAKING $\angle CEA$, $\angle AED$,

[I. 13]

$\angle CEA + \angle AED$, ARE EQUAL, TO TWO RIGHT ANGLES.

AGAIN, SINCE,

DE, INTERSECTS AB, MAKING $\angle AED$, $\angle DEB$,

[I. 13]

$\angle AED + \angle DEB$, ARE EQUAL, TO TWO RIGHT ANGLES.

BUT,

$\angle CEA + \angle AED$, WERE, ALSO, PROVED EQUAL, TO TWO RIGHT ANGLES;

[POST. 4 AND *C. N.* 1] THEREFORE,

$\angle CEA + \angle AED = \angle AED + \angle DEB$.

LET,

$\angle AED$, BE SUBTRACTED FROM EACH;

[*C. N.* 3]

THEREFORE,

THE REMAINS, $\angle CEA = \angle BED$.

SIMILARLY, IT CAN BE PROVED THAT,

$\angle CEB = \angle DEA$.

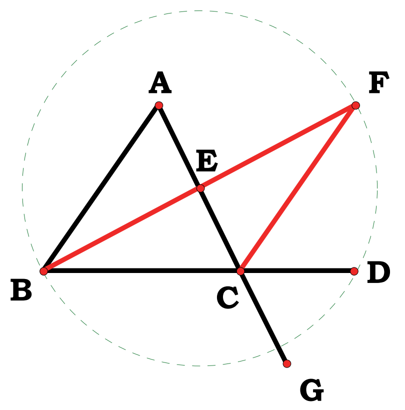
THEREFORE ETC.

Q. E. D.

[PORISM.

FROM THIS IT IS MANIFEST THAT, IF TWO STRAIGHT LINES INTERSECT ONE ANOTHER, THEY WILL MAKE THE ANGLES AT THE POINT OF SECTION EQUAL, TO FOUR RIGHT ANGLES.]

Proposition 16.



In any triangle, if one of the sides be produced, the exterior angle is greater than either of the interior and opposite angles.

Let,

ΔABC,

and let,

one side of it, BC,

be produced to D;

I say that;

the exterior $\angle ACD$, is greater than either of

the interior and opposites, $\angle CBA$, $\angle BAC$.

[I. 10] Let,

$\frac{AC}{2}$, at E.

and let,

BE, be described, and produced to F;

[I. 3] let,

$EF = BE$.

[Post. 1] let,

FC be described,

[Post. 2] and let,

AC be drawn through to G.

Then, since,

$AE = EC$, and $BE = EF$,

$AE = CE$, $EB = EF$; and $\angle AEB = \angle FEC$,

[I. 15] for,

they are vertical angles.

[I. 4] Therefore,

$AB = FC$, and $\Delta ABE = \Delta CFE$, and

the remaining angles are equal, to

the remaining angles, respectively,

namely,

those which the equal sides subtend;

therefore,

$\angle BAE = \angle ECF$.

[*C. N.* 5] BUT,

$\angle ECD > \angle ECF$; THEREFORE,

$\angle ACD > \angle BAE$.

[I. 15] SIMILARLY ALSO, IF,

BC BE BISECTED, $\angle BCG$,

THAT IS,

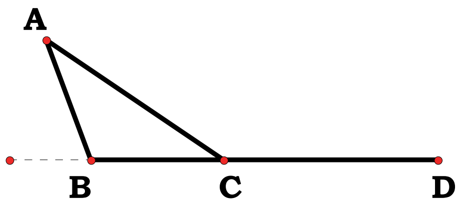
$\angle ACD$, CAN BE PROVED GREATER THAN

$\angle ABC$, AS WELL.

THEREFORE ETC.

Q. E. D.

Proposition 17.



In any triangle, two angles taken together in any manner are less than two right angles.

Let,

ΔABC;

I say that;

two angles of ΔABC, taken together,

in any manner, are less than two right angles.

[Post. 2]

For let,

BC be produced to D.

Then, since,

$\angle ACD$, is an exterior angle of ΔABC, it is greater than the interior and opposite angle, ABC.

Let,

$\angle ACB$, be added to each;

therefore,

$\angle ACD + \angle ACB > \angle ABC + \angle BCA$.

[I.13] But

$\angle ACD + \angle ACB$, are equal, to two right angles.

Therefore,

$\angle ABC + \angle BCA$, are less than two right angles.

Similarly we can prove, also, that,

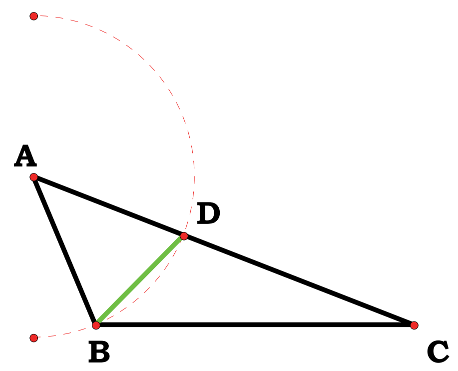
$\angle BAC + \angle ACB$, are less than two right angles, and

so are $\angle CAB + \angle ABC$, as well.

Therefore etc.

Q. E. D.

PROPOSITION 18.



IN ANY TRIANGLE THE GREATER SIDE SUBTENDS THE GREATER ANGLE.

FOR LET,

ΔABC HAVE $AC > AB$;

I SAY THAT;

$\angle ABC > \angle BCA$.

FOR, SINCE,

$AC > AB$,

[I. 3] LET,

$AD = AB$.

AND LET,

BD BE DESCRIBED.

[I. 16] THEN, SINCE,

$\angle ADB$, IS AN EXTERIOR ANGLE OF

ΔBCD, IT IS GREATER THAN THE INTERIOR AND OPPOSITE,

$\angle DCB$.

BUT,

$\angle ADB = \angle ABD$,

SINCE,

$AB = AD$;

THEREFORE,

$\angle ABD > \angle ACB$;

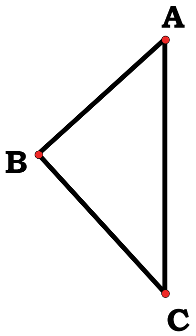
THEREFORE,

$\angle ABC$, IS MUCH GREATER THAN $\angle ACB$.

THEREFORE ETC.

Q. E. D.

PROPOSITION 19.



IN ANY TRIANGLE THE GREATER ANGLE IS SUBTENDED BY THE GREATER SIDE.

LET,

ΔABC HAVE $\angle ABC > \angle BCA$;

I SAY THAT;

$AC > AB$.

FOR, IF NOT,

$AC \leq AB$.

NOW,

$AC \neq AB$;

[I. 5] FOR THEN,

$\angle ABC = \angle ACB$; BUT, IT IS NOT;

THEREFORE,

$AC \neq AB$. NEITHER IS

$AC < AB$,

[I. 18] FOR THEN,

$\angle ABC < \angle ACB$, BUT,

IT IS NOT;

THEREFORE,

$AC \nless AB$. AND,

IT WAS PROVED THAT IT IS NOT EQUAL EITHER.

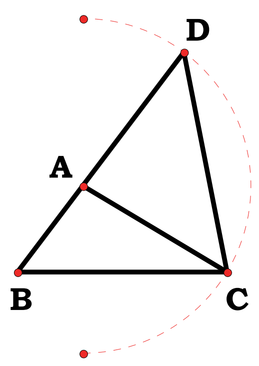
THEREFORE,

$AC > AB$.

THEREFORE ETC.

Q. E. D.

PROPOSITION 20.



IN ANY TRIANGLE TWO SIDES TAKEN TOGETHER IN ANY MANNER ARE GREATER THAN THE REMAINING ONE.

FOR LET,

ΔABC;

I SAY THAT;

IN ΔABC, TWO SIDES TAKEN TOGETHER, IN ANY MANNER, ARE GREATER THAN THE REMAINING ONE,

NAMELY,

$BA + AC > BC$,
$AB + BC > AC$,
$BC + CA > AB$.

FOR LET,

BA BE DRAWN THROUGH TO D,

LET,

$DA = CA$,

AND LET,

DC BE DESCRIBED.

[I. 5] THEN, SINCE,

$DA = AC$, $\angle ADC = \angle ACD$;

[*C. N.* 5] THEREFORE,

$\angle BCD > \angle ADC$.

[I. 19] AND, SINCE,

ΔDCB HAS $\angle BCD > \angle BDC$, AND

THE GREATER ANGLE IS SUBTENDED BY THE GREATER SIDE,

THEREFORE,

$DB > BC$.

BUT,

$DA = AC$;

THEREFORE,

$BA + AC > BC$.

SIMILARLY, ALSO, WE CAN PROVE THAT,

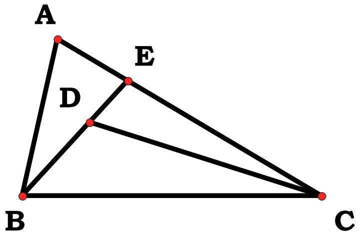
$AB + BC > CA$, AND $BC + CA > AB$.

THEREFORE ETC.

Q. E. D.

PROPOSITION 21.

IF ON ONE OF THE SIDES OF A TRIANGLE, FROM ITS EXTREMITIES, THERE BE CONSTRUCTED TWO STRAIGHT LINES MEETING WITHIN THE TRIANGLE, THE STRAIGHT LINES SO CONSTRUCTED WILL BE LESS THAN THE REMAINING TWO SIDES OF THE TRIANGLE, BUT WILL CONTAIN A GREATER ANGLE.



LET,

ON BC, ONE OF THE SIDES OF ΔABC, FROM,
ITS EXTREMITIES, B, C, BD, DC, BE CONSTRUCTED,
MEETING WITHIN THE TRIANGLE;

I SAY THAT;

BD, $DC < BA$, AC,

BUT,

CONTAIN AN $\angle BDC > \angle BAC$.

FOR LET,

BD BE DRAWN THROUGH TO E.

[I. 20]

THEN, SINCE,

IN ANY TRIANGLE,
TWO SIDES ARE GREATER THAN THE REMAINING ONE,

THEREFORE,

IN ΔABE, $AB + AE > BE$.

LET,

EC BE ADDED TO EACH;

THEREFORE,

$BA + AC > BE + EC$.

AGAIN, SINCE,

IN ΔCED, $CE + ED > CD$,

LET,

DB BE ADDED TO EACH;

THEREFORE,

$CE + EB > CD + DB$.

BUT,

$BA + AC > BE + EC$;

THEREFORE,

$BA + AC$ ARE MUCH GREATER THAN $BD + DC$.

[I. 16]

AGAIN, SINCE,

IN ANY TRIANGLE,
THE EXTERIOR ANGLE IS GREATER THAN
THE INTERIOR AND OPPOSITE ANGLE,

THEREFORE,

IN ΔCDE, THE EXTERIOR $\angle BDC > \angle CED$.

FOR THE SAME REASON, MOREOVER,

IN ΔABE, ALSO, THE EXTERIOR $\angle CEB > \angle BAC$.

BUT,

$\angle BDC > \angle CEB$;

THEREFORE,

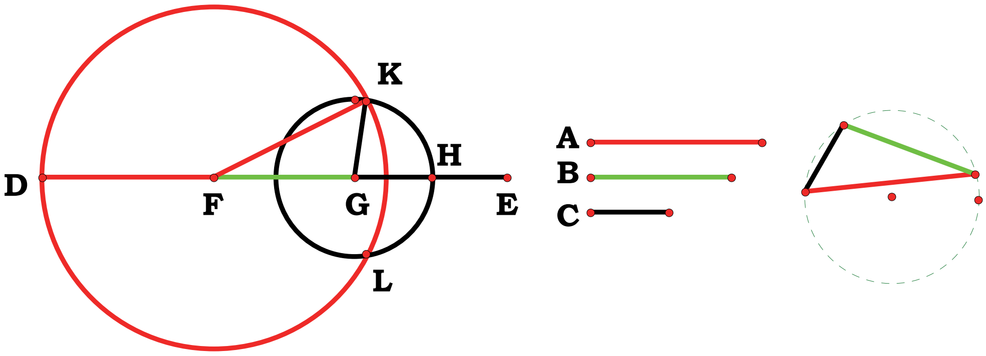
THE $\angle BDC$ IS MUCH GREATER THAN $\angle BAC$.

THEREFORE ETC.

Q. E. D.

Proposition 22.

Out of three straight lines, which are equal, to three given straight lines, to construct a triangle: thus it is necessary that two of the straight lines taken together in any manner should be greater than the remaining one. [I. 20]



Let,

BE GIVEN A, B, C,

AND LET,

OF THESE, TWO TAKEN TOGETHER, IN ANY MANNER,

BE GREATER THAN THE REMAINING ONE,

NAMELY,

$A + B > C$, $A + C > B$, AND, $B + C > A$;

THUS IT IS REQUIRED,

TO CONSTRUCT A TRIANGLE,

OUT OF LINES, EQUAL TO, A, B, C.

Let,

THERE BE SET OUT DE, TERMINATED AT D,

BUT,

OF CONVENIENT LENGTH IN THE DIRECTION OF E,

[I. 3]

AND LET,

$DF = A$, $FG = B$, AND $GH = C$.

LET,

WITH CENTRE, F, AND DISTANCE, FD,

$\odot FDK$, BE DESCRIBED;

AGAIN, LET,

WITH CENTRE, G, AND DISTANCE, GH,

$\odot GHK$, BE DESCRIBED;

AND LET,

KF, KG, BE DESCRIBED;

I SAY THAT;

ΔKFG, HAS BEEN CONSTRUCTED,

OUT OF THREE LINES, EQUAL TO, A, B, C.

FOR, SINCE,

F, IS THE CENTRE OF $\odot FDK$, $FD = FK$.

BUT,

$FD = A$;

THEREFORE,

$KF = A$.

AGAIN, SINCE,

G, IS THE CENTRE OF $\odot GHK$, $GH = GK$.

BUT,

$GH = C$;

THEREFORE,

$KG = C$. AND, $FG = B$;

THEREFORE,

THE THREE, KF, FG, GK, ARE EQUAL, TO
THE THREE, A, B, C.

THEREFORE,

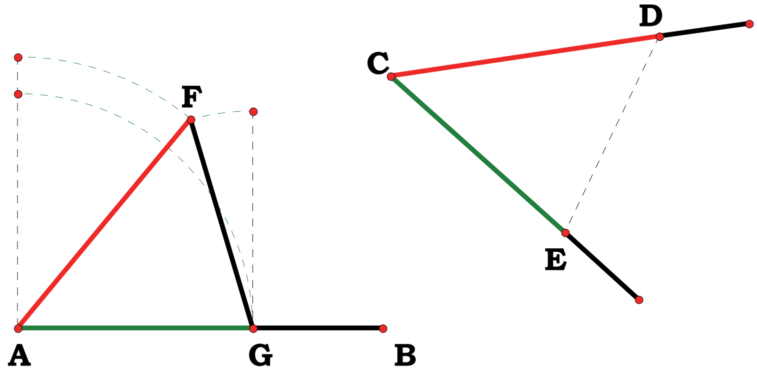
OUT OF KF, FG, GK, WHICH, ARE EQUAL, TO, A, B, C,

ΔKFG, HAS BEEN CONSTRUCTED.

Q. E. F.

Proposition 23.

On a given straight line and at a point on it to construct a rectilineal angle equal, to a given rectilineal angle.



Let,

AB be given, A the point on it, and

$\angle DCE$, the given angle;

thus it is required,

to construct on AB, and at A, on it,

an angle equal, to $\angle DCE$.

at random, let,

on CD, CE, D, E, be taken; respectively

let,

DE be described,

[I. 22] and,

out of three lines, which are equal, to
the three lines, CD, DE, CE,

let,

ΔAFG, be constructed,

in such a way, that,

$CD = AF$, $CE = AG$, and further, $DE = FG$

[I. 8] Then, since,

$DC = FA$, $CE = AG$, and $DE = FG$,

$\angle DCE = \angle FAG$,

Therefore,

on AB, and at A, on it,

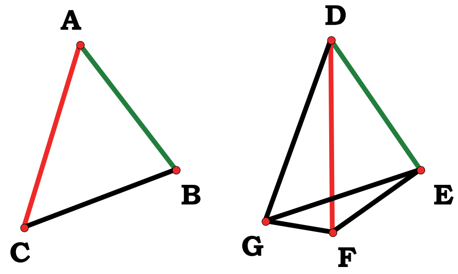
$\angle FAG$, has been constructed,

equal to, the given $\angle DCE$.

Q E F

Proposition 24.

If two triangles have the two sides equal, to two sides respectively, but have the one of the angles contained by the equal straight lines greater than the other, they will, also, have the base greater than the base.



Let,

ΔABC, ΔDEF have

$AB = DE$, and $AC = DF$,

and let,

$\angle$ at $A > \angle$ at D;

I say that;

$BC > EF$.

For, since,

$\angle BAC > \angle EDF$,

let,

there be constructed, on DE,

[I. 23] and,

at D, on it, $\angle EDG = \angle BAC$;

let,

DG be made equal, to either of AC, or DF,

and let,

EG, FG be described.

[I. 4]

Then, since ,

$AB = DE$, and, $AC = DG$,

$BA = ED$, $AC = DG$; and

$\angle BAC = \angle EDG$;

therefore,

$BC = EG$.

[I. 5] Again, since,

$DF = DG$, $\angle DGF = \angle DFG$; therefore,

$\angle DFG > \angle EGF$. Therefore,

$\angle EFG$, IS MUCH GREATER THAN $\angle EGF$.

AND, SINCE,

ΔEFG HAS $\angle EFG > \angle EGF$,

[I. 19] AND,

THE GREATER ANGLE IS SUBTENDED BY THE GREATER SIDE,
$EG > EF$. BUT,
$EG = BC$.

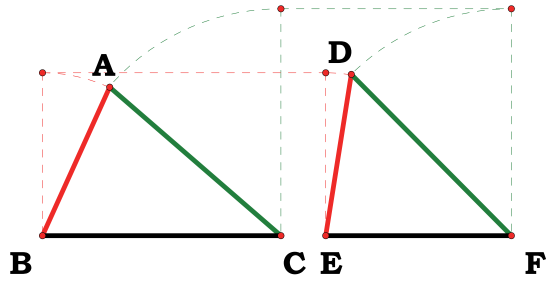
THEREFORE,
$BC > EF$.

THEREFORE ETC.

Q. E. D.

PROPOSITION 25.

IF TWO TRIANGLES HAVE THE TWO SIDES EQUAL, TO TWO SIDES RESPECTIVELY, BUT HAVE THE BASE GREATER THAN THE BASE, THEY WILL, ALSO, *HAVE THE ONE OF THE ANGLES CONTAINED BY THE EQUAL STRAIGHT LINES GREATER THAN THE OTHER.*



LET,

ΔABC, ΔDEF HAVE $AB = DE$, AND $AC = DF$,

AND LET,

$BC > EF$;

I SAY THAT;

$\angle BAC > \angle EDF$. FOR,

IF NOT, THEN,

$\angle BAC \leq \angle EDF$.

NOW,

$\angle BAC \neq \angle EDF$;

[I. 4] FOR THEN,

$BC = EF$, BUT,

IT IS NOT;

THEREFORE,

$\angle BAC \neq \angle EDF$. AGAIN, NEITHER IS

$\angle BAC < \angle EDF$;

[I. 24] FOR THEN,

$BC < EF$, BUT, IT IS NOT;

THEREFORE,

$\angle BAC \nless \angle EDF$.

BUT,

IT WAS PROVED THAT IT IS NOT EQUAL EITHER;

THEREFORE,

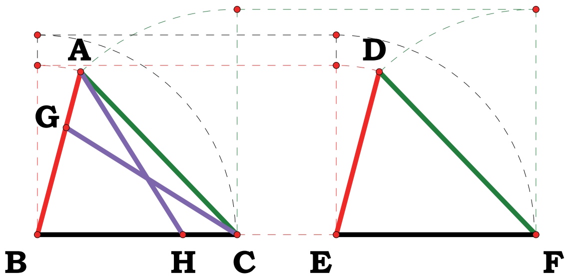
$\angle BAC > \angle EDF$.

THEREFORE ETC.

Q. E. D.

PROPOSITION 26.

IF TWO TRIANGLES HAVE THE TWO ANGLES EQUAL, TO TWO ANGLES RESPECTIVELY, AND ONE SIDE EQUAL, TO ONE SIDE, NAMELY, EITHER THE SIDE ADJOINING THE EQUAL ANGLES, OR THAT SUBTENDING ONE OF THE EQUAL ANGLES, THEY WILL, ALSO, *HAVE THE REMAINING SIDES EQUAL, TO THE REMAINING SIDES AND THE REMAINING ANGLE TO THE REMAINING ANGLE.*



LET,

ΔABC, ΔDEF HAVE

$\angle ABC = \angle DEF$, AND $\angle BCA = \angle EFD$;

AND LET,

THEM, ALSO, HAVE ONE SIDE EQUAL, TO ONE SIDE,
FIRST THAT ADJOINING THE EQUAL ANGLES, NAMELY,
$BC = EF$;

I SAY THAT;

THEY WILL, ALSO, HAVE THE REMAINING SIDES EQUAL, TO THE REMAINING SIDES, RESPECTIVELY, NAMELY,
$AB = DE$, AND
$AC = DF$, AND
THE REMAINING ANGLE TO THE REMAINING ANGLE, NAMELY,

$\angle BAC = \angle EDF$.

FOR, IF,

$AB \neq DE$,
ONE OF THEM IS GREATER.

LET

AB BE GREATER,

AND LET,

$BG = DE$;

AND LET,

GC BE DESCRIBED.

THEN, SINCE,

$BG = DE$, AND $BC = EF$,
$GB = DE$, $BC = EF$; AND

$\angle GBC = \angle DEF$;

[I. 4]

THEREFORE,

$GC = DF$, AND $\Delta GBC = \Delta DEF$, AND

THE REMAINING ANGLES WILL BE EQUAL
TO THE REMAINING ANGLES,

NAMELY,
THOSE WHICH THE EQUAL SIDES SUBTEND;

THEREFORE,

$\angle GCB = \angle DFE$. BUT BY HYPOTHESIS,

$\angle DFE = \angle BCA$; THEREFORE,

$\angle BCG = \angle BCA$, THE LESS TO THE GREATER:

WHICH,
IS IMPOSSIBLE.

THEREFORE,
$AB = DE$, BUT,
$BC = EF$;

THEREFORE,

$AB = DE$, $BC = EF$, AND $\angle ABC = \angle DEF$;

[I. 4]

THEREFORE,

$AC = DF$, AND THE REMAINING, $\angle BAC = \angle EDF$.

AGAIN, LET,
THE SIDES SUBTENDING EQUAL ANGLES BE EQUAL, AS
$AB = DE$;

I SAY AGAIN THAT,
THE REMAINING SIDES WILL BE EQUAL, TO
THE REMAINING SIDES,

NAMELY,
$AC = DF$, AND $BC = EF$,

AND FURTHER,

THE REMAINING, $\angle BAC = \angle EDF$.

FOR, IF,
$BC \neq EF$, THEN,
ONE OF THEM IS GREATER.

LET, IF POSSIBLE,
BC BE GREATER,

AND LET,

$BH = EF$;

LET,

AH BE DESCRIBED.

THEN, SINCE,

$BH = EF$, AND $AB = DE$,

$AB = DE$, $BH = EF$, AND, THEY CONTAIN EQUAL ANGLES;

[I. 4]

THEREFORE,

$AH = DF$, AND

$\Delta ABH = \Delta DEF$, AND

THE REMAINING ANGLES WILL BE EQUAL, TO
THE REMAINING ANGLES, NAMELY,
THOSE WHICH THE EQUAL SIDES SUBTEND;

THEREFORE,

$\angle BHA = \angle EFD$. BUT,

$\angle EFD = \angle BCA$;

[I. 16]

THEREFORE,

IN ΔAHC, THE EXTERIOR

$\angle BHA$ = THE INTERIOR, AND OPPOSITE $\angle BCA$:

WHICH,

IS IMPOSSIBLE.

THEREFORE,

$BC = EF$,

BUT,

$AB = DE$;

THEREFORE,

$AB = DE$, $BC = EF$, AND THEY CONTAIN EQUAL ANGLES;

[I. 4]

THEREFORE,

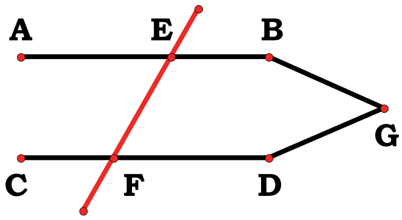
$AC = DF$, $\Delta ABC = \Delta DEF$, AND

THE REMAINS, $\angle BAC = \angle EDF$.

THEREFORE ETC.

Q. E. D.

PROPOSITION 27.



IF A STRAIGHT LINE INTERSECTING TWO STRAIGHT LINES MAKE THE ALTERNATE ANGLES EQUAL, TO ONE ANOTHER, THE STRAIGHT LINES WILL BE PARALLEL TO ONE ANOTHER.

FOR LET,

EF, INTERSECTING AB, CD, MAKE THE ALTERNATES,

$\angle AEF$, $\angle EFD$, EQUAL, TO ONE ANOTHER;

I SAY THAT;

$AB \parallel CD$. FOR,

IF NOT,

THEN,

AB, CD, WHEN PRODUCED, WILL MEET EITHER IN THE DIRECTION OF B, D, OR TOWARDS A, C.

LET,

THEM BE PRODUCED AND MEET,
IN THE DIRECTION OF B, D, AT G.

[I. 16] THEN,

IN ΔGEF, THE EXTERIOR

$\angle AEF$ = THE INTERIOR AND OPPOSITE $\angle EFG$:

WHICH,

IS IMPOSSIBLE.

THEREFORE,

AB, CD WHEN PRODUCED WILL NOT MEET IN THE DIRECTION OF B, D.

SIMILARLY IT CAN BE PROVED THAT,

NEITHER WILL THEY MEET TOWARDS A, C.

[DEF. 23]

BUT

STRAIGHT LINES,
WHICH DO NOT MEET IN EITHER DIRECTION, ARE PARALLEL;

THEREFORE,

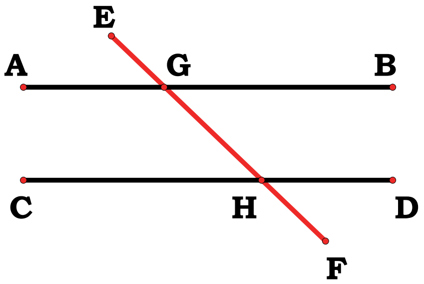
$AB \parallel CD$.

THEREFORE ETC.

Q. E. D.

Proposition 28.

If a straight line intersecting two straight lines make the exterior angle equal, to the interior and opposite angle on the same side, or the interior angles on the same side equal, to two right angles, the straight lines will be parallel to one another.



For let,

EF, intersecting AB, CD, make

the exterior $\angle EGB$ equal to

the interior and opposite $\angle GHD$, or,

the interior angles on the same side,

namely,

$\angle BGH + \angle GHD$, equal, to two right angles;

I say that;

$AB \parallel CD$. For, since,

$\angle EGB = \angle GHD$, [I. 15] while,

$\angle EGB = \angle AGH$, $\angle AGH = \angle GHD$; and they are alternate;

[I. 27] therefore,

$AB \parallel CD$. Again, since ,

$\angle BGH + \angle GHD$, are equal, to two right angles,

[I. 13] and,

$\angle AGH + \angle BGH$, are, also, equal, to two right angles,

$\angle AGH + \angle BGH = \angle BGH + \angle GHD$.

Let,

$\angle BGH$, be subtracted from each; therefore,

the remains, $\angle AGH = \angle GHD$; and

they are alternate;

[I. 27]

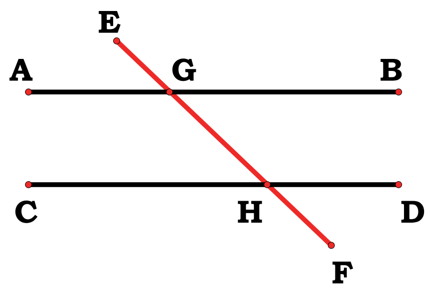
therefore,

$AB \parallel CD$.

Therefore etc.

Q. E. D.

PROPOSITION 29.



A STRAIGHT LINE INTERSECTING PARALLEL STRAIGHT LINES MAKES THE ALTERNATE ANGLES EQUAL, TO ONE ANOTHER, THE EXTERIOR ANGLE EQUAL, TO THE INTERIOR AND OPPOSITE ANGLE, AND THE INTERIOR ANGLES ON THE SAME SIDE EQUAL, TO TWO RIGHT ANGLES.

FOR LET,

EF, INTERSECT THE PARALLEL STRAIGHT LINES, AB, CD;

I SAY THAT;

IT MAKES THE ALTERNATES, $\angle AGH = \angle GHD$,

THE EXTERIOR TO THE INTERIOR AND OPPOSITE,

$\angle EGB$, = $\angle GHD$, AND

THE INTERIOR ANGLES ON THE SAME SIDE,

NAMELY,

$\angle BGH + \angle GHD$, EQUAL, TO TWO RIGHT ANGLES.

FOR, IF,

$\angle AGH \neq \angle GHD$, ONE OF THEM IS GREATER.

LET,

$\angle AGH$, BE GREATER.

LET,

$\angle BGH$ BE ADDED TO EACH;

THEREFORE,

$\angle AGH + \angle BGH$, > $\angle BGH + \angle GHD$.

[I. 13] BUT

$\angle AGH + \angle BGH$, ARE EQUAL, TO TWO RIGHT ANGLES;

THEREFORE,

$\angle BGH + \angle GHD$, ARE LESS THAN TWO RIGHT ANGLES.

[POST 5] BUT,

STRAIGHT LINES, PRODUCED INDEFINITELY, FROM,
ANGLES LESS THAN TWO RIGHT ANGLES, MEET;

THEREFORE,

AB, CD, IF PRODUCED INDEFINITELY, WILL MEET;

BUT,

THEY DO NOT MEET, BECAUSE,
BY HYPOTHESIS, THEY ARE PARALLEL.

THEREFORE,

$\angle AGH = \angle GHD$,

[I. 15] AGAIN,

$\angle AGH = \angle EGB$;

[*C. N.* 1] THEREFORE,

$\angle EGB = \angle GHD$.

LET,

$\angle BGH$, BE ADDED TO EACH;

[*C. N.* 2] THEREFORE,

$\angle EGB + \angle BGH = \angle BGH + \angle GHD$.

[I. 13] BUT,

$\angle EGB + \angle BGH$, ARE EQUAL, TO TWO RIGHT ANGLES;

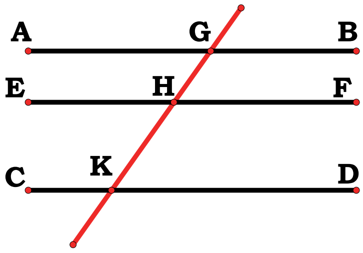
THEREFORE,

$\angle BGH + \angle GHD$, ARE, ALSO, EQUAL, TO TWO RIGHT ANGLES.

THEREFORE ETC.

Q. E. D.

PROPOSITION 30.



STRAIGHT LINES PARALLEL TO THE SAME STRAIGHT LINE ARE, ALSO, *PARALLEL TO ONE ANOTHER.*

LET,

$AB \parallel EF$, $CD \parallel EF$;

I SAY THAT;

$AB \parallel CD$.

FOR LET,

GK, INTERSECT THEM.

[I. 29]

THEN, SINCE,

$GK \cap (AB \parallel EF)$, $\angle AGK = \angle GHF$.

[I. 29] AGAIN, SINCE,

$GK \cap (EF \parallel CD)$, $\angle GHF = \angle GKD$. BUT

$\angle AGK = \angle GHF$;

[*C. N.* 1] THEREFORE,

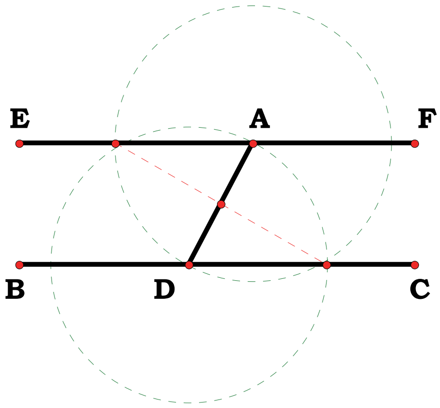
$\angle AGK = \angle GKD$; AND THEY ARE ALTERNATE.

THEREFORE,

$AB \parallel CD$.

Q. E. D.

PROPOSITION 31.



THROUGH A GIVEN *POINT TO DRAW A STRAIGHT LINE PARALLEL TO* A GIVEN *STRAIGHT LINE.*

LET,

A, BE THE GIVEN POINT, AND,

BC, THE GIVEN STRAIGHT LINE;

THUS IT IS REQUIRED, TO DRAW

THROUGH THE POINT, *A*,

A LINE PARALLEL TO *BC*.

LET, AT RANDOM,

ON *BC, D* BE TAKEN,

AND LET,

AD BE DESCRIBED;

[I. 23]

LET,

ON *DA*, AND *A*, ON IT, $\angle DAE = \angle ADC$;

AND LET,

AF, BE PRODUCED, COLLINEAR WITH *EA*.

THEN, SINCE,

AD,∩ (*BC*, *EF*), HAS MADE THE ALTERNATES,

$\angle EAD = \angle ADC$.

[I. 27] THEREFORE,

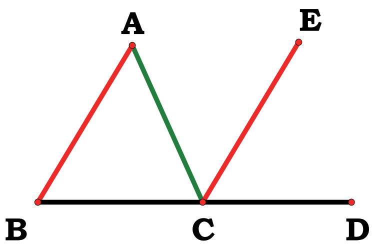
$EAF \parallel BC$,

THEREFORE,

THROUGH *A*, AND PARALLEL TO *BC*; *EAF*, HAS BEEN DRAWN.

Q. E. F.

PROPOSITION 32.



IN ANY TRIANGLE, IF ONE OF THE SIDES BE PRODUCED, THE EXTERIOR ANGLE IS EQUAL, TO THE TWO INTERIOR AND OPPOSITE ANGLES, AND THE THREE INTERIOR ANGLES OF THE TRIANGLE ARE EQUAL, TO TWO RIGHT ANGLES.

LET,

ΔABC,

AND LET,

ONE SIDE OF IT, BC, BE PRODUCED TO D;

I SAY THAT;

THE EXTERIOR, $\angle ACD =$

THE TWO INTERIOR AND OPPOSITES, $\angle CAB + \angle ABC$, AND

THE THREE INTERIOR ANGLES OF THE TRIANGLE,

$\angle ABC + \angle BCA + \angle CAB = 2\llcorner$.

[I. 31] FOR LET,

$CE \parallel AB$, THROUGH THE POINT, C,

[I. 29] THEN, SINCE,

$(AB \parallel CE) \cap AC$, THE ALTERNATE ANGLES, $\angle BAC = \angle ACE$.

[I. 29] AGAIN, SINCE,

$(AB \parallel CE) \cap BD$, THE EXTERIOR,

$\angle ECD = \angle ABC$, THE INTERIOR AND OPPOSITE.

BUT,

$\angle ACE = \angle BAC$;

THEREFORE,

THE WHOLE, $\angle ACD = \angle BAC + \angle ABC$,

THE TWO INTERIOR AND OPPOSITE ANGLES.

LET,

$\angle ACB$, BE ADDED TO EACH;

THEREFORE,

$\angle ACD + \angle ACB = \angle ABC + \angle BCA + \angle CAB$.

[I. 13] BUT,

$\angle ACD + \angle ACB$, ARE EQUAL, TO TWO RIGHT ANGLES;

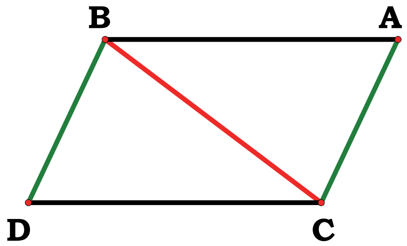
THEREFORE,

$\angle ABC + \angle BCA + \angle CAB$, ARE, ALSO, EQUAL, TO TWO RIGHT ANGLES.

THEREFORE ETC.

Q. E. D.

PROPOSITION 33.



THE STRAIGHT LINES JOINING EQUAL AND PARALLEL STRAIGHT LINES (AT THE EXTREMITIES WHICH ARE) IN THE SAME DIRECTIONS (RESPECTIVELY) ARE THEMSELVES, ALSO, *EQUAL AND PARALLEL.*

LET,

$AB = CD$, $AB \parallel CD$,

AND LET,

AC, BD, JOIN THEM (AT THE EXTREMITIES WHICH ARE) IN THE SAME DIRECTIONS (RESPECTIVELY);

I SAY THAT;

$AC = BD$, AND $AC \parallel BD$.

LET,

BC BE DESCRIBED.

[I. 29] THEN, SINCE,

$AB \parallel CD$, AND BC INTERSECTS THEM,

THE ALTERNATES, $\angle ABC = \angle BCD$.

AND, SINCE,

$AB = CD$, AND BC IS COMMON,

$AB = DC$, $BC = CB$; AND $\angle ABC = \angle BCD$;

[I. 4]THEREFORE,

$AC = BD$, AND $\Delta ABC = \Delta DCB$, AND

THE REMAINING ANGLES WILL BE EQUAL, TO THE REMAINING ANGLES, RESPECTIVELY, NAMELY, THOSE WHICH THE EQUAL SIDES SUBTEND;

THEREFORE,

$\angle ACB = \angle CBD$.

[I. 27] AND, SINCE,

$BC \cap (AC, BD)$ HAS MADE

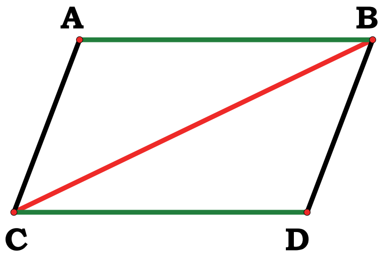
THE ALTERNATE ANGLES EQUAL, TO ONE ANOTHER,

$AC \parallel BD$, $AC = BD$

THEREFORE ETC.

Q. E. D.

Proposition 34.



In parallelogrammic areas the opposite sides and angles are equal, to one another, and the diameter bisects the areas.

Let,

⊟$ACDB$, and BC its diameter;

I say that;

the opposite sides and angles of ⊟$ACDB$, are equal, to one another, and the diameter, BC, bisects it.

[I. 29] For, since,

$AB \parallel CD$, and BC, intersects them,

the alternates, $\angle ABC = \angle BCD$.

[I. 29] Again, since,

$AC \parallel BD$, and BC intersects them,

the alternates, $\angle ACB = \angle CBD$.

[I. 26] Therefore,

ΔABC, ΔDCB have

$\angle ABC = \angle DCB$, $\angle BCA = \angle CBD$, and

one side equal, to one side,

namely,

that adjoining the equal angles, and common to both of them, BC;

therefore,

they will, also, have the remaining sides equal, to the remaining sides, respectively, and, the remaining angle to the remaining angle;

therefore,

$AB = CD$, $AC = BD$, and further,

$\angle BAC = \angle CDB$.

[*C. N.* 2] And, since,

$\angle ABC = \angle BCD$, $\angle CBD$, to $\angle ACB$,

the whole, $\angle ABD = \angle ACD$. And,

$\angle BAC = \angle CDB$.

THEREFORE,

IN PARALLELOGRAMMIC AREAS, THE OPPOSITE SIDES AND ANGLES ARE EQUAL, TO ONE ANOTHER.

I SAY, NEXT, THAT;

THE DIAMETER, ALSO, BISECTS THE AREAS.

FOR, SINCE,

$AB = CD$, AND BC IS COMMON,

THE TWO SIDES,

$AB = DC$, $BC = CB$; AND $\angle ABC = \angle BCD$;

[I. 4] THEREFORE,

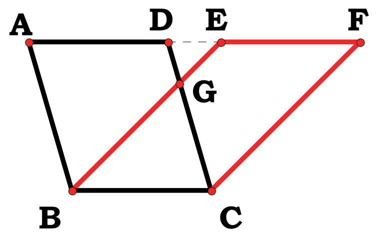
$AC = DB$, AND $\Delta ABC = \Delta DCB$.

THEREFORE,

THE DIAMETER, BC, BISECTS THE PARALLELOGRAM, $ACDB$.

Q. E. D.

PROPOSITION 35.



PARALLELOGRAMS WHICH ARE ON THE SAME BASE AND IN THE SAME PARALLELS ARE EQUAL, TO ONE ANOTHER.

LET,

⊟*ABCD*, ⊟*EBCF*, BE ON THE SAME BASE, *BC*, AND IN THE SAME PARALLELS, *AF*, *BC*;

I SAY THAT;

⊟*ABCD* = ⊟*EBCF*.

[I. 34] FOR, SINCE,

⊟*ABCD*M, *AD* = *BC*.

[*C. N.* 1] FOR THE SAME REASON ALSO,

EF = *BC*,

SO THAT,

AD = *EF*; AND

DE IS COMMON;

[*C. N.* 2] THEREFORE,

THE WHOLES, *AE* = *DF*.

[I. 34] BUT,

AB = *DC*;

[I. 29] THEREFORE,

EA = *FD*, *AB* = *DC*, AND

∠*FDC* = ∠*EAB*, THE EXTERIOR TO THE INTERIOR;

[I. 4] THEREFORE,

EB = *FC*, AND Δ*EAB*, = Δ*FDC*.

LET,

DGE BE SUBTRACTED FROM EACH;

[*C. N.* 3] THEREFORE,

THE TRAPEZIUMS WHICH REMAIN, *ABGD* = *EGCF*.

LET,

Δ*GBC*, BE ADDED TO EACH;

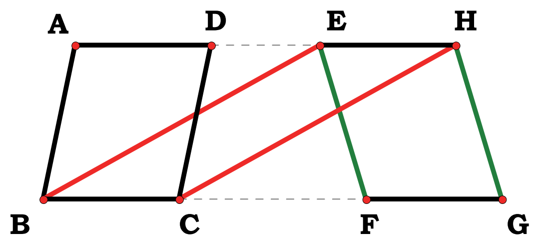
[*C. N.* 2] THEREFORE,

THE WHOLES, ⊟*ABCD* = ⊟*EBCF*.

THEREFORE ETC.

Q. E. D.

PROPOSITION 36.



PARALLELOGRAMS WHICH ARE ON EQUAL BASES AND IN THE SAME PARALLELS ARE EQUAL, TO ONE ANOTHER.

LET,

⊟$ABCD$, ⊟$EFGH$,

WHICH ARE ON EQUAL BASES, BC, FG, AND
IN THE SAME PARALLELS, AH, BG;

I SAY THAT;

⊟$ABCD$ = ⊟$EFGH$.

FOR LET,

BE, CH BE DESCRIBED.

[*C. N.* 1] THEN, SINCE,

$BC = FG$, WHILE ,
$FG = EH$, $BC = EH$.

BUT,

THEY ARE, ALSO, PARALLEL. AND,
EB, HC, JOIN THEM;

[I. 33] BUT,

STRAIGHT LINES JOINING EQUAL AND PARALLEL STRAIGHT LINES (AT THE EXTREMITIES WHICH ARE) IN THE SAME DIRECTIONS (RESPECTIVELY) ARE EQUAL AND PARALLEL.

[I. 34] [I. 35] THEREFORE,

⊟$EBCH$ = ⊟$ABCD$;

FOR,

IT HAS THE SAME BASE, BC, WITH IT, AND
IS IN THE SAME PARALLELS, BC, AH WITH IT.

[I. 35] FOR THE SAME REASON ALSO,

⊟$EFGH$ = ⊟$EBCH$

[*C. N.* 1]

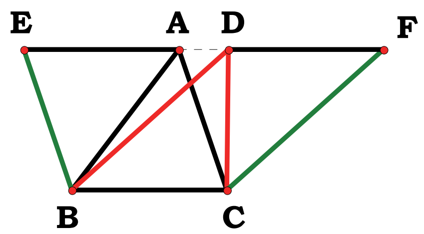
SO THAT

⊟$ABCD$ = ⊟$EFGH$.

THEREFORE ETC.

Q. E. D.

PROPOSITION 37.



TRIANGLES WHICH ARE ON THE SAME BASE AND IN THE SAME PARALLELS ARE EQUAL, TO ONE ANOTHER.

LET,

ΔABC, ΔDBC BE ON THE SAME BASE, BC, AND

IN $AD \parallel BC$;

I SAY THAT;

$\Delta ABC = \Delta DBC$.

LET,

AD BE PRODUCED IN BOTH DIRECTIONS TO E, F;

[I. 31] LET,

THROUGH B, $BE \parallel CA$,

[I. 31] AND LET,

THROUGH C, $CF \parallel BD$. THEN,

EACH, OF THE FIGURES;

$\boxminus EBCA = \boxminus DBCF$,

[I. 35] FOR,

THEY ARE ON THE SAME BASE, BC, AND $BC \parallel EF$. MOREOVER,

$\Delta ABC = \dfrac{\boxminus EBCA}{2}$; [I. 34] FOR,

THE DIAMETER, AB, BISECTS IT. AND,

$\Delta DBC = \dfrac{\boxminus DBCF}{2}$;

[I. 34] FOR,

THE DIAMETER, DC, BISECTS IT.

[BUT THE HALVES OF EQUAL THINGS ARE EQUAL, TO ONE ANOTHER.]

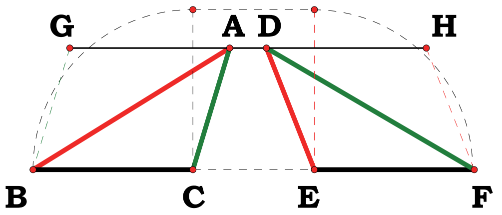
THEREFORE,

$\Delta ABC = \Delta DBC$.

THEREFORE ETC.

Q. E. D.

Proposition 38.



Triangles which are on equal bases and in the same parallels are equal, to one another.

Let,

ΔABC, ΔDEF on equal bases, $BC = EF$, and $BF \parallel AD$;

I say that;

$\Delta ABC = \Delta DEF$.

For let,

AD be produced, in both directions, to G, H;

[I. 31] Let,

through B, $BG \parallel CA$,

and let,

through F, $FH \parallel DE$.

Then,

$\boxminus GBCA = \boxminus DEFH$;

[I. 36] for,

$BC = EF$, and $BF \parallel GH$.

[I. 34] Moreover,

$$\Delta ABC = \frac{\boxminus GBCA}{2};$$

for,

the diameter AB bisects it.

[I. 34] And,

$\Delta FED = \dfrac{\boxminus DEFH}{2}$, for

the diameter, DF, bisects it.

[But the halves of equal things are equal, to one another.]

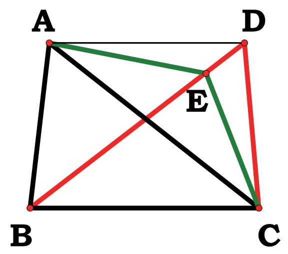
Therefore,

$\Delta ABC = \Delta DEF$.

Therefore etc.

Q. E. D.

PROPOSITION 39.



EQUAL TRIANGLES WHICH ARE ON THE SAME BASE AND ON THE SAME SIDE ARE, ALSO, *IN THE SAME PARALLELS.*

LET,

$\Delta ABC = \Delta DBC$, ON THE SAME BASE, BC, AND

ON THE SAME SIDE OF IT;

[I SAY THAT;

THEY ARE, ALSO, IN THE SAME PARALLELS.]

AND [FOR] LET,

AD BE DESCRIBED;

I SAY THAT;

$AD \parallel BC$. FOR, IF NOT,

[I. 31] LET,

$AE \parallel BC$, BE DRAWN THROUGH THE POINT, A, AND LET,

EC BE DESCRIBED. THEREFORE,

$\Delta ABC = \Delta EBC$;

[I. 37] FOR,

IT IS ON THE SAME BASE, BC, WITH IT, AND,

IN THE SAME PARALLELS.

BUT,

$\Delta ABC = \Delta DBC$;

[*C. N.* 1] THEREFORE,

$\Delta DBC = \Delta EBC$,

THE GREATER TO THE LESS: WHICH,

IS IMPOSSIBLE.

THEREFORE,

$AE \nparallel BC$.

SIMILARLY, WE CAN PROVE THAT;

NEITHER IS ANY OTHER STRAIGHT LINE EXCEPT AD;

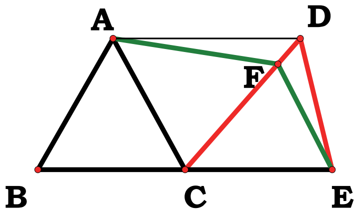
THEREFORE,

$AD \parallel BC$.

THEREFORE ETC.

Q. E. D.

[**PROPOSITION 40.**



EQUAL TRIANGLES WHICH ARE ON EQUAL BASES AND ON THE SAME SIDE ARE, ALSO, *IN THE SAME PARALLELS.*

LET,

$\Delta ABC = \Delta CDE,$

ON BASES, $BC = CE$, AND ON THE SAME SIDE.

I SAY THAT;

THEY ARE, ALSO, IN THE SAME PARALLELS.

FOR LET,

AD BE DESCRIBED;

I SAY THAT;

$AD \parallel BE$. FOR, IF NOT,

[I. 31] LET

$AF \parallel BE$, BE DRAWN, THROUGH A,

AND LET

FE BE DESCRIBED.

THEREFORE,

$\Delta ABC = \Delta FCE;$

[I. 38] FOR

$BC = CE$, AND $BE \parallel AF$. BUT

$\Delta ABC = \Delta DCE;$

[$C\,N.$ 1] THEREFORE,

$\Delta DCE = \Delta FCE,$

THE GREATER TO THE LESS: WHICH IS IMPOSSIBLE. THEREFORE,

$AF \nparallel BE.$

SIMILARLY, WE CAN PROVE THAT;

NEITHER IS ANY OTHER STRAIGHT LINE EXCEPT AD;

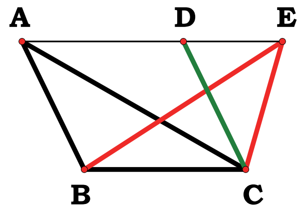
THEREFORE,

$AD \parallel BE.$

THEREFORE ETC.

Q. E. D.]

Proposition 41.



If a parallelogram have the same base with a triangle and be in the same parallels, the parallelogram is double of the triangle.

For let,

$\boxminus ABCD$, have the same base, BC with ΔEBC,

and let,

$BC \parallel AE$;

I say that;

$\boxminus ABCD = 2\Delta BEC$.

For let,

AC be described.

Then,

$\Delta ABC = \Delta EBC$;

[I. 37] for,

it is on the same base, BC, with it, and $BC \parallel AE$.

[I. 34] But,

$\boxminus ABCD = 2\Delta ABC$;

for,

the diameter, AC, bisects it; so,

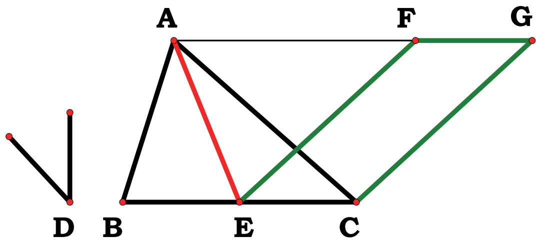
$\boxminus ABCD = 2\Delta EBC$.

Therefore etc.

Q. E. D.

PROPOSITION 42.

TO CONSTRUCT,



IN A GIVEN RECTILINEAL ANGLE, A PARALLELOGRAM EQUAL, TO A GIVEN TRIANGLE.

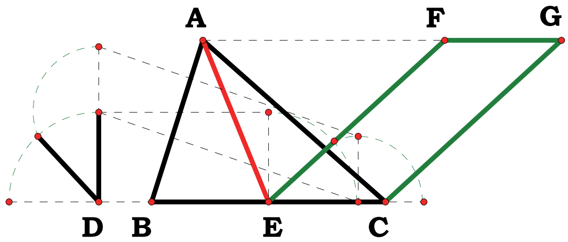
LET,

ΔABC, AND $\angle D$;

THUS IT IS REQUIRED,

TO CONSTRUCT IN THE RECTILINEAL $\angle D$,

A PARALLELOGRAM EQUAL, TO ΔABC.



LET,

BC BE BISECTED AT E,

AND LET,

AE BE DESCRIBED;

[I. 23]

ON,

EC, AND AT E, ON IT,

LET,

$\angle CEF = \angle D$,

[I. 31] LET

THROUGH A, $AG \parallel EC$,

AND LET

THROUGH C, $CG \parallel EF$. THEN

$\boxminus FECG$.

[I. 38] AND, SINCE,

$BE = EC$, $\Delta ABE = \Delta AEC$,

FOR,

$BE = EC$, AND $BC \parallel AG$;

THEREFORE,

$\Delta ABC = 2\Delta AEC$.

[I. 41] BUT,

$\boxminus FECG = 2\Delta AEC$,

FOR,

IT HAS THE SAME BASE WITH IT, AND
IS IN THE SAME PARALLELS WITH IT;

THEREFORE,

$\boxminus FECG = \Delta ABC$.

AND,

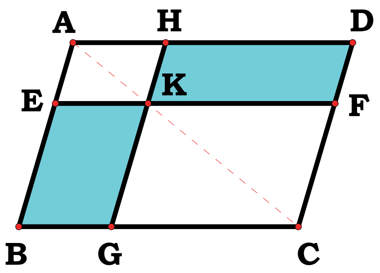
IT HAS $\angle CEF = \angle D$.

THEREFORE,

$\boxminus FECG = \Delta ABC$, $\angle CEF = D$.

Q. E. F.

PROPOSITION 43.



IN ANY PARALLELOGRAM THE COMPLEMENTS OF THE PARALLELOGRAMS ABOUT THE DIAMETER ARE EQUAL, TO ONE ANOTHER.

LET,

⊟*ABCD*, AND

AC, ITS DIAMETER; AND ABOUT *AC*,

LET,

⊟*EH*, ⊟*FG*, AND *BK*, *KD*, BE THE SO-CALLED COMPLEMENTS;

I SAY THAT;

THE COMPLEMENTS, $BK = KD$.

[I. 34]

FOR, SINCE,

⊟*ABCD*, AND *AC* ITS DIAMETER, $\Delta ABC = \Delta ACD$.

AGAIN, SINCE,

⊟*EH*, AND *AK* IS ITS DIAMETER, $\Delta AEK = \Delta AHK$.

FOR THE SAME REASON,

$\Delta KFC = \Delta KGC$.

NOW, SINCE,

$\Delta AEK = \Delta AHK$, AND $\Delta KFC = \Delta KGC$,

[*C. N.* 2]

$\Delta AEK + \Delta KGC = \Delta AHK + \Delta KFC$.

AND,

THE WHOLES, $\Delta ABC = \Delta ADC$;

[*C. N.* 3]

THEREFORE,

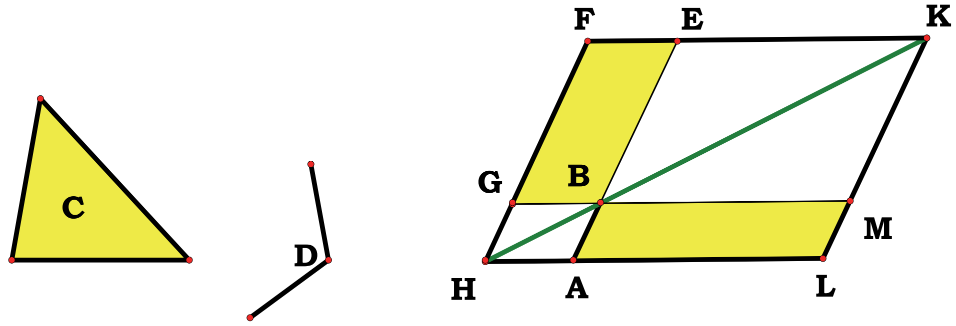
THE COMPLEMENTS WHICH REMAIN, *BK*, = *KD*.

THEREFORE ETC.

Q. E. D.

PROPOSITION 44.

TO A GIVEN STRAIGHT LINE TO APPLY, IN A GIVEN RECTILINEAL ANGLE, A PARALLELOGRAM EQUAL, TO A GIVEN TRIANGLE.



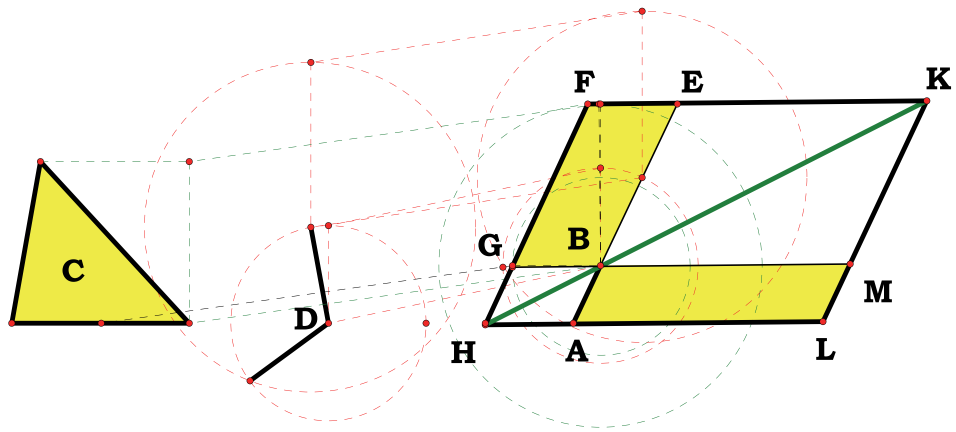
LET,

AB, ΔC, AND $\angle D$ BE GIVEN;

THUS IT IS REQUIRED,

TO APPLY TO AB,

IN AN ANGLE EQUAL, TO $\angle D$, A PARALLELOGRAM EQUAL, TO ΔC.



[I. 42] LET,

$\boxminus BEFG = \Delta C$, $\angle EBG = \angle D$;

LET, IT BE PLACED, SO THAT;

BE IS COLLINEAR WITH AB;

LET,

FG BE DRAWN THROUGH, TO H,

[I. 31] AND LET,

$AH \parallel$ TO EITHER BG OR EF, THROUGH A.

LET,

HB BE DESCRIBED.

[I. 29] THEN, SINCE,

HF, $\cap$ ($AH \parallel EF$),

$\angle AHF + \angle HFE$, ARE EQUAL, TO TWO RIGHT ANGLES.

THEREFORE,

$\angle BHG + \angle GFE$, ARE LESS THAN TWO RIGHT ANGLES;

[POST. 5] AND,
STRAIGHT LINES, PRODUCED INDEFINITELY,
FROM ANGLES LESS THAN TWO RIGHT ANGLES, MEET;

THEREFORE,
HB, FE, WHEN PRODUCED, WILL MEET.

LET,
THEM BE PRODUCED, AND MEET AT K;

[I. 31] LET THROUGH,

K, KL, ∥ TO EITHER, EA OR FH,

AND LET,
HA, GB BE PRODUCED TO THE POINTS, L, M.

THEN,

⊟$HLKF$, HK IS ITS DIAMETER, AND

⊟AG, ⊟ME, AND

LB, BF, THE SO-CALLED COMPLEMENTS, ABOUT HK;

[I. 43] THEREFORE,
$LB = BF$. BUT,

⊟$BF = \Delta C$;

[C. N. 1] THEREFORE,

⊟$LB = \Delta C$.

[I. 15] AND, SINCE,

$\angle GBE = \angle ABM$, WHILE,

$\angle GBE = \angle D$, $\angle ABM = \angle D$.

THEREFORE,

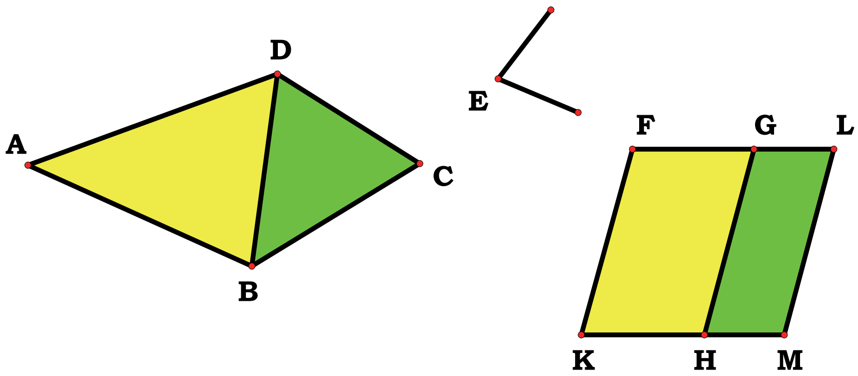
⊟$LB = \Delta C$,

HAS BEEN APPLIED TO AB, IN $\angle ABM$, $= \angle D$.

Q. E. F.

Proposition 45.

To construct, in a given rectilineal angle, a parallelogram equal, to a given rectilineal figure.

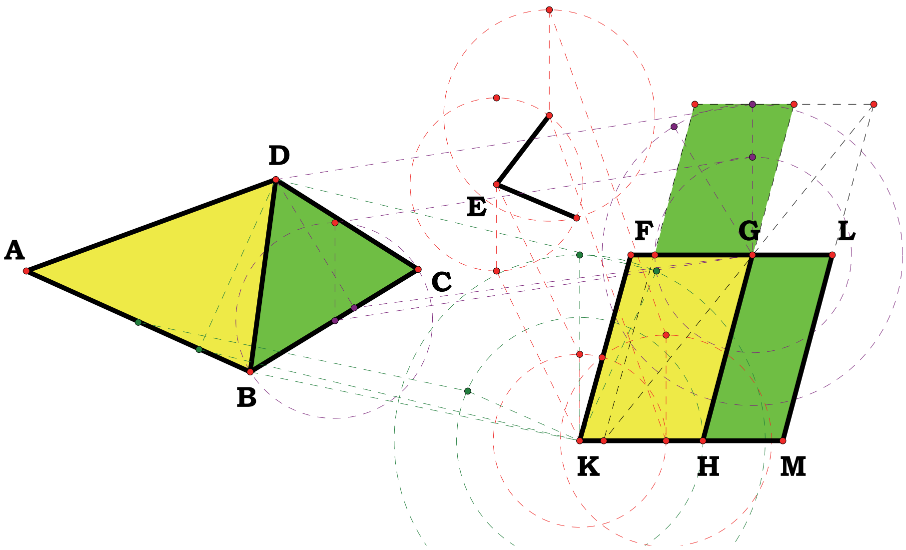


Let,

ABCD be the given rectilineal figure, and $\angle E$;

thus it is required,

to construct, in $\angle E$, a parallelogram equal, to the rectilineal figure, *ABCD*.



[I. 42] Let,

DB be described,

and let,

$\boxminus FH = \Delta ABD$, in $\angle HKF = \angle E$;

[I. 44] let,

$\boxminus GM = \Delta DBC$, be applied to *GH*, in $\angle GHM = \angle E$.

[*C. N.* 1] Then, since,

$\angle E = \angle HKF$, $\angle E = \angle GHM$,

$\angle HKF = \angle GHM$.

LET,

$\angle KHG$, BE ADDED TO EACH;

THEREFORE,

$\angle FKH + \angle KHG = \angle KHG + \angle GHM$.

[I. 29] BUT,

$\angle FKH + \angle KHG$, ARE EQUAL, TO TWO RIGHT ANGLES;

THEREFORE,

$\angle KHG + \angle GHM$, ARE, ALSO, EQUAL, TO TWO RIGHT ANGLES.

[I. 14] THUS,
WITH GH, AND AT H, ON IT, KH, HM,
NOT LYING ON THE SAME SIDE,
MAKE THE ADJACENT ANGLES EQUAL, TO TWO RIGHT ANGLES;

THEREFORE,
KH IS COLLINEAR WITH HM.

[I. 29] AND, SINCE,

$HG \cap (KM \parallel FG)$, THE ALTERNATES, $\angle MHG = \angle HGF$.

LET,

$\angle HGL$ BE ADDED TO EACH;

[*C. N.* 2] THEREFORE,

$\angle MHG + \angle HGL = \angle HGF + \angle HGL$.

[I. 29] BUT,

$\angle MHG + \angle HGL$, ARE EQUAL, TO TWO RIGHT ANGLES;

[*C. N.* 1] THEREFORE,

$\angle HGF + \angle HGL$, ARE, ALSO, EQUAL, TO TWO RIGHT ANGLES.

[I. 14] THEREFORE,
FG IS COLLINEAR WITH GL.

[I. 34] AND, SINCE,

$FK = HG$, $FK \parallel HG$, AND $HG = ML$, $HG \parallel ML$,

[*C. N.* 1; I. 30] ALSO,

$KF = ML$, $KF \parallel ML$; AND

KM, FL, JOIN THEM, (AT THEIR EXTREMITIES);

[I. 33] THEREFORE,

$KM = FL$, $KM \parallel FL$.

THEREFORE,

$\boxminus KFLM$.

AND, SINCE,

$\Delta ABD = \boxminus FH$, AND $\Delta DBC = \boxminus GM$,

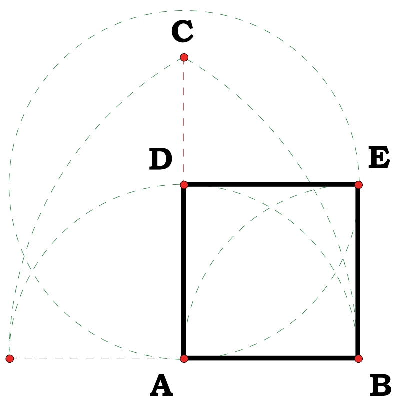
THE WHOLE RECTILINEAL FIGURE $ABCD = \boxminus KFLM$.

THEREFORE,

$\boxminus KFLM$, HAS BEEN CONSTRUCTED EQUAL, TO
THE GIVEN RECTILINEAL FIGURE, $ABCD$,
IN $\angle FKM = \angle E$.

Q. E. F.

Proposition 46.



On a given straight line to describe a square.

Let,

AB be the given straight line;

thus it is required,

to describe a square, on AB.

[I. 11]

Let,

AC be drawn at right angles to AB, from A on it,

and let,

$AD = AB$;

let,

through D, $DE \parallel AB$,

[I. 31] and let,

through the point, B, $BE \parallel AD$.

Therefore,

$\boxminus ADEB$;

[I. 34] therefore,

$AB = DE$, and, $AD = BE$.

But,

$AB = AD$;

therefore,

the four straight lines,

BA, AD, DE, EB, are equal, to one another;

therefore,

$\boxminus ADEB$, is equilateral.

I say next that;

it is, also, right-angled.

[I. 29] For, since,

$AD \cap (AB \parallel DE)$,

$\angle BAD + \angle ADE$, are equal, to two right angles.

But,

$\llcorner BAD$ is right;

THEREFORE,

∟*ADE* IS, ALSO, RIGHT.

[I. 34] AND,

IN PARALLELOGRAMMIC AREAS THE OPPOSITE SIDES, AND, ANGLES ARE EQUAL, TO ONE ANOTHER;

THEREFORE,

EACH, OF THE OPPOSITES, ∟*ABE*, ∟*BED*, IS, ALSO, RIGHT.

THEREFORE,

⊟*ADEB* IS RIGHT-ANGLED.

AND,

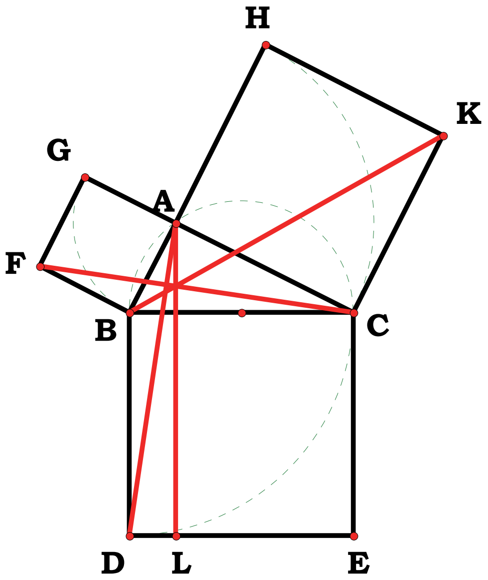
IT WAS, ALSO, PROVED EQUILATERAL.

THEREFORE,

IT IS A SQUARE; AND IT IS DESCRIBED ON *AB*.

Q. E. F.

PROPOSITION 47.



IN RIGHT-ANGLED TRIANGLES THE SQUARE, ON THE SIDE SUBTENDING THE RIGHT ANGLE IS EQUAL, TO THE SQUARES ON THE SIDES CONTAINING THE RIGHT ANGLE.

LET,

⊿*AB*, HAVING, ∟*BAC*;

I SAY THAT;

⊡*BC* = ⊡*BA* + ⊡*AC*.

[I. 46] FOR LET,

THERE BE DESCRIBED, ON *BC*, ⊡*BDEC*, AND

ON *BA*, *AC*, ⊡*GB*, ⊡*HC*;

LET,

THROUGH *A*,

AL ‖ TO EITHER, *BD* OR *CE*,

AND LET,

AD, *FC* BE DESCRIBED.

THEN, SINCE,

∟*BAC*, ∟*BAG*,

IT FOLLOWS THAT;

BA, AND AT *A* ON IT, *AC*, *AG*,
NOT LYING ON THE SAME SIDE, MAKE
THE ADJACENT ANGLES EQUAL, TO TWO RIGHT ANGLES;

[I. 14] THEREFORE,

CA IS IN A STRAIGHT LINE WITH *AG*.

FOR THE SAME REASON,

BA IS, ALSO, IN A STRAIGHT LINE WITH *AH*.

AND, SINCE,

∟*DBC* = ∟*FBA*: FOR, EACH IS RIGHT:

LET,

∠*ABC*, BE ADDED TO EACH;

[C. N. 2] THEREFORE,

THE WHOLE, $\angle DBA = \angle FBC$,

AND, SINCE,

$DB = BC$, AND $FB = BA$,

[I. 4] THEREFORE,

$AB = FB$, $BD = BC$, AND $\angle ABD = \angle FBC$;

THEREFORE,

$AD = FC$, AND $\Delta ABD = \Delta FBC$.

[I. 41] NOW,

$\boxminus BL = 2\ \Delta ABD$,

FOR,

THEY HAVE THE SAME BASE, BD, AND
ARE IN THE SAME PARALLELS, BD, AL.

[I. 41] AND,

$\boxdot GB = 2\Delta FBC$,

FOR,

THEY AGAIN HAVE THE SAME BASE, FB, AND
ARE IN THE SAME PARALLELS, FB, GC,
[BUT THE DOUBLES OF EQUALS ARE EQUAL, TO ONE ANOTHER.]

THEREFORE,

$\boxminus BL = \boxdot GB$.

SIMILARLY,

IF AE, BK BE DESCRIBED,

$\boxminus CL$, CAN, ALSO, BE PROVED EQUAL, TO $\boxdot HC$;

[C. N. 2] THEREFORE,

THE WHOLE $\boxdot BDEC = \boxdot GB + \boxdot HC$.

AND,

$\boxdot BDEC$, IS DESCRIBED, ON BC, AND

$\boxdot GB$, $\boxdot HC$, ON BA, AC.

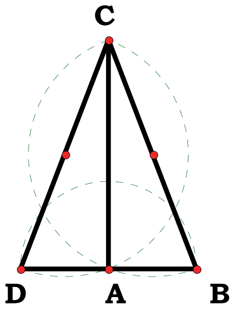
THEREFORE,

$\boxdot BC = \boxdot BA + \boxdot AC$.

THEREFORE ETC.

Q. E. D.

PROPOSITION 48.



IF IN A TRIANGLE THE SQUARE, ON ONE OF THE SIDES BE EQUAL, TO THE SQUARES ON THE REMAINING TWO SIDES OF THE TRIANGLE, THE ANGLE CONTAINED BY THE REMAINING TWO SIDES OF THE TRIANGLE IS RIGHT.

FOR,

IN ΔABC,

LET,

$\boxdot BC = \boxdot BA + \boxdot AC$;

I SAY THAT;

$\angle BAC$, IS RIGHT.

FOR LET,

AD, BE DRAWN FROM A, AT RIGHT ANGLES TO AC,

LET,

$AD = BA$,

AND LET,

DC BE DESCRIBED.

SINCE,

$DA = AB$,

$\boxdot DA = \boxdot AB$.

LET,

$\boxdot AC$, BE ADDED TO EACH;

THEREFORE,

$\boxdot DA + \boxdot AC = \boxdot BA + \boxdot AC$.

[I. 47] BUT,

$\boxdot DC = \boxdot DA + \boxdot AC$,

FOR,

$\llcorner DAC$, IS RIGHT; AND $\boxdot BC = \boxdot BA + \boxdot AC$,

FOR,

THIS IS THE HYPOTHESIS;

THEREFORE,

$\boxdot DC = \boxdot BC$,

SO THAT,

$DC = BC$.

[I. 8]

AND, SINCE,

$DA = AB$, AND
AC IS COMMON,
$DA = BA$, $AC = AC$; AND
$DC = BC$;

THEREFORE,

$\angle DAC = \angle BAC$,

BUT,

$\llcorner DAC$;

THEREFORE,

$\llcorner BAC$.

THEREFORE ETC.

Q. E. D

BOOK II.

OF

EUCLID'S ELEMENTS

TRANSLATED FROM THE TEXT OF HEIBERG

BY

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REFORMATTED AND ABRIDGED FOR STUDY OF *THE ELEMENTS*.

BY JOHN CLARK.

BOOK II.

DEFINITIONS.

1. ANY RECTANGULAR PARALLELOGRAM IS SAID TO BE **CONTAINED** BY THE TWO STRAIGHT LINES CONTAINING THE RIGHT ANGLE.

2. AND IN ANY PARALLELOGRAMMIC AREA LET ANY ONE WHATEVER OF THE PARALLELOGRAMS ABOUT ITS DIAMETER WITH THE TWO COMPLEMENTS BE CALLED A **GNOMON**.

NOTES.

Definition 1. *ANY RECTANGULAR PARALLELOGRAM IS SAID TO BE* CONTAINED *BY THE TWO STRAIGHT LINES CONTAINING THE RIGHT ANGLE.*

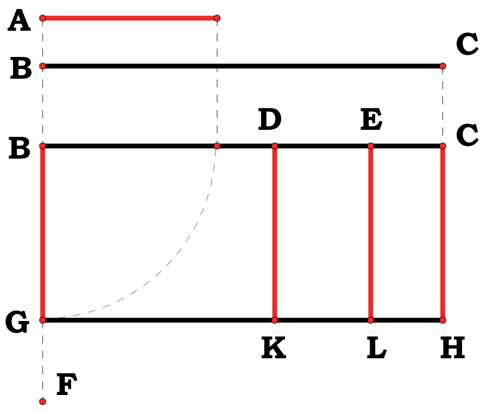
NOTES.

Definition 2. *AND IN ANY PARALLELOGRAMMIC AREA LET ANY ONE WHATEVER OF THE PARALLELOGRAMS ABOUT ITS DIAMETER WITH THE TWO COMPLEMENTS BE CALLED A* GNOMON.

BOOK II.

PROPOSITIONS.

Proposition 1.



If there be two straight lines, and one of them be cut into any number of segments whatever, the rectangle contained by the two straight lines is equal, to the rectangles contained by the uncut straight line and each, of the segments.

Let,

A, BC,

and, at random, let,

BC be divided at D, E;

I say that;

$A \times BC = A \times BD + A \times DE + A \times EC.$

[I. 11] For let,

BF be drawn, from B, at right angles, to BC;

[I. 3] let,

$BG = A$,

[I. 31] let,

through G, $GH \parallel BC$,

let,

through, D, E, C, $(DK, EL, CH) \parallel BG$.

Then,

$BH = BK + DL + EH.$

Now,

$BH = A \boxtimes BC$,

for,

it is contained by GB, BC, and

$BG = A$; $BK = A \boxtimes BD$,

for,

it is contained by GB, BD, and

$BG = A$; and $DL = A \boxtimes DE$,

[I. 34] FOR,

DK, THAT IS $BG = A$.

SIMILARLY ALSO,

$EH = A \boxtimes EC$.

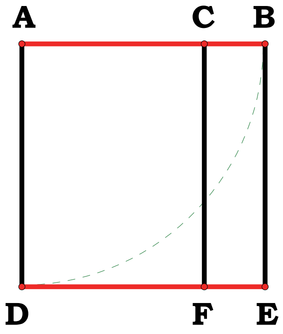
THEREFORE,

$A \boxtimes BC = A \boxtimes BD + A \boxtimes DE + A \boxtimes EC$.

THEREFORE ETC.

Q. E. D.

Proposition 2.



If a straight line be cut at random, the rectangle contained by the whole and both of the segments is equal, to the square, on the whole.

For, at random, let,

AB be divided at C;

I say that;

$AB \boxtimes BC + BA \boxtimes AC = \boxdot AB$.

[I. 46] For let,

$\boxdot ADEB$, be described, on AB,

[I. 31] and let,

CF be drawn, through C, parallel to either, AD or BE.

Then,

$\boxdot AE = AF \boxtimes CE$.

Now,

$\boxdot AE = \boxdot AB$;

$\boxtimes AF = BA \boxtimes AC$,

for,

$BA \boxtimes AC = DA \boxtimes AC$, and

$AD = AB$; and

$\boxtimes CE = AB \boxtimes BC$,

for,

$BE = AB$.

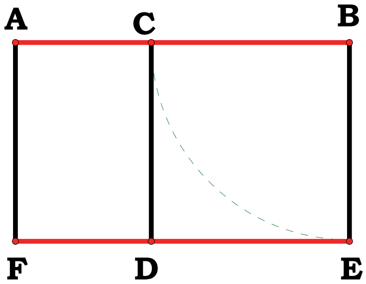
Therefore,

$BA \boxtimes AC + AB \boxtimes BC = \boxdot AB$.

Therefore etc.

Q. E. D.

Proposition 3.



If a straight line be cut at random, the rectangle contained by the whole and one of the segments is equal, to the rectangle contained by the segments and the square, on the aforesaid segment.

For, at random, let,

AB be divided at C;

I say that;

$AB \boxtimes BC = AC \boxtimes CB + \boxdot BC.$

[I. 46] For let,

$\boxdot CDEB$, be described, on CB;

let,

ED be drawn through to F,

[I. 31] and let,

through A,

AF be drawn, parallel to either, CD or BE.

Then,

$\boxtimes AE = \boxtimes AD + \boxdot CE.$

Now,

$AE = AB \boxtimes BC,$

for,

$AB \boxtimes BC = AB \boxtimes BE$, and

$BE = BC;$

$\boxtimes AD = AC \boxtimes CB,$

for,

$DC = CB;$

and,

$\boxdot DB = \boxdot CB.$

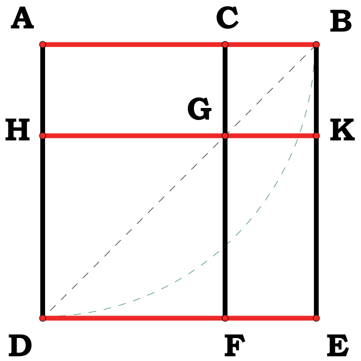
Therefore,

$AB \boxtimes BC = AC \boxtimes CB + \boxdot BC.$

Therefore etc.

Q. E. D.

Proposition 4.



If a straight line be cut at random, the square, on the whole is equal, to the squares on the segments and twice the rectangle contained by the segments.

For, at random, let,

AB be divided at C;

I say that;

$\boxdot AB = \boxdot AC + \boxdot CB + 2AC \boxtimes CB.$

[I. 46] For let,

$\boxdot ADEB = \boxdot AB,$

let,

BD be joined;

let,

through C,

CF be drawn, parallel to either, AD or EB,

[I. 31]

and let,

through G,

HK be drawn, parallel to either, AB or DE.

[I. 29] Then, since,

$CF \parallel AD$, and

BD intersects them,

the exterior $\angle CGB$ = the interior and opposite $\angle ADB$.

But,

$\angle ADB = \angle ABD,$

[I. 5] since,

$BA = AD;$

therefore,

$\angle CGB = \angle GBC,$

[I. 6] so that;

$BC = CG.$

[I. 34] But,

$CB = GK$, and $CG = KB;$

THEREFORE,

$GK = KB$;

THEREFORE,

$CGKB$ IS EQUILATERAL.

I SAY NEXT THAT;

IT IS, ALSO, RIGHT-ANGLED.

[1. 29] FOR, SINCE,

$CG \parallel BK$,

$\angle KBC + \angle GCB = 2 \llcorner$.

BUT,

$\llcorner KBC$, IS RIGHT;

THEREFORE,

$\llcorner BCG$, IS, ALSO, RIGHT,

[I. 34] SO THAT,

THE OPPOSITES, $\llcorner CGK$, $\llcorner GKB$, ARE, ALSO, RIGHT.

THEREFORE,

$CGKB$ IS RIGHT-ANGLED; AND

IT WAS, ALSO, PROVED EQUILATERAL;

THEREFORE,

IT IS A SQUARE; AND

IT IS DESCRIBED, $\boxdot CB$.

[I. 34] FOR THE SAME REASON,

HF IS, ALSO, A SQUARE; AND

IT IS DESCRIBED, $\boxdot HG$, THAT IS $\boxdot AC$.

THEREFORE,

$\boxdot HF$, $\boxdot KC$, ARE THE SQUARES, $\boxdot AC$, $\boxdot CB$.

NOW, SINCE,

$\boxtimes AG = \boxtimes GE$, AND

$\boxtimes AG = AC \boxtimes CB$,

FOR,

$GC = CB$,

THEREFORE,

$\boxtimes GE = AC \boxtimes CB$.

THEREFORE,

$\boxtimes AG + \boxtimes GE = 2AC \boxtimes CB.$

BUT,

$\boxdot HF$, $\boxdot CK$, ARE, ALSO, THE SQUARES, $\boxdot AC$, $\boxdot CB$;

THEREFORE,

$\boxdot HF + \boxdot CK + \boxtimes AG + \boxtimes GE = \boxdot AC + \boxdot CB + 2\ AC \boxtimes CB.$

BUT,

$\boxdot HF + \boxdot CK + \boxtimes AG + \boxtimes GE = \boxdot ADEB,$

WHICH,

IS $\boxdot AB$.

THEREFORE,

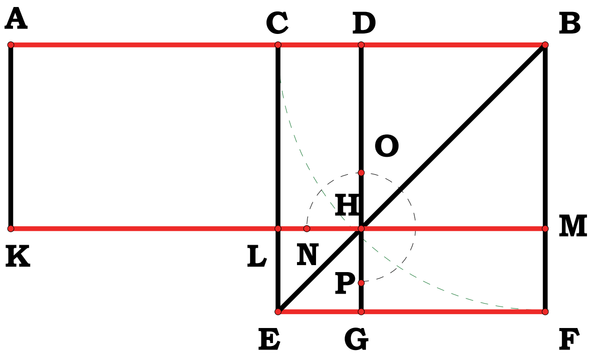
$\boxdot AB = \boxdot AC + \boxdot CB + 2\ AC \boxtimes CB.$

THEREFORE ETC.

Q. E. D.

PROPOSITION 5.

IF A STRAIGHT LINE BE CUT INTO EQUAL AND UNEQUAL SEGMENTS, THE RECTANGLE CONTAINED BY THE UNEQUAL SEGMENTS OF THE WHOLE TOGETHER WITH THE SQUARE, ON THE STRAIGHT LINE BETWEEN THE POINTS OF SECTION IS EQUAL, TO THE SQUARE, ON THE HALF.



FOR LET,

AB,

BE DIVIDED INTO EQUAL SEGMENTS, AT *C*, AND,

INTO UNEQUAL SEGMENTS, AT *D*;

I SAY THAT;

AD⊠*DB* + ⊡ *CD* = ⊡*CB*.

[I. 46]

FOR LET,

⊡*CEFB*, BE DESCRIBED, ⊡*CB*,

AND LET,

BE BE JOINED;

LET,

THROUGH *D*,

DG BE DRAWN, PARALLEL TO EITHER, *CE* OR *BF*,

LET AGAIN,

THROUGH *H*,

KM BE DRAWN, PARALLEL TO EITHER, *AB* OR *EF*,

[I. 31]

LET AGAIN,

THROUGH *A*,

AK BE DRAWN, PARALLEL TO EITHER, *CL* OR *BM*.

[I. 43]

THEN, SINCE,

THE COMPLEMENTS, ⊠*CH* = ⊠*HF*,

LET,

⊡*DM* BE ADDED TO EACH;

THEREFORE,

THE WHOLES, $\boxtimes CM = \boxtimes DF$.

BUT,

$\boxtimes CM = \boxtimes AL$,

[I. 36] SINCE,
$AC = CB$;

THEREFORE,

$\boxtimes AL = \boxtimes DF$.

LET,

$\boxtimes CH$ BE ADDED TO EACH;

THEREFORE,

THE WHOLES, $\boxtimes AH$ = THE GNOMON, NOP.

BUT,

$\boxtimes AH = AD \boxtimes DB$,

FOR,
$DH = DB$,

THEREFORE,

THE GNOMON, $NOP = AD \boxtimes DB$.

$\boxdot LG = \boxdot\ CD$,

LET,

$\boxdot LG$ BE ADDED TO EACH;

THEREFORE,

THE GNOMON, NOP, AND $\boxdot LG = AD \boxtimes DB + \boxdot CD$.

BUT,

THE GNOMON, $NOP + \boxdot LG = \boxdot CEFB$,

WHICH,

IS DESCRIBED, $\boxdot CB$;

THEREFORE,

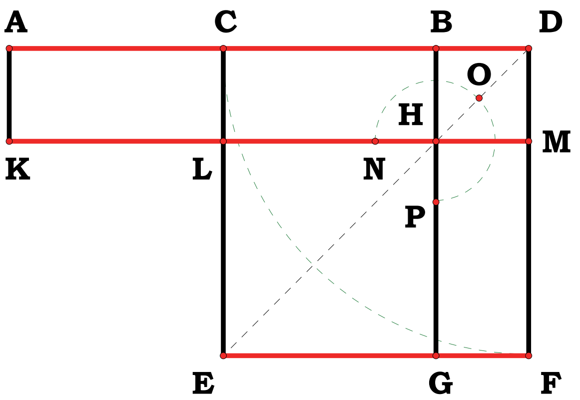
$AD \boxtimes DB + \boxdot\ CD = \boxdot\ B$.

THEREFORE ETC.

Q. E. D.

Proposition 6.

If a straight line be bisected and a straight line be added to it in a straight line, the rectangle contained by the whole with the added straight line and the added straight line together with the square, on the half is equal, to the square, on the straight line made up of the half and the added straight line.



For let,

AB, be bisected at C,

and let,

BD, be added to it in a straight line;

I say that;

$AD \boxtimes DB + \boxdot CB = \boxdot CD$.

[I. 46] For let,

$\boxdot CEFD$, be described, $\boxdot CD$,

and let,

DE be joined;

let,

through B,

BG be drawn, parallel to either, EC or DF,

let,

through H,

KM be drawn, parallel to either, AB or EF,

[I. 31] let,

through A,

AK be drawn, parallel to either, CL or DM.

[I. 36]

Then, since,

$AC = CB$, $\boxtimes AL = \boxtimes CH$.

[I. 43] But,

$\boxtimes CH = \boxtimes HF$.

Therefore,

$\boxtimes AL = \boxtimes HF$.

LET,

$\boxtimes CM$ BE ADDED TO EACH;

THEREFORE,

THE WHOLE, $\boxtimes AM$ = THE GNOMON, NOP.

BUT,

$\boxtimes AM = AD \boxtimes DB$,

FOR,

$DM = DB$;

THEREFORE,

THE GNOMON, $NOP = AD \boxtimes DB$.

$\boxdot LG = \boxdot BC$,

LET,

$\boxdot LG$ BE ADDED TO EACH;

THEREFORE,

$AD \boxtimes DB + \boxdot CB$ = THE GNOMON, $NOP + \boxdot LG$.

BUT,

THE GNOMON, $NOP + \boxdot LG = \boxdot CEFD$,

WHICH IS DESCRIBED, $\boxdot CD$;

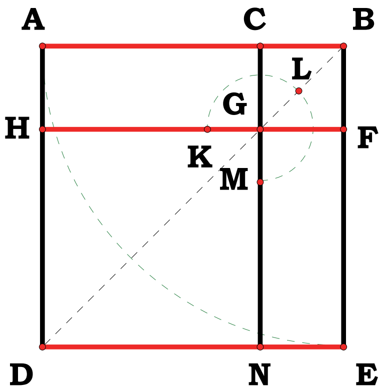
THEREFORE,

$AD \boxtimes DB + \boxdot CB = \boxdot CD$.

THEREFORE ETC.

Q. E. D.

PROPOSITION 7.



IF A STRAIGHT LINE BE CUT AT RANDOM, THE SQUARE, ON THE WHOLE AND THAT ON ONE OF THE SEGMENTS BOTH TOGETHER ARE EQUAL, TO TWICE THE RECTANGLE CONTAINED BY THE WHOLE AND THE SAID SEGMENT AND THE SQUARE, ON THE REMAINING SEGMENT.

FOR, AT RANDOM, LET,

AB, BE DIVIDED AT C;

I SAY THAT;

$\boxdot AB + \boxdot BC = 2AB \boxtimes BC + \boxdot CA.$

[I. 46] FOR LET,

$\boxdot ADEB$, BE DESCRIBED, $\boxdot AB$,

LET,

THE FIGURE BE DRAWN.

[I. 43] THEN, SINCE,

$\boxtimes AG = \boxtimes GE$,

LET,

$\boxdot CF$ BE ADDED TO EACH;

THEREFORE,

THE WHOLES, $\boxtimes AF = \boxtimes CE.$

THEREFORE,

$\boxtimes AF + \boxtimes CE = 2\boxtimes AF.$

BUT,

$\boxtimes AF + \boxtimes CE =$ THE GNOMON, $KLM + \boxdot CF$;

THEREFORE,

THE GNOMON, $KLM + \boxdot CF = 2\boxdot AF.$

BUT,

$2AB \boxtimes BC = 2\boxtimes AF$;

FOR,

$BF = BC$;

THEREFORE,

THE GNOMON, $KLM + \boxdot CF = 2AB \boxtimes BC$.

$\boxdot DG = \boxdot AC$,

LET,

$\boxdot DG$ BE ADDED TO EACH;

THEREFORE,

THE GNOMON, $KLM + \boxdot BG + \boxdot GD = 2AB \boxtimes BC + \boxdot AC$.

BUT,

THE GNOMON, $KLM + \boxdot BG + \boxdot GD =$

$\boxdot ADEB + \boxdot CF = \boxdot AB + \boxdot BC$;

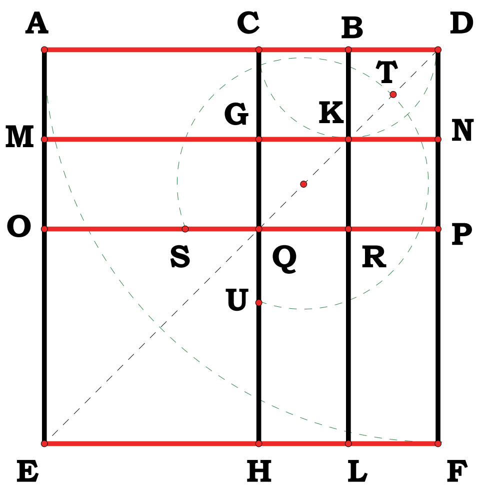
THEREFORE,

$\boxdot AB + \boxdot BC = 2AB \boxtimes BC + \boxdot AC$.

THEREFORE ETC.

Q. E. D.

Proposition 8.



If a straight line be cut at random, four times the rectangle contained by the whole and one of the segments together with the square, on the remaining segment is equal, to the square described on the whole and the aforesaid segment as on one straight line.

For, at random, let,

AB be divided at C;

I say that;

$4AB \boxtimes BC + \boxdot\, AC = \boxdot(AB + BC)$.

For let,

BD be produced collinear [with AB]

and let,

$BD = CB$;

let,

$\boxdot AEFD$, be described, $\boxdot AD$,

and let,

the figure be drawn double.

Then, since,

$CB = BD$, while

$CB = GK$, and

BD to KN,

therefore,

$GK = KN$.

For the same reason,

$QR = RP$.

And, since,

$BC = BD$, and

GK to KN,

[I. 36] therefore,

$\boxdot CK = \boxdot KD$, and

$\boxdot GR$ to $\boxdot RN$.

But,

$\boxdot CK = \boxdot RN$,

[I. 43] for,

they are complements of $\boxminus CP$;

therefore,

$\boxdot KD = \boxdot GR$;

therefore,

$\boxdot DK$, $\boxdot CK$, $\boxdot GR$, $\boxdot RN$, are equal, to one another.

Therefore,

the four are quadruple of $\boxdot CK$.

Again, since,

$CB = BD$, while
$BD = BK$, that is CG, and
$CB = GK$, that is GQ,

therefore,

$CG = GQ$.

[I. 36] And, since,

$CG = GQ$, and QR to RP,

$\boxtimes AG = \boxtimes MQ$, and $\boxtimes QL$ to $\boxtimes RF$.

But,

$\boxtimes MQ = \boxtimes QL$,

[I. 43]for,

they are complements of $\boxminus ML$;

therefore,

$\boxtimes AG = \boxtimes RF$;

therefore,

$\boxtimes AG$, $\boxtimes MQ$, $\boxtimes QL$, $\boxtimes RF$, are equal, to one another.

Therefore,

the four are quadruple of $\boxtimes AG$.

But,

$\boxdot CK, \boxdot KD, \boxdot GR, \boxdot RN = 4\boxdot CK$;

therefore,

THE EIGHT AREAS, WHICH CONTAIN

THE GNOMON, $STU = 4 \boxtimes AK$.

NOW, SINCE,

$\boxtimes AK = AB \boxtimes BD$,

FOR,

$BK = BD$,

THEREFORE,

$4AB \boxtimes BD = 4 \boxtimes AK$.

BUT,

THE GNOMON, STU, WAS, ALSO, PROVED TO BE $4 \boxtimes AK$;

THEREFORE,

$4AB \boxtimes BD =$ THE GNOMON, STU.

$\boxdot OH = \boxdot AC$,

LET,

$\boxdot OH$ BE ADDED TO EACH;

THEREFORE,

$4AB \boxtimes BD + \boxdot AC =$ THE GNOMON, $STU + \boxdot OH$.

BUT,

THE GNOMON, $STU + \boxdot OH = \boxdot AEFD$,

WHICH IS DESCRIBED, $\boxdot AD$;

THEREFORE,

$4AB \boxtimes BD + \boxdot AC = \boxdot AD$.

BUT,

$BD = BC$;

THEREFORE,

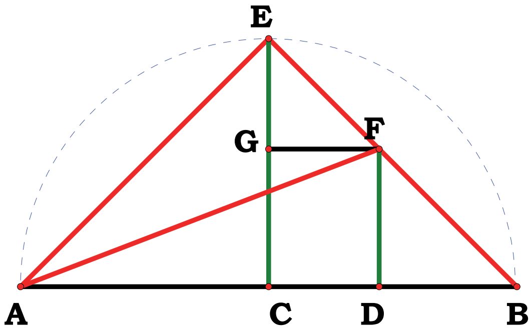
$4AB \boxtimes BC + \boxdot AC = \boxdot AD$, THAT IS TO

THE SQUARE, DESCRIBED, $\boxdot (AB + BC)$

THEREFORE ETC.

Q. E. D.

PROPOSITION 9.



IF A STRAIGHT LINE BE CUT INTO EQUAL AND UNEQUAL SEGMENTS, THE SQUARES ON THE UNEQUAL SEGMENTS OF THE WHOLE ARE DOUBLE OF THE SQUARE, ON THE HALF AND OF THE SQUARE, ON THE STRAIGHT LINE BETWEEN THE POINTS OF SECTION.

FOR LET,

AB, BE DIVIDED INTO EQUAL SEGMENTS AT, *C*,

AND,

INTO UNEQUAL SEGMENTS, AT *D*;

I SAY THAT;

$\boxdot AD + \boxdot DB = 2(\boxdot AC + \boxdot CD)$.

FOR LET,

CE BE DRAWN, FROM *C*, AT RIGHT ANGLES, TO *AB*,

AND LET,

IT BE MADE EQUAL, TO EITHER, *AC* OR *CB*;

LET,

EA, *EB* BE JOINED,

LET,

DF BE DRAWN, THROUGH *D*, PARALLEL TO *EC*, AND
FG, THROUGH *F*, PARALLEL TO *AB*,

AND LET,

AF BE JOINED.

THEN, SINCE,

$AC = CE$, $\angle EAC = \angle AEC$.

[I. 32] AND, SINCE,

$\angle$AT *C*, IS RIGHT, THE REMAINING, $\angle EAC$, $\angle AEC$,
ARE EQUAL, TO ONE RIGHT ANGLE. AND,
THEY ARE EQUAL;

THEREFORE,

EACH, $\angle CEA$, $\angle CAE$, IS HALF A RIGHT ANGLE.

FOR THE SAME REASON,

EACH, $\angle CEB$, $\angle EBC$, IS, ALSO, HALF A RIGHT ANGLE;

THEREFORE,

THE WHOLE $\llcorner AEB$, IS RIGHT.

AND, SINCE,

$\angle GEF$, IS HALF A RIGHT ANGLE, AND $\angle EGF$, IS RIGHT,

[I. 29] FOR,

IT IS EQUAL TO THE INTERIOR AND OPPOSITE $\angle ECB$,

[I. 32]

THE REMAINING $\angle EFG$, IS HALF A RIGHT ANGLE;

THEREFORE,

$\angle GEF = \angle EFG$,

[I. 6] SO THAT,
$EG = GF$.

[I. 29] AGAIN, SINCE,

$\angle$AT B, IS HALF A RIGHT ANGLE, AND $\llcorner FDB$ IS RIGHT, FOR,

IT IS AGAIN EQUAL, TO

THE INTERIOR AND OPPOSITE $\angle ECB$,

[I. 32]

THE REMAINING $\angle BED$, IS HALF A RIGHT ANGLE;

THEREFORE,

$\angle$AT $B = \angle DFB$,

[I. 6] SO THAT,
THE SIDES, $FD = DB$.

NOW, SINCE,
$AC = CE$,

$\boxdot AC = \boxdot CE$;

THEREFORE,

$\boxdot AC + \boxdot CE = 2\boxdot AC$.

[I. 47] BUT,

$\boxdot EA = \boxdot AC + \boxdot CE$,

FOR,

$\llcorner ACE$ IS RIGHT;

THEREFORE,

$\boxdot EA = 2\boxdot AC$.

AGAIN, SINCE,

$EG = GF$, $\boxdot EG = \boxdot GF$;

THEREFORE,

$\boxdot EG + \boxdot GF = 2\boxdot GF$.

BUT,

$\boxdot EF = \boxdot EG + \boxdot GF$;

THEREFORE,

$\boxdot EF = 2\boxdot GF$.

[I. 34] BUT,

$GF = CD$;

THEREFORE,

$\boxdot EF = 2\boxdot CD$.

BUT,

$\boxdot EA = 2\boxdot AC$;

THEREFORE,

$\boxdot AE + \boxdot EF = 2(\boxdot AC + \boxdot CD)$.

AND,

$\boxdot AF = \boxdot AE + \boxdot EF$,

[I. 47] FOR,

$\llcorner AEF$ IS RIGHT;

THEREFORE,

$\boxdot AF = 2(\boxdot AC + \boxdot CD)$.

BUT,

$\boxdot AD + \boxdot DF = \boxdot AF$,

[I. 47] FOR,

THE ANGLE AT D IS RIGHT;

THEREFORE,

$\boxdot AD + \boxdot DF = 2(\boxdot AC + \boxdot CD)$. AND

$DF = DB$;

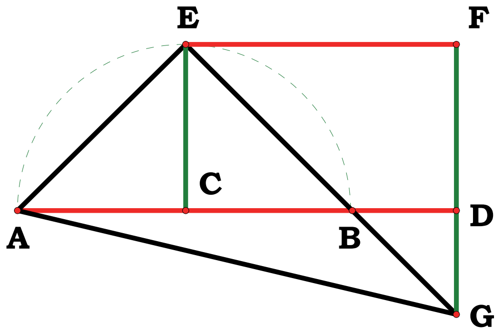
THEREFORE,

$\boxdot AD + \boxdot DB = 2(\boxdot AC + \boxdot CD)$.

THEREFORE ETC.

Q. E. D.

PROPOSITION 10.



IF A STRAIGHT LINE BE BISECTED, AND A STRAIGHT LINE BE ADDED TO IT IN A STRAIGHT LINE, THE SQUARE, ON THE WHOLE WITH THE ADDED STRAIGHT LINE AND THE SQUARE, ON THE ADDED STRAIGHT LINE BOTH TOGETHER ARE DOUBLE OF THE SQUARE, ON THE HALF AND OF THE SQUARE DESCRIBED ON THE STRAIGHT LINE MADE UP OF THE HALF AND THE ADDED STRAIGHT LINE AS ON ONE STRAIGHT LINE.

FOR LET,

AB, BE BISECTED, AT C,

AND LET,

BD, BE ADDED TO IT IN A STRAIGHT LINE;

I SAY THAT;

$⊡AD$, $⊡DB = 2(⊡AC + ⊡CD)$.

[I. 11] FOR LET,

CE BE DRAWN FROM C, AT RIGHT ANGLES, TO AB,

[I. 3] AND LET,

IT BE MADE EQUAL, TO EITHER, AC OR CB;

LET,

EA, EB BE JOINED;

LET,

THROUGH E, $EF \parallel AD$,

[I. 31] AND LET,

THROUGH D, $FD \parallel CE$.

THEN, SINCE,

$EF \cap$, $EC \parallel FD$,

[I. 29] $\angle CEF + \angle EFD$, ARE EQUAL, TO TWO RIGHT ANGLES;

THEREFORE,

$\angle FEB + \angle EFD$, ARE LESS THAN TWO RIGHT ANGLES.

[I. POST. 5] BUT,

STRAIGHT LINES PRODUCED

FROM ANGLES LESS THAN TWO RIGHT ANGLES MEET;

THEREFORE,

EB, FD, IF PRODUCED IN THE DIRECTION B, D, WILL MEET.

LET,

THEM BE PRODUCED AND MEET AT G,

AND LET,

AG BE JOINED.

[I. 5] THEN, SINCE,

$AC = CE$, $\angle EAC = \angle AEC$;

AND,

$\angle$AT C, IS RIGHT;

[I. 32] THEREFORE,

EACH, $\angle EAC$, $\angle AEC$, IS HALF A RIGHT ANGLE.

FOR THE SAME REASON, EACH, OF

$\angle CEB$, $\angle EBC$, IS, ALSO, HALF A RIGHT ANGLE;

THEREFORE,

$\llcorner AEB$, IS RIGHT.

[I. 15] AND, SINCE,

$\angle EBC$, IS HALF A RIGHT ANGLE,

$\angle DBG$, IS, ALSO, HALF A RIGHT ANGLE.

BUT,

$\llcorner BDG$ IS, ALSO, RIGHT,

[I. 29]

FOR,

$\llcorner BDG = \llcorner DCE$, THEY BEING ALTERNATE;

[I. 32] THEREFORE,

THE REMAINING, $\angle DGB$, IS HALF A RIGHT ANGLE;

THEREFORE,

$\angle DGB = \angle DBG$,

[I. 6] SO,

THAT THE SIDES, $BD = GD$.

AGAIN, SINCE,

$\angle EGF$, IS HALF A RIGHT ANGLE, AND

$\angle$AT F, IS RIGHT,

[I. 34] FOR,

$\angle$AT F = THE OPPOSITE ANGLE, $\angle$AT C,

[I. 32]

THE REMAINING, $\angle FEG$, IS HALF A RIGHT ANGLE;

THEREFORE,

$\angle EGF = \angle FEG$,

[I. 6] SO THAT,
THE SIDES, $GF = EF$.

NOW, SINCE,

$\boxdot EC = \boxdot CA$,

$\boxdot EC + \boxdot CA = 2\boxdot CA$.

[I. 47] BUT,

$\boxdot EA = \boxdot EC + \boxdot CA$;

[*C. N.* 1]

THEREFORE,

$\boxdot EA = 2\boxdot AC$.

AGAIN, SINCE,
$FG = EF$,

$\boxdot FG = \boxdot FE$;

THEREFORE,

$\boxdot GF + \boxdot FE = 2\boxdot EF$.

[I. 47] BUT,

$\boxdot EG = \boxdot GF + \boxdot FE$;

THEREFORE,

$\boxdot EG = 2\boxdot EF$.

[I. 34] AND $EF = CD$;

THEREFORE,

$\boxdot EG = 2\boxdot CD$.

BUT,

$\boxdot EA = 2\boxdot AC$;

THEREFORE,

$\boxdot AE + \boxdot EG = 2(\boxdot AC + \boxdot CD)$.

[I. 47] AND,

$$\boxdot AG = \boxdot AE + \boxdot EG;$$

THEREFORE,

$$\boxdot AG = 2(\boxdot AC + \boxdot CD).$$

[I. 47] BUT,

$$\boxdot AD + \boxdot DG = \boxdot AG;$$

THEREFORE,

$$\boxdot AD + \boxdot DG = 2(\boxdot AC + \boxdot CD2).$$

AND,

$$DG = DB;$$

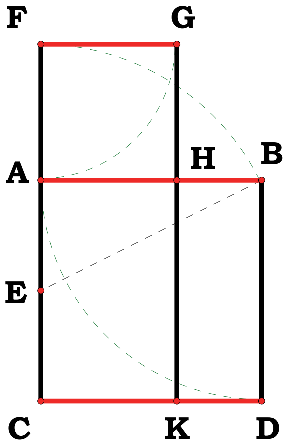
THEREFORE,

$$\boxdot AD + \boxdot DB = 2(\boxdot AC + \boxdot CD2).$$

THEREFORE ETC.

Q. E. D.

Proposition 11.



To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal, to the square, on the remaining segment.

LET,

AB BE GIVEN;

THUS IT IS REQUIRED,

TO DIVIDE, AB, SO THAT;

THE RECTANGLE CONTAINED BY

THE WHOLE AND

ONE OF THE SEGMENTS EQUALS

THE SQUARE, ON THE REMAINING SEGMENT.

[I. 46] FOR LET,

$\boxdot ABDC$, BE DESCRIBED, $\boxdot AB$;

LET,

AC BE BISECTED AT E,

AND LET,

BE, BE JOINED;

LET,

CA BE DRAWN, THROUGH TO F, AND LET,

$EF = BE$;

LET,

$\boxdot FH$, BE DESCRIBED, $\boxdot AF$,

AND LET,

GH BE DRAWN, THROUGH TO K.

I SAY THAT;

AB HAS BEEN DIVIDED, AT H,

SO AS,

TO MAKE, $AB \boxtimes BH = \boxdot AH$.

FOR, SINCE,

AC, HAS BEEN BISECTED, AT E,

[II. 6] AND,

FA IS ADDED TO IT,

$CF \boxtimes FA + \boxdot AE = \boxdot EF$.

BUT,

$EF = EB$;

THEREFORE,

$CF \boxtimes FA + \boxdot AE = \boxdot EB.$

[I. 47] BUT,

$\boxdot BA + \boxdot AE = \boxdot EB,$

FOR,

$\angle$AT A, IS RIGHT:

THEREFORE,

$CF \boxtimes FA + \boxdot AE = \boxdot BA + \boxdot AE.$

LET,

$\boxdot AE$, BE SUBTRACTED FROM EACH;

THEREFORE,

$CF \boxtimes FA = \boxdot AB.$

NOW,

$CF \boxtimes FA = \boxtimes FK,$

FOR,

$AF = FG$; AND $\boxdot AB = \boxdot AD$;

THEREFORE,

$\boxtimes FK = \boxdot AD.$

LET,

$\boxtimes AK$ BE SUBTRACTED FROM EACH;

THEREFORE,

$\boxdot FH = \boxtimes HD.$

AND,

$HD = AB \boxtimes BH,$

FOR,

$AB = BD$; AND $\boxdot FH = \boxdot AH$;

THEREFORE,

$AB \boxtimes BH = \boxdot HA,$

THEREFORE,

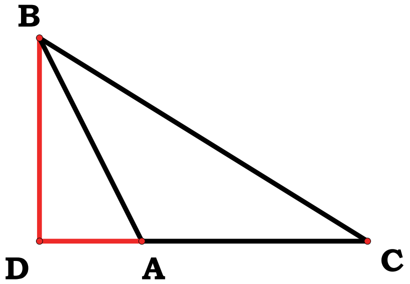
AB, HAS BEEN DIVIDED, AT H, SO AS TO MAKE

$AB \boxtimes BH = \boxdot HA.$

Q. E. F.

PROPOSITION 12.

IN OBTUSE-ANGLED TRIANGLES, THE SQUARE, ON THE SIDE SUBTENDING THE OBTUSE ANGLE IS GREATER THAN THE SQUARES ON THE SIDES CONTAINING THE OBTUSE ANGLE BY TWICE THE RECTANGLE CONTAINED BY ONE OF THE SIDES ABOUT THE OBTUSE ANGLE, NAMELY THAT ON WHICH THE PERPENDICULAR FALLS, AND THE STRAIGHT LINE CUT OFF OUTSIDE BY THE PERPENDICULAR TOWARDS THE OBTUSE ANGLE.



LET,

ΔABC BE AN OBTUSE-ANGLED TRIANGLE HAVING

$\angle BAC$, OBTUSE,

AND LET,

BD BE DRAWN, FROM B, PERPENDICULAR TO
CA, PRODUCED;

I SAY THAT;

$\boxdot BC > \boxdot BA + \boxdot AC$, BY $2CA \boxtimes AD$.

[II. 4] FOR, SINCE, AT RANDOM,
CD, HAS BEEN DIVIDED AT A,

$\boxdot DC = \boxdot CA + \boxdot AD + 2CA \boxtimes AD$.

LET,

$\boxdot DB$, BE ADDED TO EACH;

THEREFORE,

$\boxdot CD + \boxdot DB = \boxdot CA + \boxdot AD + \boxdot DB + 2CA \boxtimes AD$.

[I. 47] BUT,

$\boxdot CB = \boxdot CD + \boxdot DB$,

FOR,

$\angle$AT D, IS RIGHT;

[I. 47] AND,

$\boxdot AB = \boxdot AD + \boxdot DB$;

THEREFORE,

$\boxdot CB = \boxdot CA + \boxdot AB + 2CA \boxtimes AD$;

SO THAT,

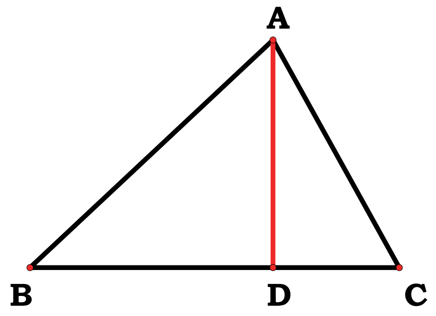
$$\boxdot CB > \boxdot CA + \boxdot AB \text{ BY } 2CA \boxtimes AD.$$

THEREFORE ETC.

Q. E. D.

Proposition 13.

In acute-angled triangles the square, on the side subtending the acute angle is less than the squares on the sides containing the acute angle by twice the rectangle contained by one of the sides about the acute angle, namely that on which the perpendicular falls, and the straight line cut off within by the perpendicular towards the acute angle.



Let,

ΔABC be an acute-angled triangle having

$\angle$at B, acute,

and let,

AD be drawn from A, perpendicular to BC;

I say that;

$\boxdot AC < \boxdot CB + \boxdot BA$, by $2CB \boxtimes BD$.

[II. 7] For, since,

CB, has been divided at random, at D,

$\boxdot CB + \boxdot BD = 2CB \boxtimes BD + \boxdot DC$.

Let,

$\boxdot DA$, be added to each;

therefore,

$\boxdot CB + \boxdot BD + \boxdot DA = 2CB \boxtimes BD + \boxdot AD + \boxdot DC$.

But,

$\boxdot AB = \boxdot BD + \boxdot DA$,

[I. 47] for,

$\angle$at D, is right; and $\boxdot AC = \boxdot AD + \boxdot DC$;

therefore,

$\boxdot CB + \boxdot BA = \boxdot AC + 2CB \boxtimes BD$, so that,

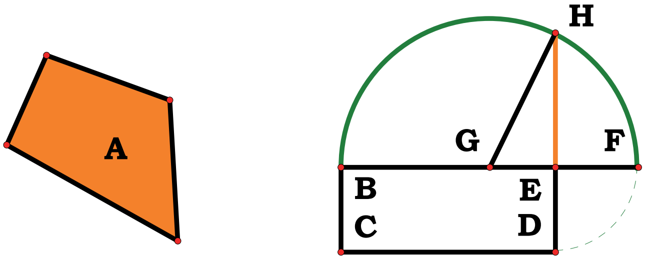
$\boxdot AC < \boxdot CB + \boxdot BA$, by $2CB \boxtimes BD$.

Therefore etc.

Q. E. D.

Proposition 14.

To construct a square equal, to a given rectilineal figure.

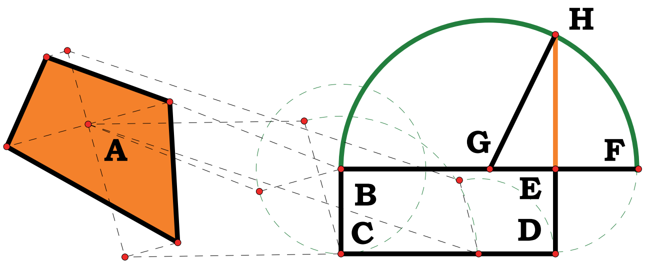


LET,

A BE THE GIVEN RECTILINEAL FIGURE;

THUS IT IS REQUIRED,

TO CONSTRUCT A SQUARE EQUAL, TO
THE RECTILINEAL FIGURE, *A*.



[I. 45] FOR LET,

THERE BE CONSTRUCTED

THE RECTANGULAR ⊟*BD*, EQUAL, TO

THE RECTILINEAL FIGURE, *A*.

THEN, IF,

BE = *ED*,

THAT,

WHICH WAS ENJOINED WILL HAVE BEEN DONE;

FOR,

⊡*BD*, HAS BEEN CONSTRUCTED EQUAL, TO

THE RECTILINEAL FIGURE, *A*.

BUT, IF NOT,

ONE OF *BE* OR *ED*, IS GREATER.

LET,

BE, BE GREATER, AND LET,
IT BE PRODUCED, TO *F*; LET,
EF = *ED*,

AND LET,

BF be bisected, at G.

With,

centre G, and

distance of GB or GF,

let,

the semicircle, BHF, be described;

let,

DE be produced, to H,

and let,

GH be joined.

[II. 5]

Then, since,

BF, has been divided into equal segments, at G, and

into unequal segments, at E,

$BE \boxtimes EF + \boxdot EG = \boxdot GF$.

But,

$GF = GH$;

therefore,

$BE \boxtimes EF + \boxdot GE = \boxdot GH$.

[I. 47] But,

$\boxdot HE + \boxdot EG = \boxdot GH$;

therefore,

$BE \boxtimes EF + \boxdot GE = \boxdot HE + \boxdot EG$.

Let,

$\boxdot GE$, be subtracted from each;

therefore,

$BE \boxtimes EF = \boxdot EH$.

But,

$BE \boxtimes EF = \boxtimes BD$,

for,

$EF = ED$;

therefore,

$\boxminus BD = \boxdot HE$.

And,

BD = THE RECTILINEAL FIGURE, A.

THEREFORE,

THE RECTILINEAL FIGURE, A = ⊡EH.

THEREFORE,

A SQUARE, NAMELY,

THAT WHICH CAN BE DESCRIBED, ⊡EH,

HAS BEEN CONSTRUCTED EQUAL, TO
THE GIVEN RECTILINEAL FIGURE, A.

Q. E. F.

BOOK III.

OF

EUCLID'S ELEMENTS

TRANSLATED FROM THE TEXT OF HEIBERG

BY

SIR THOMAS L. HEATH,

K. C. B. K. C. V. O. F. R. S.

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**HONORARY FELLOW (SOMETIME FELLOW) OF
TRINITY COLLEGE CAMBRIDGE**

2013 *EDITION*

REVISED WITH SUBTRACTIONS

**REFORMATTED AND ABRIDGED
FOR STUDY OF
THE ELEMENTS.
BY JOHN CLARK.**

BOOK III.

DEFINITIONS.

1. **EQUAL CIRCLES** ARE THOSE THE DIAMETERS OF WHICH ARE EQUAL, OR THE RADII OF WHICH ARE EQUAL.

2. A STRAIGHT LINE IS SAID TO **TOUCH A CIRCLE** WHICH, MEETING THE CIRCLE AND BEING PRODUCED, DOES NOT CUT THE CIRCLE.

3. **CIRCLES** ARE SAID TO **TOUCH ONE ANOTHER** WHICH, MEETING ONE ANOTHER, DO NOT CUT ONE ANOTHER.

4. IN A CIRCLE STRAIGHT LINES ARE SAID **TO BE EQUALLY DISTANT FROM THE CENTRE** WHEN THE PERPENDICULARS DRAWN TO THEM FROM THE CENTRE ARE EQUAL.

5. AND THAT STRAIGHT LINE IS SAID TO BE **AT A GREATER DISTANCE** ON WHICH THE GREATER PERPENDICULAR FALLS.

6. A **SEGMENT OF A CIRCLE** IS THE FIGURE CONTAINED BY A STRAIGHT LINE AND A CIRCUMFERENCE OF A CIRCLE.

7. AN **ANGLE OF A SEGMENT** IS THAT CONTAINED BY A STRAIGHT LINE AND A CIRCUMFERENCE OF A CIRCLE.

8. AN **ANGLE IN A SEGMENT** IS THE ANGLE WHICH, WHEN A POINT IS TAKEN ON THE CIRCUMFERENCE OF THE SEGMENT AND STRAIGHT LINES ARE JOINED FROM IT TO THE EXTREMITIES OF THE STRAIGHT LINE WHICH IS THE **BASE OF THE SEGMENT**, IS CONTAINED BY THE STRAIGHT LINES SO JOINED.

9. AND, WHEN THE STRAIGHT LINES CONTAINING THE ANGLE CUT OFF A CIRCUMFERENCE, THE ANGLE IS SAID TO **STAND UPON** THAT CIRCUMFERENCE.

10. A **SECTOR OF A CIRCLE** IS THE FIGURE WHICH, WHEN AN ANGLE IS CONSTRUCTED AT THE CENTRE OF THE CIRCLE, IS CONTAINED BY THE STRAIGHT LINES CONTAINING THE ANGLE AND THE CIRCUMFERENCE CUT OFF BY THEM.

11. **SIMILAR SEGMENTS OF CIRCLES** ARE THOSE WHICH ADMIT EQUAL ANGLES, OR IN WHICH THE ANGLES ARE EQUAL, TO ONE ANOTHER.

NOTES.

DEFINITION 1. EQUAL CIRCLES *ARE THOSE THE DIAMETERS OF WHICH ARE EQUAL, OR THE RADII OF WHICH ARE EQUAL*.

NOTES.

DEFINITION 2. *A STRAIGHT LINE IS SAID TO* TOUCH A CIRCLE *WHICH, MEETING THE CIRCLE AND BEING PRODUCED, DOES NOT CUT THE CIRCLE.*

Notes.

Definition 3. Circles *are said to* touch one another *which, meeting one another, do not cut one another.*

NOTES.

DEFINITION 4. *IN A CIRCLE STRAIGHT LINES ARE SAID* TO BE EQUALLY DISTANT FROM THE CENTRE *WHEN THE PERPENDICULARS DRAWN TO THEM FROM THE CENTRE ARE EQUAL.*

Notes.

Definition 5. *And that straight line is said to be* at a greater distance *on which the greater perpendicular falls.*

Notes.

Definition 6. *A* segment of a circle *is the figure contained by a straight line and a circumference of a circle.*

NOTES.

DEFINITION 7. *AN* ANGLE OF A SEGMENT *IS THAT CONTAINED BY A STRAIGHT LINE AND A CIRCUMFERENCE OF A CIRCLE.*

NOTES.

DEFINITION 8. *AN* ANGLE IN A SEGMENT *IS THE ANGLE WHICH, WHEN A POINT IS TAKEN ON THE CIRCUMFERENCE OF THE SEGMENT AND STRAIGHT LINES ARE JOINED FROM IT TO THE EXTREMITIES OF THE STRAIGHT LINE WHICH IS THE* BASE OF THE SEGMENT, *IS CONTAINED BY THE STRAIGHT LINES SO JOINED.*

NOTES.

DEFINITION 9. *AND, WHEN THE STRAIGHT LINES CONTAINING THE ANGLE CUT OFF A CIRCUMFERENCE, THE ANGLE IS SAID TO* STAND UPON *THAT CIRCUMFERENCE.*

Notes.

Definition 10. *A* sector of a circle *is the figure which, when an angle is constructed at the centre of the circle, is contained by the straight lines containing the angle and the circumference cut off by them.*

NOTES.

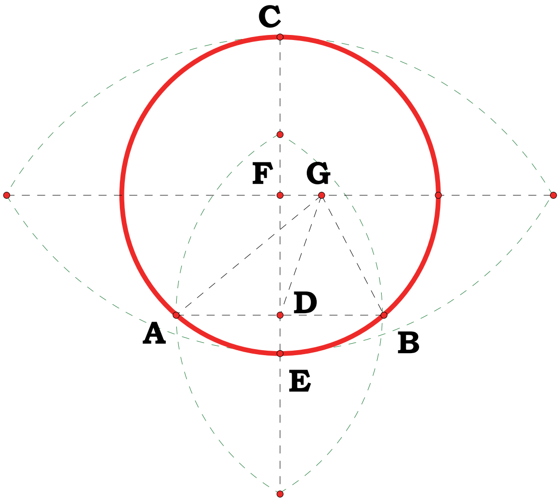
DEFINITION 11. SIMILAR SEGMENTS OF CIRCLES *ARE THOSE WHICH ADMIT EQUAL ANGLES, OR IN WHICH THE ANGLES ARE EQUAL, TO ONE ANOTHER.*

BOOK III.

PROPOSITIONS.

Proposition 1.

To find the centre of a given circle.



Let,

$\odot ABC$ be given;

thus it is required,

to find the centre of

$\odot ABC$.

Let, at random,

AB,

be drawn through it

and let,

it be bisected at D;

let,

from D, $DC \perp AB$

and let,

it be drawn, through, to E;

let,

CE be bisected, at F;

I say that;

F is the centre of $\odot ABC$.

For,

suppose it is not,

but, if possible, let,

G be the centre,

and let,

GA, GD, GB be joined.

Then, since,

$AD = DB$, and DG is common,

$AD = BD$, $DG = DG$; and

the bases, $GA = GB$,

for,

they are radii;

[I. 8] THEREFORE,

$\angle ADG = \angle GDB$.

[I. DEF. 10] BUT,

WHEN A STRAIGHT LINE SET UP ON A STRAIGHT LINE MAKES THE ADJACENT ANGLES EQUAL, TO ONE ANOTHER, EACH, OF THE EQUAL ANGLES IS RIGHT;

THEREFORE,

$\angle GDB$, IS RIGHT. BUT,

$\angle FDB$, IS, ALSO, RIGHT;

THEREFORE,

$\angle FDB = \angle GDB$,

GREATER TO THE LESS: WHICH,
IS IMPOSSIBLE.

THEREFORE,

G IS NOT THE CENTRE OF $\odot ABC$.

SIMILARLY, WE CAN PROVE THAT;
NEITHER IS ANY OTHER POINT,
EXCEPT F.

THEREFORE,

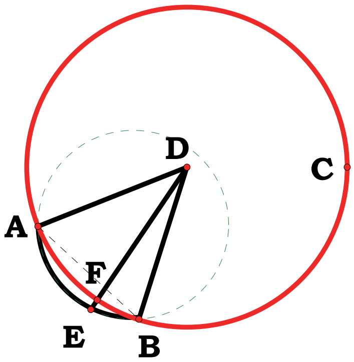
F, IS THE CENTRE OF $\odot ABC$.

PORISM.

FROM THIS IT IS MANIFEST THAT, IF IN A CIRCLE A STRAIGHT LINE CUT A STRAIGHT LINE INTO TWO EQUAL PARTS AND AT RIGHT ANGLES, THE CENTRE OF THE CIRCLE IS ON THE CUTTING STRAIGHT LINE.

Q. E. F.

PROPOSITION 2.



IF ON THE CIRCUMFERENCE OF A CIRCLE TWO POINTS BE TAKEN AT RANDOM, THE STRAIGHT LINE JOINING THE POINTS WILL FALL WITHIN THE CIRCLE.

LET,

$\odot ABC$,

AND LET, AT RANDOM

A AND B, BE TAKEN ON ITS CIRCUMFERENCE;

I SAY THAT;

THE LINE, JOINED, FROM A TO B,

WILL FALL WITHIN THE CIRCLE.

FOR SUPPOSE,

IT DOES NOT, BUT,

IF POSSIBLE, LET,

IT FALL OUTSIDE, AS AEB;

[III. 1] LET,

THE CENTRE OF $\odot ABC$, BE TAKEN, AND LET,

IT BE D;

LET,

DA, DB BE JOINED,

AND LET,

DFE BE DRAWN THROUGH.

[I. 5] THEN, SINCE,

$DA = DB$,

$\angle DAE = \angle DBE$.

[I. 16] AND, SINCE,

ONE SIDE, AEB, OF ΔDAE, IS PRODUCED, $\angle DEB > \angle DAE$.

BUT,

$\angle DAE = \angle DBE$;

THEREFORE,

$\angle DEB > \angle DBE$.

[I. 19] AND,

THE GREATER ANGLE IS SUBTENDED BY THE GREATER SIDE;

THEREFORE,

$DB > DE$. BUT,

$DB = DF$;

THEREFORE,

$DF > DE$,

THE LESS THAN THE GREATER: WHICH,

IS IMPOSSIBLE.

THEREFORE,

THE STRAIGHT LINE, JOINED,

FROM A TO B, WILL NOT FALL OUTSIDE THE CIRCLE.

SIMILARLY WE CAN PROVE,

THAT NEITHER WILL IT FALL ON THE CIRCUMFERENCE ITSELF;

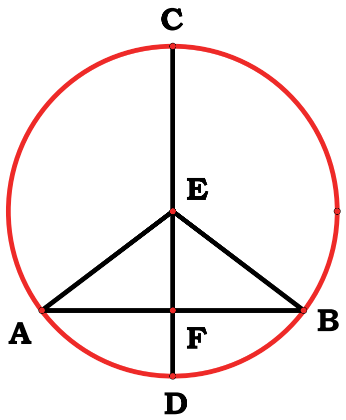
THEREFORE,

IT WILL FALL WITHIN.

THEREFORE ETC.

Q. E. D.

PROPOSITION 3.



IF IN A CIRCLE, A STRAIGHT LINE THROUGH THE CENTRE BISECT A STRAIGHT LINE NOT THROUGH THE CENTRE, IT, ALSO, *CUTS IT AT RIGHT ANGLES; AND IF IT CUT IT AT RIGHT ANGLES, IT, ALSO, BISECTS IT.*

LET,

$\odot ABC$,

AND LET,

IN IT, A STRAIGHT LINE CD,

THROUGH THE CENTRE,

BISECT AB, NOT THROUGH THE CENTRE,

AT THE POINT, F;

I SAY THAT;

IT, ALSO, DIVIDES IT AT RIGHT ANGLES.

FOR LET,

THE CENTRE OF $\odot ABC$, BE TAKEN,

AND LET,

IT BE E;

LET,

EA, EB, BE JOINED.

THEN, SINCE,

$AF = FB$,

AND,

FE IS COMMON,

TWO SIDES ARE EQUAL, TO TWO SIDES; AND

THE BASES, $EA = EB$;

[I. 8] THEREFORE,

$\angle AFE = \angle BFE$.

[I. DEF. 10] BUT,

WHEN A STRAIGHT LINE SET UP ON A STRAIGHT LINE MAKES THE ADJACENT ANGLES EQUAL, TO ONE ANOTHER, EACH, OF THE EQUAL ANGLES IS RIGHT;

THEREFORE,

EACH, $\llcorner AFE$, $\llcorner BFE$, IS RIGHT.

THEREFORE,

CD, WHICH IS THROUGH THE CENTRE, AND
BISECTS AB, WHICH IS NOT THROUGH THE CENTRE, ALSO DIVIDES IT AT RIGHT ANGLES.

AGAIN, LET,

CD DIVIDE AB, AT RIGHT ANGLES;

I SAY THAT;

IT, ALSO, BISECTS IT,

THAT IS, THAT;

$AF = FB$.

FOR,

WITH THE SAME CONSTRUCTION,

[I. 5] SINCE,

$EA = EB$,

$\angle EAF = \angle EBF$.

BUT,

$\llcorner AFE = \llcorner BFE$,

[I. 26] THEREFORE,

ΔEAF, ΔEBF ARE TWO TRIANGLES HAVING
TWO ANGLES EQUAL, TO TWO ANGLES, AND
ONE SIDE EQUAL, TO ONE SIDE, NAMELY,
EF, WHICH IS COMMON TO THEM, AND
SUBTENDS ONE OF THE EQUAL ANGLES;

THEREFORE,

THEY WILL, ALSO, HAVE THE REMAINING SIDES EQUAL, TO THE REMAINING SIDES;

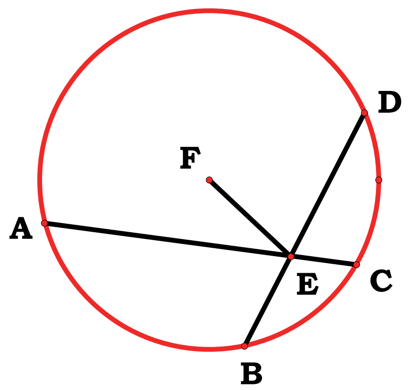
THEREFORE,

$AF = FB$.

THEREFORE ETC.

Q. E. D.

PROPOSITION 4.



IF IN A CIRCLE TWO STRAIGHT LINES CUT ONE ANOTHER WHICH ARE NOT THROUGH THE CENTRE, THEY DO NOT BISECT ONE ANOTHER.

LET,

⊙*ABCD*,

AND IN IT LET,

AC, *BD*,

WHICH ARE NOT THROUGH THE CENTRE,
INTERSECT ONE ANOTHER, AT *E*;

I SAY THAT;

THEY DO NOT BISECT ONE ANOTHER.

FOR, IF POSSIBLE, LET,

THEM BISECT ONE ANOTHER,

SO THAT,

AE = *EC*, AND *BE* TO *ED*;

[III. 1] LET,

THE CENTRE OF ⊙*ABCD*, BE TAKEN,

AND LET,

IT BE *F*;

LET,

FE BE JOINED.

[III. 3] THEN, SINCE,

FE, THROUGH THE CENTRE BISECTS
AC, NOT THROUGH THE CENTRE,
IT, ALSO, INTERSECTS IT AT RIGHT ANGLES;

THEREFORE,

∟*FEA*, IS RIGHT.

[III. 3] AGAIN, SINCE,

FE, BISECTS *BD*,
IT, ALSO, INTERSECTS IT AT RIGHT ANGLES;

THEREFORE,

∟*FEB*, IS RIGHT.

BUT,

∟*FEA*, WAS, ALSO, PROVED RIGHT;

THEREFORE,

$\llcorner FEA = \llcorner FEB$,

THE LESS TO THE GREATER: WHICH,
IS IMPOSSIBLE.

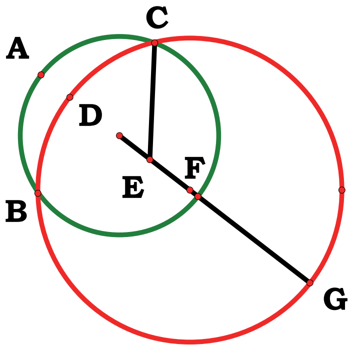
THEREFORE,

AC, BD, DO NOT BISECT ONE ANOTHER.

THEREFORE ETC.

Q. E. D.

PROPOSITION 5.



IF TWO CIRCLES CUT ONE ANOTHER, THEY WILL NOT HAVE THE SAME CENTRE.

FOR LET,

$\odot ABC$, $\odot CDG$,

INTERSECT ONE ANOTHER AT

B AND C;

I SAY THAT;

THEY WILL NOT HAVE THE SAME CENTRE.

FOR, IF POSSIBLE, LET,

IT BE E;

LET,

EC BE JOINED,

AND, AT RANDOM, LET,

EFG BE DRAWN THROUGH.

[I. DEF. 15] THEN, SINCE,

E, IS THE CENTRE OF $\odot ABC$,

$EC = EF$.

AGAIN, SINCE,

E, IS THE CENTRE OF $\odot CDG$,

$EC = EG$.

BUT,

$EC = EF$ ALSO;

THEREFORE,

$EF = EG$,

THE LESS TO THE GREATER: WHICH,

IS IMPOSSIBLE.

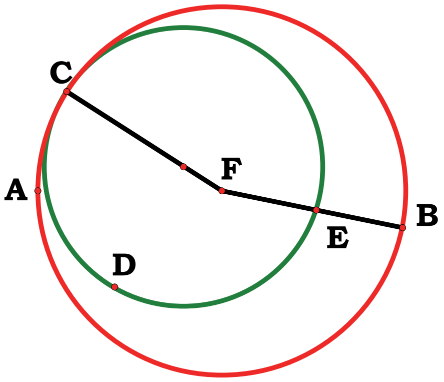
THEREFORE,

E, IS NOT THE CENTRE OF $\odot ABC$, $\odot CDG$.

THEREFORE ETC.

Q. E. D.

PROPOSITION 6.



IF TWO CIRCLES TOUCH ONE ANOTHER, THEY WILL NOT HAVE THE SAME CENTRE.

FOR LET,

$\odot ABC$, $\odot CDE$,

TOUCH ONE ANOTHER AT C;

I SAY THAT;

THEY WILL NOT HAVE THE SAME CENTRE.

FOR, IF POSSIBLE, LET,

IT BE F

LET,

FC BE JOINED,

AND, AT RANDOM, LET,

FEB BE DRAWN THROUGH.

THEN, SINCE,

F, IS THE CENTRE OF $\odot ABC$,

$FC = FB$.

AGAIN, SINCE,

F, IS THE CENTRE OF $\odot CDE$,

$FC = FE$.

BUT,

$FC = FB$;

THEREFORE,

$FE = FB$,

THE LESS TO THE GREATER: WHICH,

IS IMPOSSIBLE.

THEREFORE,

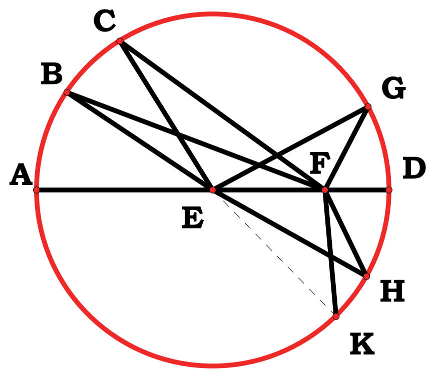
F IS NOT THE CENTRE OF $\odot ABC$ AND $\odot CDE$.

THEREFORE ETC.

Q. E. D.

PROPOSITION 7.

IF, ON THE DIAMETER OF A CIRCLE, A POINT BE TAKEN WHICH IS NOT THE CENTRE OF THE CIRCLE, AND FROM THE POINT STRAIGHT LINES FALL UPON THE CIRCLE, THAT WILL BE GREATEST ON WHICH THE CENTRE IS, THE REMAINDER OF THE SAME DIAMETER WILL BE LEAST, AND OF THE REST THE NEARER TO THE STRAIGHT LINE THROUGH THE CENTRE IS ALWAYS GREATER THAN THE MORE REMOTE, AND ONLY TWO EQUAL STRAIGHT LINES WILL FALL FROM THE POINT ON THE CIRCLE, ONE ON EACH SIDE OF THE LEAST STRAIGHT LINE.



LET,

$\odot ABCD$, AND LET,

AD BE A DIAMETER OF IT;

LET,

ON AD, A POINT, F, BE TAKEN WHICH IS NOT THE CENTRE OF THE CIRCLE,

LET,

E BE THE CENTRE OF THE CIRCLE,

AND LET,

FROM F, FB, FC, FG, INTERSECT $\odot ABCD$;

I SAY THAT;

FA IS GREATEST,

FD IS LEAST, AND, OF THE REST,

$FB > FC$, AND $FC > FG$.

FOR LET,

BE, CE, GE, BE JOINED.

[I. 20] THEN, SINCE,

IN ANY TRIANGLE TWO SIDES ARE GREATER THAN THE REMAINING ONE,

$EB + EF > BF$.

BUT,

$AE = BE$; THEREFORE,

$AF > BF$.

AGAIN, SINCE,

$BE = CE$, AND FE IS COMMON,

$BE + EF = CE + EF$.

BUT,

$\angle BEF > \angle CEF$;

[I. 24] THEREFORE,
THE BASES, $BF > CF$.

FOR THE SAME REASON,
$CF > FG$.

AGAIN, SINCE,
$GF + FE > EG$, AND $EG = ED$,
$GF + FE > ED$.

LET,
EF BE SUBTRACTED FROM EACH;

THEREFORE,
THE REMAINDERS, $GF > FD$.

THEREFORE,
FA IS GREATEST, FD IS LEAST, AND
$FB > FC$, AND $FC > FG$.

I SAY, ALSO, THAT;
FROM F,
ONLY TWO EQUAL STRAIGHT LINES WILL FALL ON
$\odot ABCD$,
ONE ON EACH SIDE OF THE LEAST, FD.

FOR,
ON EF, AND AT E, ON IT,

[I. 23] LET,
$\angle FEH = \angle GEF$, AND LET,
FH BE JOINED.

[I. 4] THEN, SINCE,
$GE = EH$, AND
EF IS COMMON,
$GE + EF = HE + EF$; AND
$\angle GEF = \angle HEF$;

THEREFORE,
THE BASES, $FG = FH$.

I SAY AGAIN THAT;
ANOTHER STRAIGHT LINE EQUAL, TO
FG, WILL NOT FALL ON THE CIRCLE FROM F.

FOR, IF POSSIBLE, LET,
FK SO FALL.

THEN, SINCE,

$FK = FG$, AND

$FH = FG$,

$FK = FH$,

THE NEARER TO
THE STRAIGHT LINE THROUGH THE CENTRE BEING
THUS EQUAL, TO THE MORE REMOTE: WHICH,
IS IMPOSSIBLE.

THEREFORE,

ANOTHER STRAIGHT LINE, EQUAL, TO
GF, WILL NOT FALL FROM F, UPON THE CIRCLE;

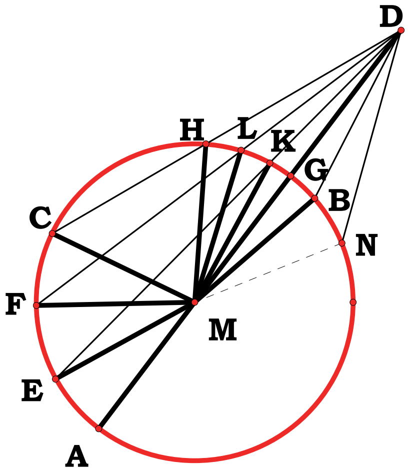
THEREFORE,

ONLY ONE STRAIGHT LINE WILL SO FALL.

THEREFORE ETC.

Q. E. D.

Proposition 8.



If a point be taken outside a circle and from the point straight lines be drawn through to the circle, one of which is through the centre and the others are drawn at random, then, of the straight lines which fall on the concave circumference, that through the centre is greatest, while of the rest the nearer to that through the centre is always greater than the more remote, but, of the straight lines falling on the convex circumference, that between the point and the diameter is least, while of the rest the nearer to the least is always less than the more remote, and only two equal straight lines will fall on the circle from the point, one on each side of the least.

Let,

⊙ABC, and let,

D, be taken outside ABC; let,

there be drawn through from it

DA, DE, DF, DC, and let,

DA be through the centre;

I say that;

of the straight lines falling on

the concave circumference, $AEFC$,

DA, through the centre, is greatest, while,

$DE > DF$ and $DF > DC$;

but,

of the straight lines falling on

the convex circumference, $HLKG$,

DG, between the point and

the diameter, AG, is least; and

the nearer to the least, DG, is always less than

the more remote, namely,

$DK < DL$, and $DL < DH$.

[III. 1] For let,

the centre of ⊙ABC, be taken, and let,

it be M;

LET,

ME, MF, MC, MK, ML, MH, BE JOINED.

THEN, SINCE,

$AM = EM$, LET,

MD BE ADDED TO EACH;

THEREFORE,

$AD = EM + MD$.

[I. 20] BUT,

$EM + MD > ED$;

THEREFORE, ALSO,

$AD > ED$.

AGAIN, SINCE,

$ME = MF$, AND MD IS COMMON,

THEREFORE,

$EM + MD = FM + MD$; AND $\angle EMD > \angle FMD$;

[I. 24] THEREFORE,

THE BASES, $ED > FD$.

SIMILARLY WE CAN PROVE THAT,

$FD > CD$;

THEREFORE,

DA IS GREATEST, WHILE

$DE > DF$, AND $DF > DC$.

[I. 20] NEXT, SINCE,

$MK + KD > MD$, AND $MG = MK$,

THEREFORE,

THE REMAINDERS, $KD > GD$, SO THAT,

$GD < KD$.

[I. 21] AND, SINCE,

ON MD, ONE OF THE SIDES OF

ΔMLD, MK, KD,

WERE CONSTRUCTED MEETING WITHIN THE TRIANGLE,

THEREFORE,

$MK + KD < ML + LD$; AND $MK = ML$;

THEREFORE,

THE REMAINDERS, $DK < DL$.

SIMILARLY WE CAN PROVE, ALSO, THAT,

$DL < DH$;

THEREFORE,

DG IS LEAST, WHILE,
$DK < DL$, AND $DL < DH$.

I SAY, ALSO, THAT;
ONLY TWO EQUAL STRAIGHT LINES WILL FALL FROM
D, ON THE CIRCLE,
ONE ON EACH SIDE OF THE LEAST, DG.

ON,
MD, AND AT M, ON IT, LET,
$\angle DMB = \angle KMD$,

AND LET,
DB, BE JOINED.

THEN, SINCE,
$MK = MB$, AND
MD IS COMMON, THE TWO SIDES,
$KM + MD = BM + MD$; AND
$\angle KMD = \angle BMD$;

[I. 4] THEREFORE,
THE BASES, $DK = DB$.

I SAY THAT;
NO OTHER STRAIGHT LINE EQUAL, TO
DK, WILL FALL ON THE CIRCLE FROM D.

FOR, IF POSSIBLE, LET,
A STRAIGHT LINE SO FALL,

AND LET,
IT BE DN.

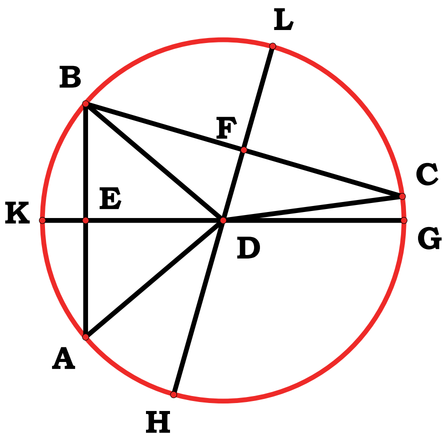
THEN, SINCE,
$DK = DN$, WHILE
$DK = DB$,
$DB = DN$,

THAT IS,
THE NEARER TO THE LEAST, DG, EQUAL, TO
THE MORE REMOTE: WHICH,
WAS PROVED IMPOSSIBLE.

THEREFORE,
NO MORE THAN TWO EQUAL STRAIGHT LINES WILL FALL ON
$\odot ABC$, FROM D, ONE ON EACH SIDE OF DG, THE LEAST.

THEREFORE ETC.

Proposition 9.



If a point be taken within a circle, and more than two equal straight lines fall from the point on the circle, the point taken is the centre of the circle.

Let,

⊙ABC, and

D, a point within it,

and let,

from D, more than two equal straight lines,

namely,

DA, DB, DC, fall on ⊙ABC;

I say that;

D, is the centre of ⊙ABC.

For let,

AB, BC be joined, and bisected at E, F,

and let,

ED, FD be joined, and

drawn through to G, K, H, L.

[I. 8] Then, since,

$AE = EB$, and ED is common,

$AE + ED = BE + ED$; and

the bases, $DA = DB$;

therefore,

$\angle AED = \angle BED$.

[I. Def. 10] Therefore,

each, of the angles, ∟AED, ∟BED, is right;

therefore,

GK divides AB into two equal parts, and

at right angles.

[III. 1, Por.] And since,

if in a circle a straight line cut

a straight line into two equal parts, and

at right angles,

the centre of the circle is on

THE CUTTING STRAIGHT LINE,
THE CENTRE OF THE CIRCLE IS ON GK.

FOR THE SAME REASON,

THE CENTRE OF $\odot ABC$, IS, ALSO, ON HL.

AND,

THE STRAIGHT LINES,
GK, HL, HAVE NO OTHER POINT COMMON, BUT D;

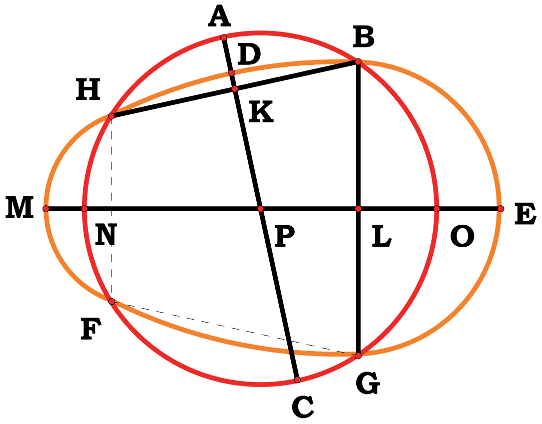
THEREFORE,

D, IS THE CENTRE OF $\odot ABC$.

THEREFORE ETC.

Q. E. D.

Proposition 10.



A circle does not cut a circle at more points than two.

For, if possible, let,

$\odot ABC$,

divide $\odot DEF$, at

more points than two,

namely,

B, G, F, H;

let,

BH, BG, be joined, and bisected at K, L,

and let,

from K, L,

KC, LM, be drawn, at right angles, to BH, BG, and carried through to A, E.

[III. 1, Por.] Then, since,

in $\odot ABC$, AC divides BH, into

two equal parts, and at right angles,

the centre of $\odot ABC$, is on AC.

Again, since,

in the same $\odot ABC$, NO divides BG, into

two equal parts, and at right angles,

the centre of $\odot ABC$, is on NO.

But,

it was, also, proved to be on AC, and

$AC \cap NO$, at no point except at P;

therefore,

P is the centre of $\odot ABC$.

Similarly we can, also, prove that,

P is the centre of $\odot DEF$;

[III. 5] therefore,

$\odot ABC$, $\odot DEF$,

which intersect one another have the same centre, P:

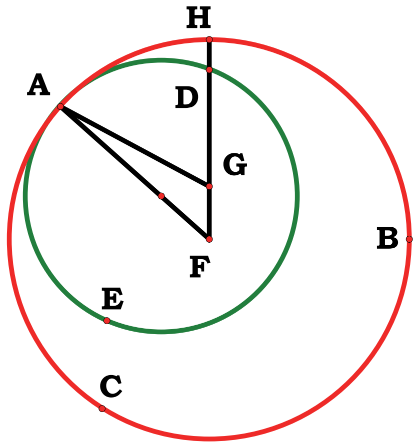
WHICH,

IS IMPOSSIBLE,

THEREFORE ETC.

Q. E. D.

PROPOSITION 11.



IF TWO CIRCLES TOUCH ONE ANOTHER INTERNALLY, AND THEIR CENTRES BE TAKEN, THE STRAIGHT LINE JOINING THEIR CENTRES, IF IT BE, ALSO, *PRODUCED, WILL FALL ON THE POINT OF CONTACT OF THE CIRCLES.*

FOR LET,

$\odot ABC \cap \odot ADE$ AT A,

AND LET,

THE CENTRE, F, OF $\odot ABC$, AND THE CENTRE, G, OF ADE, BE TAKEN;

I SAY THAT;

THE STRAIGHT LINE, JOINED, FROM G TO F, AND PRODUCED, WILL FALL ON A.

FOR SUPPOSE,

IT DOES NOT, BUT, IF POSSIBLE, LET,
IT FALL AS FGH,

AND LET,

AF, AG, BE JOINED.

THEN, SINCE,

$AG + GF > FA$, THAT IS, THAN FH,

LET,

FG BE SUBTRACTED FROM EACH;

THEREFORE,

THE REMAINDERS, $AG > GH$. BUT,
$AG = GD$;

THEREFORE, ALSO,

$GD > GH$,
THE LESS THAN THE GREATER: WHICH, IS IMPOSSIBLE.

THEREFORE,

THE STRAIGHT LINE, JOINED, FROM F TO G, WILL NOT FALL OUTSIDE;

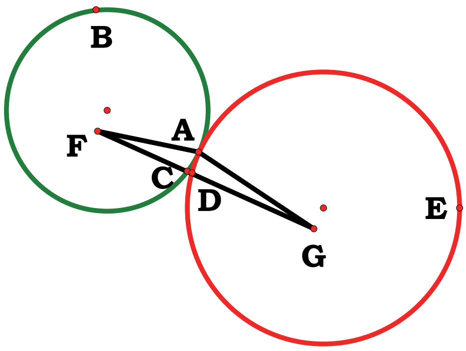
THEREFORE,

IT WILL FALL, AT A, ON THE POINT OF CONTACT.

THEREFORE ETC.

Q. E. D.

[Proposition 12.



If two circles touch one another externally, the straight line joining their centres will pass through the point of contact.

For let,

$\odot ABC$, $\odot ADE$,

touch one another externally at A,

and let,

the centre, F, of $\odot ABC$, and

the centre, G, of $\odot ADE$, be taken;

I say that;

the straight line, joined, from F to G,
will pass through the point of contact, at A.

For suppose,

it does not, but, if possible, let,
it pass as $FCDG$,

and let,

AF, AG, be joined.

Then, since,

F, is the centre of $\odot ABC$,

$FA = FC$.

Again, since,

G, is the centre of $\odot ADE$,

$GA = GD$. But,
$FA = FC$;

therefore,

$FA + AG = FC + GD$,

[I. 20] so that,

the whole, $FG > FA + AG$;
but it is, also, less: which,
is impossible.

Therefore,

the straight line, joined, from F to G,
will not fail to pass through
the point of contact, at A;

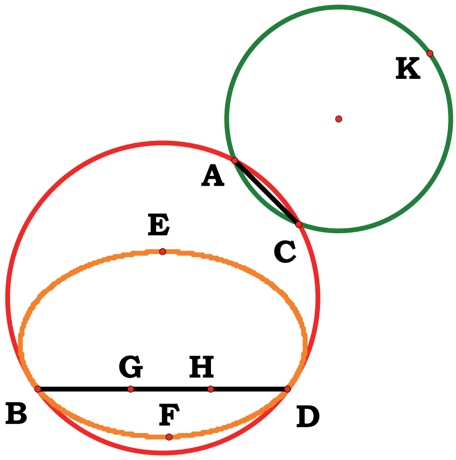
THEREFORE,

IT WILL PASS THROUGH IT.

THEREFORE ETC.

Q. E. D.]

PROPOSITION 13.



A CIRCLE DOES NOT TOUCH A CIRCLE AT MORE POINTS THAN ONE, WHETHER IT TOUCH IT INTERNALLY OR EXTERNALLY.

FOR, IF POSSIBLE, LET,

⊙*ABDC*, TOUCH

⊙*EBFD*,

FIRST,

INTERNALLY,

AT MORE POINTS THAN ONE, NAMELY, *D*, *B*.

LET,

THE CENTRE, *G*, OF ⊙*ABDC*, AND

THE CENTRE, *H*, OF ⊙*EBFD*, BE TAKEN.

[III. 11] THEREFORE,

THE STRAIGHT LINE, JOINED, FROM *G* TO *H*,
WILL FALL, ON *B*, *D*.

LET,

IT SO FALL, AS *BGHD*.

THEN, SINCE,

G, IS THE CENTRE OF ⊙*ABCD*,

$BG = GD$;

THEREFORE,

$BG > HD$;

THEREFORE,

BH IS MUCH GREATER THAN *HD*.

AGAIN, SINCE,

H, IS THE CENTRE OF ⊙*EBFD*,

$BH = HD$;

BUT,

IT WAS, ALSO, PROVED MUCH GREATER THAN IT: WHICH,
IS IMPOSSIBLE.

THEREFORE,

A CIRCLE DOES NOT TOUCH A CIRCLE INTERNALLY
AT MORE POINTS THAN ONE.

I SAY, FURTHER, THAT;

NEITHER DOES IT SO TOUCH IT EXTERNALLY.

FOR, IF POSSIBLE, LET,

$\odot ACK$, TOUCH $\odot ABDC$,

AT MORE POINTS THAN ONE, NAMELY,
A, C,

AND LET,
AC BE JOINED.

[III. 2] THEN, SINCE,
ON THE CIRCUMFERENCE OF EACH, OF THE CIRCLES,

$\odot ABDC$, $\odot ACK$, A, C, HAVE BEEN TAKEN AT RANDOM,

THE STRAIGHT LINE JOINING
THE POINTS WILL FALL WITHIN EACH CIRCLE;

[III. DEF. 3] BUT,

IT FELL WITHIN $\odot ABCD$, AND, OUTSIDE ACK: WHICH,

IS ABSURD.

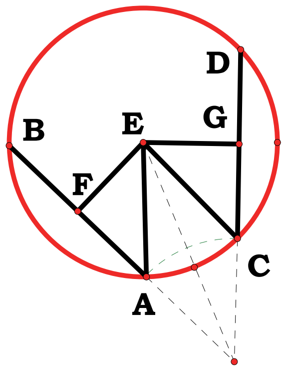
THEREFORE,
A CIRCLE DOES NOT TOUCH
A CIRCLE EXTERNALLY AT MORE POINTS THAN ONE.

AND IT WAS PROVED THAT,
NEITHER DOES IT SO TOUCH IT INTERNALLY.

THEREFORE ETC.

Q. E. D.

PROPOSITION 14.



IN A CIRCLE, EQUAL STRAIGHT LINES ARE EQUALLY DISTANT FROM THE CENTRE, AND THOSE WHICH ARE EQUALLY DISTANT FROM THE CENTRE ARE EQUAL, TO ONE ANOTHER.

LET,

$\odot ABDC$,

AND LET,

AB, CD BE EQUAL STRAIGHT LINES IN IT;

I SAY THAT;

AB, CD ARE EQUALLY DISTANT FROM THE CENTRE.

[III. 1] FOR LET,

THE CENTRE OF $\odot ABDC$, BE TAKEN, AND LET,

IT BE E;

LET,

FROM E,

EF, EG BE DRAWN PERPENDICULAR, TO AB, CD,

AND LET,

AE, EC, BE JOINED.

[III. 3] THEN, SINCE,

EF, THROUGH THE CENTRE INTERSECTS AB,

NOT THROUGH THE CENTRE,

AT RIGHT ANGLES, IT, ALSO, BISECTS IT.

THEREFORE,

$AF = FB$;

THEREFORE,

$AB = 2AF$.

FOR THE SAME REASON,

$CD = 2CG$; AND $AB = CD$;

THEREFORE,

$AF = CG$.

AND, SINCE,

$AE = EC$,

$\boxdot AE = \boxdot EC$.

BUT,

$\boxdot AF + \boxdot EF = \boxdot AE$,

FOR,

$\angle$AT F, IS RIGHT; AND

$\boxdot EG + \boxdot GC = \boxdot EC$,

[I. 47] FOR,

$\angle$AT G, IS RIGHT;

THEREFORE,

$\boxdot AF + \boxdot FE = \boxdot CG + \boxdot GE$,

OF WHICH $\boxdot AF = \boxdot CG$, FOR,

$AF = CG$;

THEREFORE, WHICH REMAINS,

$\boxdot FE = \boxdot EG$,

THEREFORE,

$EF = EG$.

[III. DEF. 4] BUT,

IN A CIRCLE,

STRAIGHT LINES ARE SAID TO BE EQUALLY DISTANT FROM THE CENTRE WHEN,

THE PERPENDICULARS DRAWN TO THEM,

FROM THE CENTRE, ARE EQUAL;

THEREFORE,

AB, CD ARE EQUALLY DISTANT FROM THE CENTRE.

NEXT, LET,

AB, CD, BE EQUALLY DISTANT FROM THE CENTRE;

THAT IS, LET,

$EF = EG$.

I SAY THAT;

$AB = CD$.

FOR, WITH THE SAME CONSTRUCTION,

WE CAN PROVE, SIMILARLY, THAT;

$AB = 2AF$, AND $CD = 2CG$.

AND, SINCE,

$AE = CE$,

$\boxdot AE = \boxdot CE$.

BUT,

$\boxdot EF + \boxdot FA = \boxdot AE$,

[I. 47] AND,

$\boxdot EG$, $\boxdot GC = \boxdot CE$.

THEREFORE,

$\boxdot EF + \boxdot FA = \boxdot EG + \boxdot GC$, OF WHICH

$\boxdot EF = \boxdot EG$, FOR,

$EF = EG$;

THEREFORE,, WHICH REMAINS

$\boxdot AF = \boxdot CG$; THEREFORE,

$AF = CG$.

AND,

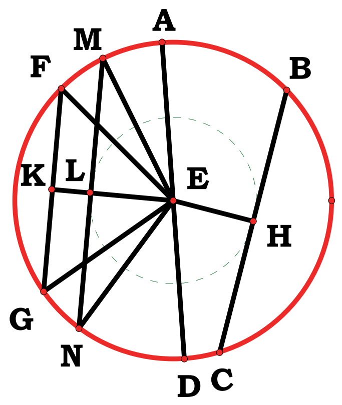
$AB = 2AF$, AND $CD = 2CG$;

THEREFORE,

$AB = CD$.

THEREFORE ETC.

Q. E. D.



PROPOSITION 15.

OF STRAIGHT LINES IN A CIRCLE THE DIAMETER IS GREATEST, AND OF THE REST THE NEARER TO THE CENTRE IS ALWAYS GREATER THAN THE MORE REMOTE.

LET,

$\odot ABCD$, LET,

AD BE ITS DIAMETER, AND

E THE CENTRE;

AND LET,

BC BE NEARER TO THE DIAMETER, AD, AND

FG MORE REMOTE;

I SAY THAT;

AD IS GREATEST, AND $BC > FG$.

FOR LET,

FROM THE CENTRE E,

EH, EK BE DRAWN, PERPENDICULAR, TO BC, FG.

[III. DEF. 5] THEN, SINCE,

BC IS NEARER TO THE CENTRE, AND

FG MORE REMOTE,

$EK > EH$.

LET,

$EL = EH$,

LET,

THROUGH L,

$LM \perp EK$, AND CARRIED THROUGH TO N,

AND LET,

ME, EN, FE, EG, BE JOINED.

[III. 14] THEN, SINCE,

$EH = EL$,

$BC = MN$.

AGAIN, SINCE,

$AE = EM$, AND

$ED = EN$,

$AD = ME + EN$.

[I. 20] BUT,

$ME + EN > MN$, AND

$MN = BC$;

THEREFORE,

$AD > BC$.

[I. 24] AND, SINCE,

$ME + EN = FE + EG$, AND

$\angle MEN < \angle FEG$,

THEREFORE,

$MN > FG$.

BUT,

$MN = BC$.

THEREFORE,

THE DIAMETER AD IS GREATEST, AND

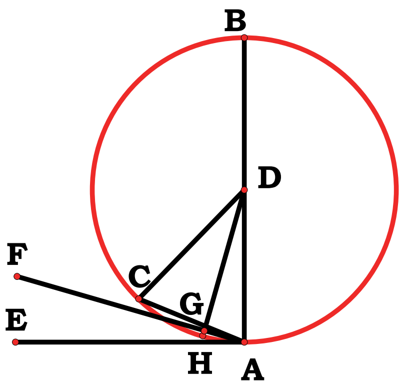
$BC > FG$.

THEREFORE ETC.

Q. E. D.

PROPOSITION 16.

THE STRAIGHT LINE DRAWN AT RIGHT ANGLES TO THE DIAMETER OF A CIRCLE *FROM ITS EXTREMITY WILL FALL OUTSIDE THE CIRCLE, AND INTO THE SPACE BETWEEN THE STRAIGHT LINE AND THE CIRCUMFERENCE ANOTHER STRAIGHT LINE CANNOT BE INTERPOSED; FURTHER THE ANGLE OF THE SEMICIRCLE IS GREATER, AND THE REMAINING ANGLE LESS, THAN ANY ACUTE RECTILINEAL ANGLE.*



LET,

⊙ABC BE ABOUT D AS CENTRE, AND AB AS DIAMETER;

I SAY THAT;

THE STRAIGHT LINE,
DRAWN FROM A, AT RIGHT ANGLES, TO AB,
FROM ITS EXTREMITY, WILL FALL OUTSIDE THE CIRCLE.

FOR SUPPOSE,

IT DOES NOT,

BUT, IF POSSIBLE, LET,

IT FALL WITHIN AS CA,

AND LET,

DC, BE JOINED.

[I. 5] SINCE,

$DA = DC$,

$\angle DAC = \angle ACD$,

[I. 17] BUT,

∟DAC, IS RIGHT;

THEREFORE,

∟ACD, IS, ALSO, RIGHT:

THUS,

IN ΔACD,

THE TWO ANGLES, ∟DAC, ∟ACD, ARE EQUAL, TO
TWO RIGHT ANGLES: WHICH,
IS IMPOSSIBLE.

THEREFORE,

THE STRAIGHT LINE DRAWN FROM

THE POINT, A, AT RIGHT ANGLES, TO BA,
WILL NOT FALL WITHIN THE CIRCLE.

SIMILARLY WE CAN PROVE THAT,
NEITHER WILL IT FALL ON THE CIRCUMFERENCE;

THEREFORE,
IT WILL FALL OUTSIDE.

LET,
IT FALL, AS AE;

I SAY, NEXT, THAT;
INTO THE SPACE BETWEEN
AE, AND THE CIRCUMFERENCE, CHA,
ANOTHER STRAIGHT LINE CANNOT BE INTERPOSED.

FOR, IF POSSIBLE, LET,
ANOTHER STRAIGHT LINE BE SO INTERPOSED,
AS FA,

AND LET,
DG BE DRAWN, FROM $D \perp FA$.

[I. 19] THEN, SINCE,
$\llcorner AGD$, IS RIGHT, AND
$\angle DAG$, IS LESS THAN A RIGHT ANGLE,
$AD > DG$. BUT,
$DA = DH$;

THEREFORE,
$DH > DG$,
THE LESS THAN THE GREATER: WHICH,
IS IMPOSSIBLE.

THEREFORE,
ANOTHER STRAIGHT LINE CANNOT BE INTERPOSED INTO
THE SPACE BETWEEN THE STRAIGHT LINE AND
THE CIRCUMFERENCE.

I SAY, FURTHER, THAT;
THE ANGLE OF THE SEMICIRCLE CONTAINED BY
BA, AND THE CIRCUMFERENCE, CHA,
IS GREATER THAN ANY ACUTE RECTILINEAL ANGLE, AND
THE REMAINING ANGLE
CONTAINED BY THE CIRCUMFERENCE, CHA, AND
AE, IS LESS THAN ANY ACUTE RECTILINEAL ANGLE.

FOR, IF,
THERE IS ANY RECTILINEAL ANGLE GREATER THAN

THE ANGLE CONTAINED BY THE STRAIGHT LINE, *BA*, AND
THE CIRCUMFERENCE, *CHA*, AND
ANY RECTILINEAL ANGLE LESS THAN
THE ANGLE CONTAINED BY THE CIRCUMFERENCE, *CHA*, AND
THE STRAIGHT LINE, *AE*,

THEN,

INTO THE SPACE BETWEEN THE CIRCUMFERENCE, AND
AE, A STRAIGHT LINE WILL BE INTERPOSED
SUCH AS WILL MAKE AN ANGLE,
CONTAINED BY STRAIGHT LINES, WHICH
IS GREATER THAN THE ANGLE CONTAINED BY
THE STRAIGHT LINE, *BA*, AND
THE CIRCUMFERENCE, *CHA*, AND
ANOTHER ANGLE CONTAINED BY STRAIGHT LINES,
WHICH IS LESS THAN
THE ANGLE CONTAINED BY THE CIRCUMFERENCE, *CHA*, AND
THE STRAIGHT LINE *AE*.

BUT,

SUCH A STRAIGHT LINE CANNOT BE INTERPOSED;

THEREFORE,

THERE WILL NOT BE ANY ACUTE ANGLE CONTAINED BY
STRAIGHT LINES WHICH IS GREATER THAN
THE ANGLE CONTAINED BY *BA*, AND
THE CIRCUMFERENCE, *CHA*,

NOR YET,

ANY ACUTE ANGLE CONTAINED BY STRAIGHT LINES,
WHICH IS LESS THAN,
THE ANGLE CONTAINED BY THE CIRCUMFERENCE, *CHA*,

AND,

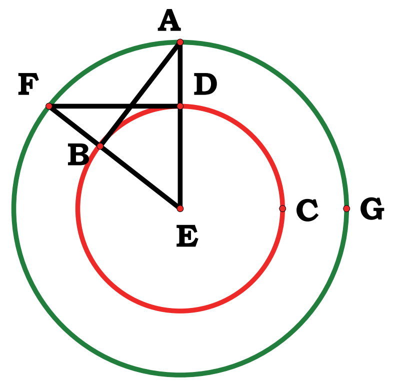
THE STRAIGHT LINE *AE*.—

PORISM.

FROM THIS IT IS MANIFEST THAT THE STRAIGHT LINE DRAWN AT RIGHT ANGLES TO THE DIAMETER OF A CIRCLE FROM ITS EXTREMITY TOUCHES THE CIRCLE.

Q. E. D.

Proposition 17.



From a given point to draw a straight line touching a given circle.

Let,

A be the given point,

and,

$\odot BCD$, the given circle;

thus it is required,

to draw from A, a straight line touching $\odot BCD$.

[III. 1]

For let,

the centre, E, of the circle be taken; let,

AE be joined,

and let,

with centre, E, and distance, EA,

$\odot AFG$, be described; let,

from D, $DF \perp EA$,

and let,

EF, AB, be joined;

I say that;

AB has been drawn, from A, touching $\odot BCD$.

For, since,

E is the centre of $\odot BCD$, $\odot AFG$,

$EA = EF$, and $ED = EB$;

therefore,

the two sides,

$AE + EB = FE + ED$: and

they contain a common angle, $\angle$at E;

[I. 4] therefore,

the bases, $DF = AB$, and

$\Delta DEF = \Delta BEA$, and

the remaining angles to the remaining angles;

therefore,

$\llcorner EDF = \llcorner EBA$. BUT,

$\llcorner EDF$, IS RIGHT;

THEREFORE,

$\llcorner EBA$, IS, ALSO, RIGHT.

NOW,

EB IS A RADIUS;

[III. 16, POR.] AND,

THE STRAIGHT LINE DRAWN AT RIGHT ANGLES TO THE DIAMETER OF A CIRCLE, FROM ITS EXTREMITY, TOUCHES THE CIRCLE;

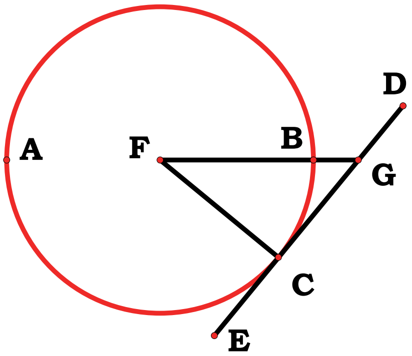
THEREFORE,

AB TOUCHES THE CIRCLE BCD.

THEREFORE,

FROM A, AB, HAS BEEN DRAWN TOUCHING $\odot BCD$.

PROPOSITION 18.



IF A STRAIGHT LINE TOUCH A CIRCLE, AND A STRAIGHT LINE BE JOINED FROM THE CENTRE TO THE POINT OF CONTACT, THE STRAIGHT LINE SO JOINED WILL BE PERPENDICULAR TO THE TANGENT.

FOR LET,

DE, TOUCH $\odot ABC$, AT C,

LET,

THE CENTRE, F, OF $\odot ABC$, BE TAKEN, AND LET,

FC BE JOINED FROM, F TO C;

I SAY THAT;

$FC \perp DE$.

FOR, IF NOT, LET,

FG BE DRAWN, FROM $F \perp DE$.

[I. 17] THEN, SINCE,

$\llcorner FGC$, IS RIGHT, $\angle FCG$, IS ACUTE;

[I. 19] THE GREATER ANGLE IS SUBTENDED BY THE GREATER SIDE;

THEREFORE,

$FC > FG$. BUT,

$FC = FB$;

THEREFORE, ALSO,

$FB > FG$,

THE LESS THAN THE GREATER: WHICH, IS IMPOSSIBLE.

THEREFORE,

FG IS NOT PERPENDICULAR TO DE.

SIMILARLY WE CAN PROVE THAT,

NEITHER IS ANY OTHER STRAIGHT LINE, EXCEPT, FC;

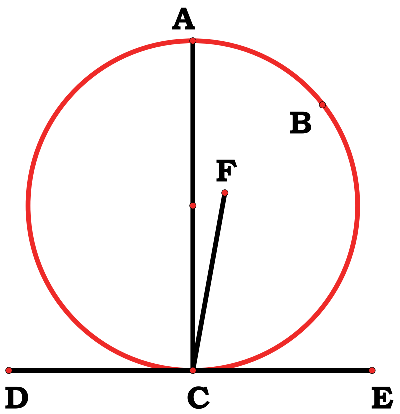
THEREFORE,

$FC \perp DE$.

THEREFORE ETC.

Q. E. D.

PROPOSITION 19.



IF A STRAIGHT LINE TOUCH A CIRCLE, AND FROM THE POINT OF CONTACT A STRAIGHT LINE BE DRAWN AT RIGHT ANGLES TO THE TANGENT, THE CENTRE OF THE CIRCLE WILL BE ON THE STRAIGHT LINE SO DRAWN.

FOR LET,

DE TOUCH $\odot ABC$, AT C,

AND LET,

FROM C, $CA \perp DE$;

I SAY THAT;

THE CENTRE OF THE CIRCLE IS ON AC.

FOR,

SUPPOSE IT IS NOT, BUT, IF POSSIBLE, LET,
F BE THE CENTRE, AND LET,
CF BE JOINED.

[III. 18] SINCE,

DE, TOUCHES $\odot ABC$, AND
FC HAS BEEN JOINED FROM
THE CENTRE TO THE POINT OF CONTACT,
$FC \perp DE$;

THEREFORE,

$\llcorner FCE$, IS RIGHT. BUT,

$\llcorner ACE$, IS, ALSO, RIGHT;

THEREFORE,

$\llcorner FCE = \llcorner ACE$,

THE LESS TO THE GREATER: WHICH,
IS IMPOSSIBLE.

THEREFORE,

F IS NOT THE CENTRE OF $\odot ABC$.

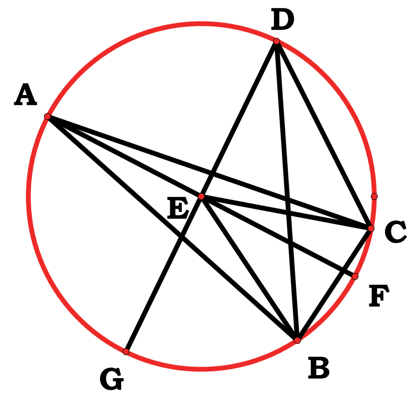
SIMILARLY WE CAN PROVE THAT,

NEITHER IS ANY OTHER POINT,
EXCEPT A POINT, ON AC.

THEREFORE ETC.

Q. E. D.

PROPOSITION 20.



IN A CIRCLE THE ANGLE AT THE CENTRE IS DOUBLE OF THE ANGLE AT THE CIRCUMFERENCE, WHEN THE ANGLES HAVE THE SAME CIRCUMFERENCE AS BASE.

LET,

$\odot ABC$, LET,

$\angle BEC$, BE AN ANGLE AT ITS CENTRE, AND

$\angle BAC$, AN ANGLE AT THE CIRCUMFERENCE,

AND LET,

THEM HAVE THE SAME CIRCUMFERENCE, BC, AS BASE;

I SAY THAT;

$\angle BEC = 2\angle BAC$.

FOR LET,

AE BE JOINED AND DRAWN, THROUGH TO F.

[I. 5] THEN, SINCE,

$EA = EB$, $\angle EAB = \angle EBA$;

THEREFORE,

$\angle EAB + \angle EBA = 2\angle EAB$.

[I. 32] BUT,

$\angle BEF = \angle EAB + \angle EBA$;

THEREFORE, ALSO,

$\angle BEF = 2\angle EAB$.

FOR THE SAME REASON,

$\angle FEC = 2\angle EAC$.

THEREFORE,

$\angle BEC = 2\angle BAC$.

AGAIN LET,

ANOTHER STRAIGHT LINE BE INFLECTED,

AND LET,

THERE BE ANOTHER ANGLE, $\angle BDC$;

LET,

DE BE JOINED AND PRODUCED, TO G.

SIMILARLY THEN WE CAN PROVE THAT,

$\angle GEC = 2\angle EDC$, OF WHICH

$\angle GEB = 2\angle EDB$;

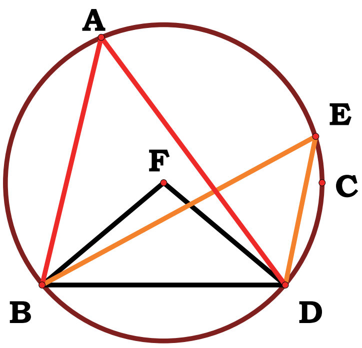
THEREFORE, WHICH REMAINS,

$\angle BEC = 2\angle BDC$.

THEREFORE ETC.

Q. E. D.

PROPOSITION 21.



IN A CIRCLE, THE ANGLES IN THE SAME SEGMENT ARE EQUAL, TO ONE ANOTHER.

LET,

$\odot ABCD$,

AND LET,

$\angle BAD$, $\angle BED$, BE IN THE SAME SEGMENT, $BAED$;

I SAY THAT;

$\angle BAD$, $\angle BED$, ARE EQUAL, TO ONE ANOTHER.

FOR LET,

THE CENTRE OF $\odot ABCD$, BE TAKEN,

AND LET,

IT BE F;

LET,

BE, ED BE JOINED.

[III. 20] NOW, SINCE,

$\angle BFD$, IS AT THE CENTRE, AND

$\angle BAD$, AT THE CIRCUMFERENCE, AND

THEY HAVE THE SAME CIRCUMFERENCE, BCD, AS BASE,

THEREFORE,

$\angle BFD = 2\angle BAD$.

FOR THE SAME REASON,

$\angle BFD = 2\angle BED$;

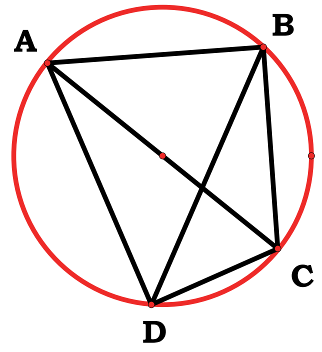
THEREFORE,

$\angle BAD = \angle BED$.

THEREFORE ETC.

Q. E. D.

PROPOSITION 22,



THE OPPOSITE ANGLES OF QUADRILATERALS IN CIRCLES ARE EQUAL, TO TWO RIGHT ANGLES.

LET,

$\odot ABCD$, AND LET,

$ABCD$ BE A QUADRILATERAL IN IT;

I SAY THAT;

THE OPPOSITE ANGLES ARE EQUAL, TO TWO RIGHT ANGLES.

LET,

AC, BD BE JOINED.

[I. 32] THEN, SINCE,

IN ANY TRIANGLE,

THE THREE ANGLES ARE EQUAL, TO TWO RIGHT ANGLES,

$\angle CAB + \angle ABC + \angle BCA$, OF

ΔABC, ARE EQUAL, TO TWO RIGHT ANGLES.

[III. 21] BUT,

$\angle CAB = \angle BDC$,

FOR,

THEY ARE IN THE SAME SEGMENT, $BADC$; AND

$\angle ACB = \angle ADB$,

FOR,

THEY ARE IN THE SAME SEGMENT, $ADCB$;

THEREFORE,

$\angle ADC = \angle BAC + \angle ACB$.

LET,

$\angle ABC$, BE ADDED TO EACH;

THEREFORE,

$\angle ABC + \angle BAC + \angle ACB = \angle ABC + \angle ADC$.

BUT,

$\angle ABC + \angle BAC + \angle ACB$,

ARE EQUAL, TO TWO RIGHT ANGLES;

THEREFORE,

$\angle ABC + \angle ADC$,

ARE, ALSO, EQUAL, TO TWO RIGHT ANGLES.

SIMILARLY WE CAN PROVE THAT,

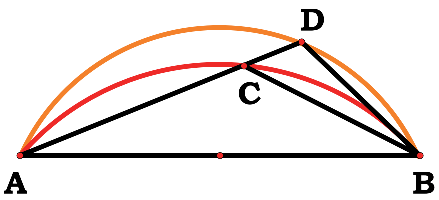
$\angle BAD + \angle DCB$,

ARE, ALSO, EQUAL, TO TWO RIGHT ANGLES.

THEREFORE ETC.

Q. E. D.

PROPOSITION 23.



ON THE SAME STRAIGHT LINE THERE CANNOT BE CONSTRUCTED TWO SIMILAR AND UNEQUAL SEGMENTS OF CIRCLES ON THE SAME SIDE.

FOR, IF POSSIBLE, LET,

ON *AB*,

TWO SIMILAR AND UNEQUAL,

⌓*ACB*, ⌓*ADB*, BE CONSTRUCTED, ON THE SAME SIDE;

LET,

ACD BE DRAWN THROUGH,

AND LET,

CB, *DB* BE JOINED.

THEN, SINCE,

⌓*ACB*, IS SIMILAR TO ⌓*ADB*,

[III. DEF. 11]

AND,

SIMILAR SEGMENTS OF CIRCLES ARE THOSE WHICH ADMIT EQUAL ANGLES,

$\angle ACB = \angle ADB$, THE EXTERIOR TO THE INTERIOR:

[I. 16] WHICH,

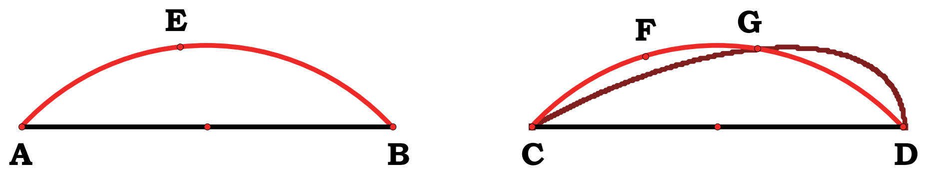
IS IMPOSSIBLE.

THEREFORE ETC.

Q. E. D.

Proposition 24.

Similar segments of circles on equal straight lines are equal, to one another.



For let,

⌓*AEB*, ⌓*CFD* be on equal straight lines, *AB*, *CD*;

I say that;

⌓*AEB* = ⌓*CFD*.

For,

⌓*AEB*, be applied, to ⌓*CFD*, and

if *A*, be placed, on *C*, and the *AB* on *CD*,
B, will, also, coincide with *D*,

because,

AB = *CD*; and *AB* coinciding with *CD*,

⌓*AEB*, will, also, coincide with ⌓*CFD*.

For,

if *AB*, coincide with *CD*, but,

⌓*AEB*, do not coincide with ⌓*CFD*,

it will either,

fall within it, or
outside it; or
it will fall awry, as *CGD*, and
a circle cuts a circle at more points than two:

[III. 10] which,

is impossible.

Therefore,

if *AB*, be applied, to *CD*,

⌓*AEB*, will not fail to coincide with ⌓*CFD* also;

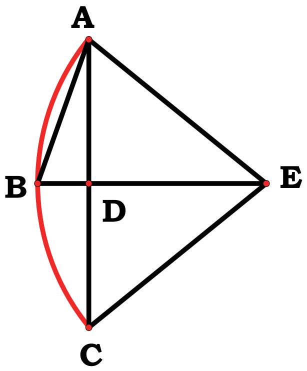
therefore,

it will coincide with it and will be equal, to it.

Therefore etc.

Q. E. D.

PROPOSITION 25.



GIVEN A SEGMENT OF A CIRCLE, TO DESCRIBE THE COMPLETE CIRCLE OF WHICH IT IS A SEGMENT.

LET,

$\frown ABC$ BE GIVEN;

THUS IT IS REQUIRED,

TO DESCRIBE $\odot ABC$,

FOR LET,

AC BE BISECTED AT D, LET,

$DB \perp AC$, AND LET,

AB BE JOINED;

$\angle ABD$ IS THEN GREATER THAN, EQUAL, TO,

OR LESS THAN, $\angle BAD$.

FIRST LET,

IT BE GREATER; AND

ON BA, AND AT A, ON IT,

LET,

$\angle BAE = \angle ABD$;

LET,

DB BE DRAWN THROUGH TO E,

AND LET,

EC BE JOINED.

[I. 6] THEN, SINCE,

$\angle ABE = \angle BAE$, $EB = EA$.

AND, SINCE,

$AD = DC$, AND

DE IS COMMON, THE TWO SIDES,

$AD + DE = CD + DE$; AND

$\llcorner ADE = \llcorner CDE$, FOR,

EACH IS RIGHT;

THEREFORE,

THE BASES, $AE = CE$. BUT,

$AE = BE$; THEREFORE,

$BE = CE$;

THEREFORE,

THE THREE STRAIGHT LINES,

AE, EB, EC, ARE EQUAL, TO ONE ANOTHER.

[III. 9] THEREFORE,

THE CIRCLE DRAWN WITH CENTRE, E, AND DISTANCE ONE OF THE STRAIGHT LINES,

AE, EB, EC, WILL, ALSO, PASS THROUGH THE REMAINING POINTS, AND

WILL HAVE BEEN COMPLETED.

THEREFORE,

GIVEN A SEGMENT OF A CIRCLE,

THE COMPLETE CIRCLE HAS BEEN DESCRIBED.

AND IT IS MANIFEST THAT,

$\frown ABC$, IS LESS THAN A SEMICIRCLE,

BECAUSE,

THE CENTRE, E, HAPPENS TO BE OUTSIDE IT.

SIMILARLY, EVEN IF,

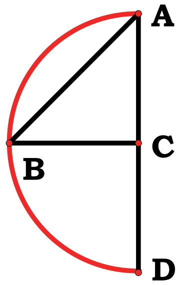
$\angle ABD = \angle BAD$,

AD BEING EQUAL, TO EACH, OF THE TWO, BD, DC,

DA, DB, DC, WILL BE EQUAL,

TO ONE ANOTHER, D, WILL BE

THE CENTRE OF THE COMPLETED CIRCLE,



AND

ABC, WILL CLEARLY BE A SEMICIRCLE.

BUT, IF,

$\angle ABD < \angle BAD$,

AND IF,

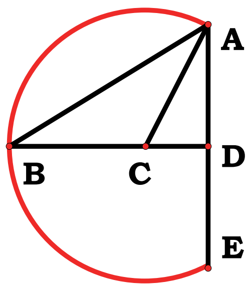
WE CONSTRUCT,

BA, AND AT A, ON IT,

AN ANGLE EQUAL, TO $\angle ABD$,

THE CENTRE WILL FALL ON DB,

WITHIN THE SEGMENT, ABC,



AND CLEARLY,

$\frown ABC$, WILL BE GREATER THAN A SEMICIRCLE.

THEREFORE,

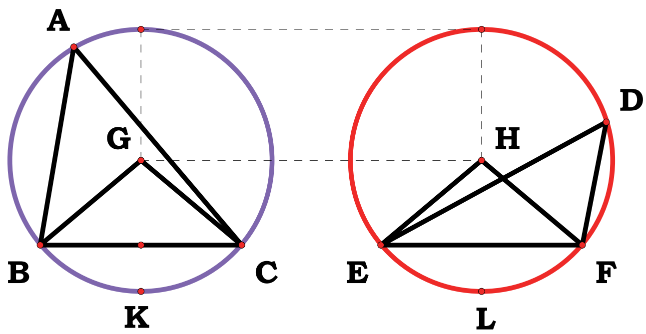
GIVEN A SEGMENT OF A CIRCLE,

THE COMPLETE CIRCLE HAS BEEN DESCRIBED.

Q. E. F.

PROPOSITION 26.

IN EQUAL CIRCLES EQUAL ANGLES STAND ON EQUAL CIRCUMFERENCES, WHETHER THEY STAND AT THE CENTRES OR AT THE CIRCUMFERENCES.



LET,

$\odot ABC = \odot DEF$, AND LET,

IN THEM, THERE BE EQUAL ANGLES, NAMELY,

AT THE CENTRES, $\angle BGC$, $\angle EHF$, AND

AT THE CIRCUMFERENCES, $\angle BAC$, $\angle EDF$;

I SAY THAT;

THE CIRCUMFERENCES, $BKC = ELF$.

FOR LET,

BC, EF BE JOINED.

NOW, SINCE,

$\odot ABC = \odot DEF$, THE RADII ARE EQUAL.

THUS,

BG, $GC = EH$, HF; AND $\angle$AT $G = \angle$AT H;

[I. 4] THEREFORE,

THE BASES, $BC = EF$.

[III. DEF. 11] AND, SINCE,

$\angle$AT $A = \angle$AT D,

$\frown BAC$ IS SIMILAR TO $\frown EDF$;

AND THEY ARE UPON EQUAL STRAIGHT LINES.

[III. 24] BUT,

SIMILAR SEGMENTS OF CIRCLES ON

EQUAL STRAIGHT LINES ARE EQUAL, TO ONE ANOTHER;

THEREFORE,

$\frown BAC = \frown EDF$.

BUT,

$\odot ABC = \odot DEF$;

THEREFORE,

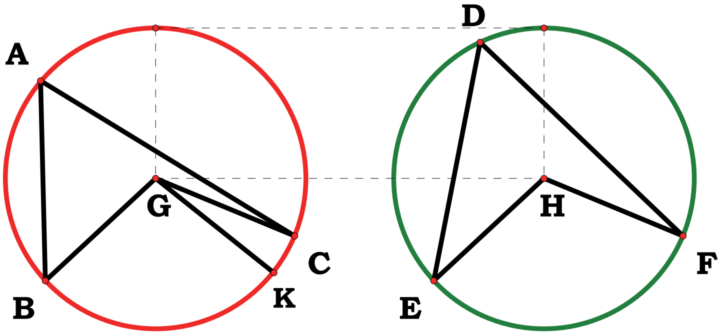
THE CIRCUMFERENCE, *BKC*,
WHICH REMAINS, IS EQUAL TO THE CIRCUMFERENCE, *ELF*.

THEREFORE ETC.

Q. E. D.

PROPOSITION 27.

IN EQUAL CIRCLES, ANGLES STANDING ON EQUAL CIRCUMFERENCES ARE EQUAL, TO ONE ANOTHER, WHETHER THEY STAND AT THE CENTRES OR AT THE CIRCUMFERENCES.



FOR,

$\odot ABC = \odot DEF$,

ON EQUAL CIRCUMFERENCES, BC, EF,

LET,

$\angle BGC$, $\angle EHF$, STAND AT THE CENTRES, G, H, AND

$\angle BAC$, $\angle EDF$, AT THE CIRCUMFERENCES;

I SAY THAT;

$\angle BGC = \angle EHF$, AND $\angle BAC = \angle EDF$.

FOR, IF,

$\angle BGC \neq \angle EHF$,

ONE OF THEM IS GREATER.

LET,

$\angle BGC$, BE GREATER: AND ON BG, AND AT G ON IT,

[I. 23] LET,

$\angle BGK = \angle EHF$.

[III. 26] NOW,

EQUAL ANGLES STAND ON EQUAL CIRCUMFERENCES, WHEN THEY ARE AT THE CENTRES; THEREFORE,
THE CIRCUMFERENCES, $BK = EF$.

BUT,

$EF = BC$; THEREFORE,
$BK = BC$,
THE LESS TO THE GREATER: WHICH,
IS IMPOSSIBLE.

THEREFORE,

$\angle BGC = \angle EHF$;

[III. 20] AND,

$\angle$AT A, IS HALF OF $\angle BGC$, AND

$\angle$AT D, HALF OF $\angle EHF$;

THEREFORE,

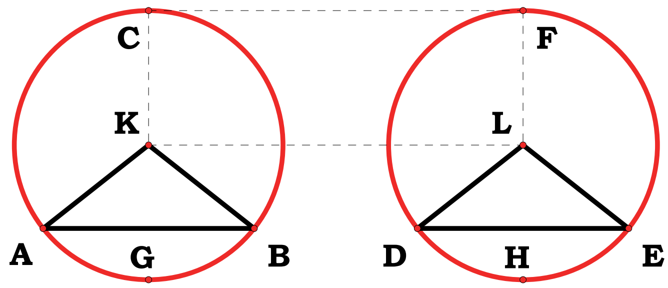
$\angle$AT $A = \angle$AT D.

THEREFORE ETC.

Q. E. D.

Proposition 28.

In equal circles equal straight lines cut off equal circumferences, the greater equal, to the greater and the less to the less.



Let,

$\odot ABC = \odot DEF$, and let,

$AB = DE$, dividing off

ACB, DFE, as greater circumferences, and

AGB, DHE, as lesser;

I say that;

the greater circumferences, $ACB = DFE$, and

the less circumferences, $AGB = DHE$.

For let,

the centres, K, L, of the circles be taken,

and let,

AK, KB, DL, LE, be joined.

Now, since,

the circles are equal,

the radii are, also, equal;

therefore,

the two sides,

AK, KB, are equal, to the two sides, DL, LE; and

the bases, $AB = DE$;

[I. 8] therefore,

$\angle AKB = \angle DLE$.

[III. 26] But,

equal angles stand on equal circumferences,

when they are at the centres;

therefore,

the circumferences, $AGB = DHE$.

And also,

$\odot ABC = \odot DEF$;

THEREFORE, WHICH REMAINS

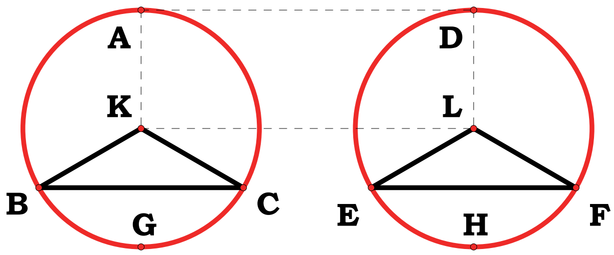
THE CIRCUMFERENCES, ACB, = DFE.

THEREFORE ETC.

Q. E. D.

PROPOSITION 29.

IN EQUAL CIRCLES EQUAL CIRCUMFERENCES ARE SUBTENDED BY EQUAL STRAIGHT LINES.



LET,

$\odot ABC = \odot DEF$,

AND LET, IN THEM,

EQUAL CIRCUMFERENCES, BGC, EHF; AND LET,
BC, EF, BE JOINED;

I SAY THAT;

$BC = EF$.

FOR LET,

THE CENTRES OF THE CIRCLES BE TAKEN,

AND LET,

THEM BE, K, L;

LET,

BK, KC, EL, LF, BE JOINED.

[III. 27] NOW, SINCE,

THE CIRCUMFERENCES, $BGC = EHF$,

$\angle BKC = \angle ELF$.

AND, SINCE,

$\odot ABC = \odot DEF$, THE RADII ARE, ALSO, EQUAL;

THEREFORE,

THE TWO SIDES, BK, KC, ARE EQUAL, TO
THE TWO SIDES, EL, LF; AND
THEY CONTAIN EQUAL ANGLES;

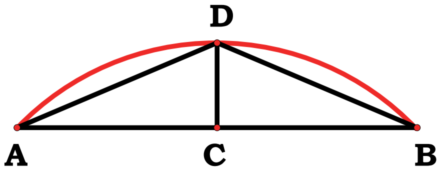
[1. 4] THEREFORE,

THE BASES, $BC = EF$.

THEREFORE ETC.

PROPOSITION 30.

TO BISECT A GIVEN CIRCUMFERENCE.



LET,

ADB BE

THE GIVEN CIRCUMFERENCE;

THUS IT IS REQUIRED,

TO BISECT THE CIRCUMFERENCE, ADB.

LET,

AB BE JOINED, AND BISECTED AT C;

LET FROM,

THE POINT C, $CD \perp AB$,

AND LET,

AD, DB BE JOINED.

THEN, SINCE,

$AC = CB$, AND

CD IS COMMON,

THE TWO SIDES, AC, CD, ARE EQUAL, TO

THE TWO SIDES, BC, CD; AND

$\llcorner ACD = \llcorner BCD$, FOR, EACH IS RIGHT;

[I. 4] THEREFORE,

THE BASES, $AD = DB$.

[III. 28] BUT,

EQUAL STRAIGHT LINES CUT OFF EQUAL CIRCUMFERENCES,

THE GREATER EQUAL, TO THE GREATER, AND,

THE LESS TO THE LESS; AND,

EACH, OF THE CIRCUMFERENCES,

AD, DB, IS LESS THAN A SEMICIRCLE;

THEREFORE,

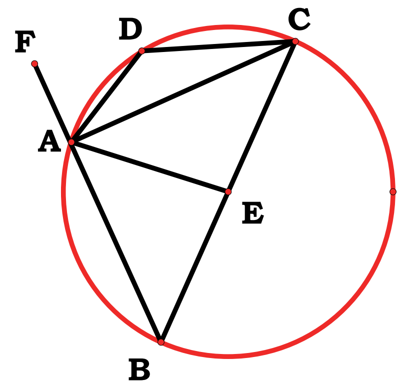
THE CIRCUMFERENCES, $AD = DB$.

THEREFORE,

THE GIVEN CIRCUMFERENCE HAS BEEN BISECTED AT D.

Q. E. F.

PROPOSITION 31.



IN A CIRCLE, THE ANGLE IN THE SEMICIRCLE IS RIGHT, THAT IN A GREATER SEGMENT LESS THAN A RIGHT ANGLE, AND THAT IN A LESS SEGMENT GREATER THAN A RIGHT ANGLE; AND FURTHER THE ANGLE OF THE GREATER SEGMENT IS GREATER THAN A RIGHT ANGLE, AND THE ANGLE OF THE LESS SEGMENT LESS THAN A RIGHT ANGLE.

LET,

$\odot ABCD$, LET,

BC BE ITS DIAMETER, AND E, ITS CENTRE,

AND LET,

BA, AC, AD, DC BE JOINED;

I SAY THAT;

$\llcorner BAC$, IN THE SEMICIRCLE, BAC, IS RIGHT,

$\angle ABC$, IN THE SEGMENT, ABC, GREATER THAN THE SEMICIRCLE IS LESS THAN A RIGHT ANGLE, AND

$\angle ADC$, IN THE SEGMENT, ADC, LESS THAN THE SEMICIRCLE IS GREATER THAN A RIGHT ANGLE.

LET,

AE BE JOINED,

AND LET,

BA BE CARRIED THROUGH TO F.

[I. 5] THEN, SINCE,

$BE = EA$, $\angle ABE = \angle BAE$.

[I. 5] AGAIN, SINCE,

$CE = EA$, $\angle ACE = \angle CAE$. THEREFORE,

$\angle BAC = \angle ABC + \angle ACB$.

[I. 32] BUT,

EXTERIOR TO ΔABC, $\angle FAC = \angle ABC + \angle ACB$;

THEREFORE,

$\llcorner BAC = \llcorner FAC$;

[I. DEF. 10] THEREFORE,

EACH IS RIGHT;

THEREFORE,

∟BAC, IN THE SEMICIRCLE, BAC, IS RIGHT.

[1. 17] NEXT, SINCE,

IN ΔABC,

$\angle ABC + \angle BAC$, ARE LESS THAN TWO RIGHT ANGLES, AND

∟BAC, IS A RIGHT ANGLE,

$\angle ABC$, IS LESS THAN A RIGHT ANGLE; AND
IT IS THE ANGLE IN THE SEGMENT, ABC,
GREATER THAN THE SEMICIRCLE.

[III. 22] NEXT, SINCE,
$ABCD$ IS A QUADRILATERAL IN A CIRCLE, AND
THE OPPOSITE ANGLES OF QUADRILATERALS
IN CIRCLES ARE EQUAL, TO TWO RIGHT ANGLES, WHILE

$\angle ABC$, IS LESS THAN A RIGHT ANGLE,

THEREFORE,

$\angle ADC$, WHICH REMAINS,

IS GREATER THAN A RIGHT ANGLE; AND
IT IS THE ANGLE IN THE SEGMENT, ADC,
LESS THAN THE SEMICIRCLE.

I SAY, FURTHER, THAT;
THE ANGLE OF THE GREATER SEGMENT, NAMELY,
THAT CONTAINED BY THE CIRCUMFERENCE, ABC, AND
AC, IS GREATER THAN A RIGHT ANGLE; AND
THE ANGLE OF THE LESS SEGMENT, NAMELY,
THAT CONTAINED BY THE CIRCUMFERENCE, ADC, AND
AC, IS LESS THAN A RIGHT ANGLE.

THIS IS AT ONCE MANIFEST.

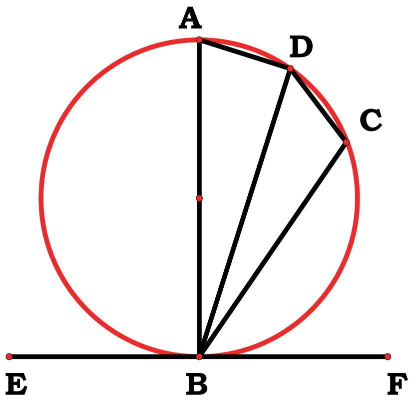
FOR, SINCE,
THE ANGLE CONTAINED BY BA, AC, IS RIGHT,
THE ANGLE CONTAINED BY THE CIRCUMFERENCE,
ABC, AND AC, IS GREATER THAN A RIGHT ANGLE.

AGAIN, SINCE,
THE ANGLE CONTAINED BY AC, AF, IS RIGHT,
THE ANGLE CONTAINED BY CA, AND
THE CIRCUMFERENCE, ADC, IS LESS THAN A RIGHT ANGLE.

THEREFORE ETC.

Q. E. D.

PROPOSITION 32.



IF A STRAIGHT LINE TOUCH A CIRCLE, AND FROM THE POINT OF CONTACT THERE BE DRAWN ACROSS, IN THE CIRCLE, A STRAIGHT LINE CUTTING THE CIRCLE, THE ANGLES WHICH IT MAKES WITH THE TANGENT WILL BE EQUAL, TO THE ANGLES IN THE ALTERNATE SEGMENTS OF THE CIRCLE.

FOR LET,

EF TOUCH $\odot ABCD$ AT B,

AND LET FROM,

B, THERE BE DRAWN ACROSS, IN $\odot ABCD$,

BD, DIVIDING IT;

I SAY THAT;

THE ANGLES, WHICH BD MAKES WITH
THE TANGENT, EF, WILL BE EQUAL, TO
THE ANGLES IN THE ALTERNATE SEGMENTS OF THE CIRCLE.

THAT IS, THAT,

$\angle FBD = \angle$CONSTRUCTED IN $\frown BAD$, AND

$\angle EBD = \angle$CONSTRUCTED IN $\frown DCB$.

FOR LET,

$BA \perp EF$,

LET, AT RANDOM,

A POINT, C, BE TAKEN, ON THE CIRCUMFERENCE, BD,

AND LET,

AD, DC, CB BE JOINED.

[III. 19]

THEN, SINCE,

EF, TOUCHES $\odot ABCD$, AT B, AND

BA HAS BEEN DRAWN, FROM THE POINT OF CONTACT,
AT RIGHT ANGLES, TO THE TANGENT,

THE CENTRE OF $\odot ABCD$, IS ON BA.

THEREFORE,

BA IS A DIAMETER OF $\odot ABCD$;

[III. 31] THEREFORE,

$\llcorner ADB$, BEING AN ANGLE IN A SEMICIRCLE, IS RIGHT.

[I. 32] THEREFORE,
THE REMAINING ANGLES,

$\angle BAD + \angle ABD$, ARE EQUAL, TO ONE RIGHT ANGLE.

BUT,

$\llcorner ABF$, IS, ALSO, RIGHT;

THEREFORE,

$\angle ABF = \angle BAD + \angle ABD$.

LET,

$\angle ABD$, BE SUBTRACTED FROM EACH;

THEREFORE, WHICH REMAINS

$\angle DBF = \angle BAD$, IN THE ALTERNATE SEGMENT OF THE CIRCLE.

[III. 22] NEXT, SINCE,
$ABCD$ IS A QUADRILATERAL IN A CIRCLE,
ITS OPPOSITE ANGLES ARE EQUAL, TO TWO RIGHT ANGLES.

BUT,

$\angle DBF + \angle DBE$, ARE, ALSO, EQUAL, TO
TWO RIGHT ANGLES;

THEREFORE,

$\angle DBF + \angle DBE = \angle BAD + \angle BCD$, OF WHICH

$\angle BAD = \angle DBF$;

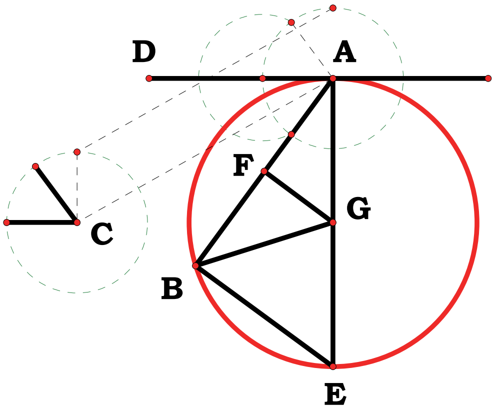
THEREFORE, WHICH REMAINS

$\angle DBE = \angle DCB$, IN
THE ALTERNATE SEGMENT, DCB, OF THE CIRCLE.

THEREFORE ETC.

Q. E. D.

Proposition 33.



On a given straight line to describe a segment of a circle admitting an angle equal, to a given rectilineal angle.

Let,

AB be given,

and,

$\angle$at C, the given rectilineal angle;

thus it is required,

to describe, on AB, a segment of a circle,

admitting an angle equal, to, $\angle$at C.

The $\angle$at C is, then,

acute, or

right, or

obtuse.

First let,

it be acute, and as in the first figure,

on AB, and let,

at A, $\angle BAD = \angle$at C;

therefore,

$\angle BAD$, is, also, acute.

Let,

$AE \perp DA$,

let,

AB be bisected at F,

let,

$FG \perp AB$,

and let,

GB be joined.

Then, since,

$AF = FB$, and FG is common,

the two sides, AF, FG, are equal, to

the two sides, BF, FG; and

$\angle AFG = \angle BFG$;

[I. 4] THEREFORE,

THE BASES, $AG = BG$.

THEREFORE,

THE CIRCLE DESCRIBED WITH CENTRE, G, AND DISTANCE, GA, WILL PASS THROUGH B ALSO.

LET,

IT BE DRAWN, AND LET,

IT BE ABE;

LET,

EB BE JOINED.

NOW, SINCE,

AD IS DRAWN, FROM A,

THE EXTREMITY OF THE DIAMETER, AE,

AT RIGHT ANGLES, TO AE,

[III. 16, POR.] THEREFORE,

AD TOUCHES $\odot ABE$.

[III. 32] SINCE THEN,

AD, TOUCHES $\odot ABE$,

AND FROM,

THE POINT OF CONTACT, AT A,

AB, IS DRAWN ACROSS IN $\odot ABE$, $\angle DAB = \angle AEB$

IN THE ALTERNATE SEGMENT OF THE CIRCLE.

BUT,

$\angle DAB = \angle$ AT C;

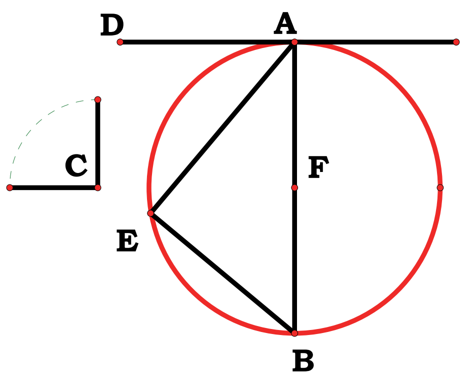
THEREFORE,

$\angle$ AT $C = \angle AEB$.

THEREFORE,

ON B, $\frown AEB$, HAS BEEN DESCRIBED

ADMITTING $\angle AEB$, $=$ $\angle$ AT C.



NEXT LET,

$\llcorner$ AT C BE RIGHT;

AND LET IT BE AGAIN REQUIRED,

TO DESCRIBE, ON AB,

A SEGMENT OF A CIRCLE,

ADMITTING AN ANGLE EQUAL, TO ∟AT C.

LET,

∟BAD = ∟AT C,

AS IS THE CASE IN THE SECOND FIGURE;

LET,

AB BE BISECTED, AT F,

AND LET,

WITH CENTRE, F, AND DISTANCE EITHER, FA OR FB,

⊙AEB, BE DESCRIBED.

[III. 16, POR.] THEREFORE,

AD, TOUCHES ⊙ABE, BECAUSE,

∠AT A, IS RIGHT.

AND,

∠BAD = THE ANGLE IN ⌓AEB,

[III. 31] FOR,

THE LATTER TOO IS ITSELF A RIGHT ANGLE,
BEING AN ANGLE IN A SEMICIRCLE.

BUT,

∠BAD = ∠AT C. THEREFORE,

∠AEB = ∠AT C.

THEREFORE AGAIN,

⌓AEB, HAS BEEN DESCRIBED,

ON AB, ADMITTING AN ANGLE EQUAL, TO ∠AT C.

NEXT, LET,

∠AT C, BE OBTUSE;

AND,

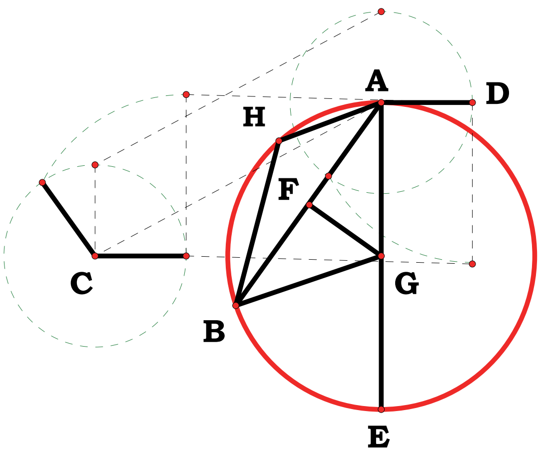
ON AB, AND AT A,

LET,

∠BAD, BE

CONSTRUCTED EQUAL, TO IT,

AS IS THE CASE IN THE THIRD FIGURE;



LET,

$AE \perp AD$,

LET,

AB BE AGAIN BISECTED, AT F,

LET,

$FG \perp AB$,

AND LET,

GB BE JOINED.

THEN, SINCE,

$AF = FB$, AND FG IS COMMON,

THE TWO SIDES, AF, FG, ARE EQUAL, TO

THE TWO SIDES, BF, FG; AND $\angle AFG = \angle BFG$;

[I. 4] THEREFORE,

THE BASES, $AG = BG$.

THEREFORE,

THE CIRCLE DESCRIBED WITH CENTRE, G, AND
DISTANCE, GA, WILL PASS THROUGH B, ALSO;

LET IT,

SO PASS, AS AEB.

[III. 16, POR.] NOW, SINCE,

$AD \perp AE$, FROM ITS EXTREMITY,

AD TOUCHES $\odot AEB$.

AND,

AB HAS BEEN DRAWN ACROSS FROM
THE POINT OF CONTACT, AT A;

[III. 32] THEREFORE,

$\angle BAD$ = THE ANGLE CONSTRUCTED IN

THE ALTERNATE $\frown AHB$.

BUT,

$\angle BAD = \angle$AT C.

THEREFORE,

$\angle$IN $\frown AHB = \angle$AT C.

THEREFORE,

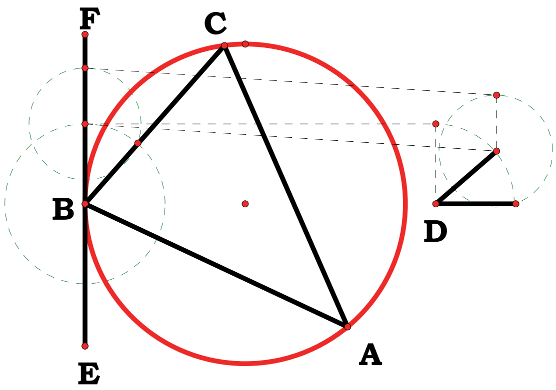
ON AB, $\frown AHB$, HAS BEEN DESCRIBED

ADMITTING AN ANGLE EQUAL, TO $\angle$AT C.

Q. E. F.

PROPOSITION 34.

FROM A GIVEN CIRCLE TO CUT OFF A SEGMENT ADMITTING AN ANGLE EQUAL, TO A GIVEN RECTILINEAL ANGLE.



LET,

⊙ABC BE GIVEN,

AND,

∠AT D, THE GIVEN ANGLE;

THUS IT IS REQUIRED,

TO DIVIDE ⊙ABC,

A SEGMENT, ADMITTING AN ANGLE, EQUAL, TO

THE GIVEN RECTILINEAL ANGLE, ∠AT D.

LET,

EF BE DRAWN TOUCHING ABC AT B, AND
ON FB, AND AT B,

[I. 23] LET,

∠FBC = ∠AT D.

[III. 32] THEN, SINCE,

EF, TOUCHES ⊙ABC, AND,

BC HAS BEEN DRAWN ACROSS FROM
THE POINT OF CONTACT, AT B,

∠FBC = ∠ CONSTRUCTED IN THE ALTERNATE ⌓BAC.

BUT,

∠FBC = ∠AT D;

THEREFORE,

∠IN ⌓BAC = ∠AT D.

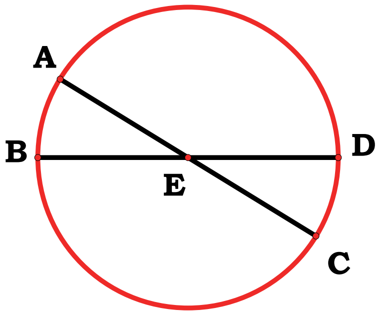
THEREFORE,

FROM ⊙ ABC, ⌓BAC, HAS BEEN DIVIDED

ADMITTING AN ANGLE EQUAL, TO THE GIVEN ANGLE, ∠AT D.

Q. E. F.

Proposition 35.



If in a circle two straight lines cut one another, the rectangle contained by the segments of the one is equal, to the rectangle contained by the segments of the other.

For let,

in ⊙$ABCD$,

AC, BD, intersect one another at E;

I say that;

$AE \boxtimes EC = DE \boxtimes EB$.

If now,

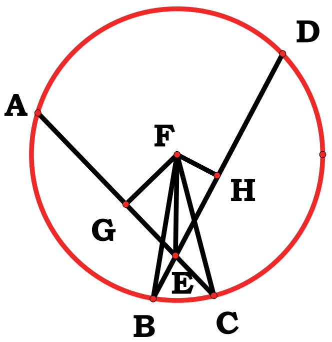
AC, BD intersect, so that,

E is the centre of ⊙$ABCD$,

it is manifest that;

AE, EC, DE, EB, being equal,

$AE \boxtimes C = DE \boxtimes EB$.



Next let,

AC, DB intersect the centre;

let,

the centre of $ABCD$ be taken,

and let,

it be F;

let from,

F,

$FG \perp AC$, $FH \perp DB$,

and let,

FB, FC, FE be joined.

[III. 3] Then, since,

GF, intersects AC, not through

the centre at right angles, it, also, bisects it;

therefore,

$AG = GC$.

[II. 5] Since, then,

AC, has been bisected at G, and intersected at E,

$AE \boxtimes EC + \boxdot EG = \boxdot GC$;

Let,

$\boxdot GF$ be added;

therefore,

$AE \boxtimes EC + \boxdot GE + \boxdot GF = \boxdot CG + \boxdot GF$.

[I. 47] But,

$\boxdot FE = \boxdot EG + \boxdot GF$, and

$\boxdot FC = \boxdot CG + GF$;

therefore,

$AE \boxtimes EC + \boxdot FE = \boxdot FC$.

And $FC = FB$;

therefore,

$AE \boxtimes EC + \boxdot EF = \boxdot FB$.

For the same reason, also,

$DE \boxtimes EB + \boxdot FE = \boxdot FB$.

But,

$AE \boxtimes EC + \boxdot FE = \boxdot FB$;

therefore,

$AE \boxtimes EC + \boxdot FE = DE \boxtimes EB + \boxdot FE$.

Let,

$\boxdot FE$, be subtracted from each;

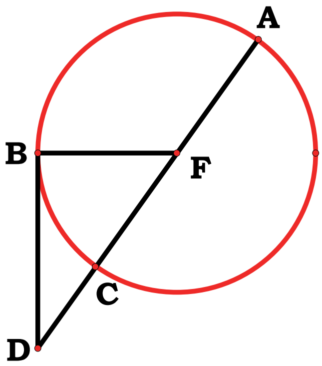
therefore, which remains

$AE \boxtimes EC = DE \boxtimes EB$.

Therefore etc.

Proposition 36.

If a point be taken outside a circle and from it there fall on the circle two straight lines, and if one of them cut the circle and the other touch it, the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference will be equal, to the square, on the tangent.



For let,

D, be taken outside $\odot ABC$,

and let,

from D, DCA, DB, fall on $\odot ABC$; let,

DCA cut $\odot ABC$, and let,

BD touch it;

I say that;

$AD \boxtimes DC = DB$.

Then,

DCA is

either,

through the centre, or

not through the centre.

First let,

it be through the centre,

and let,

F be the centre of $\odot ABC$;

let,

FB be joined;

[III. 18] therefore,

$\llcorner FBD$, is right.

[II. 6] And, since,

AC has been bisected, at F, and

CD is added to it,

$AD \boxtimes DC + \boxdot FC = \boxdot FD$. But,

$FC = FB$;

THEREFORE,

$AD \boxtimes DC + \boxdot FB = \boxdot FD$.

[I. 47] AND,

$\boxdot FB + \boxdot BD = \boxdot FD$;

THEREFORE,

$AD \boxtimes DC + \boxdot FB = \boxdot FB + \boxdot BD$.

LET,

$\boxdot FB$, BE SUBTRACTED FROM EACH;

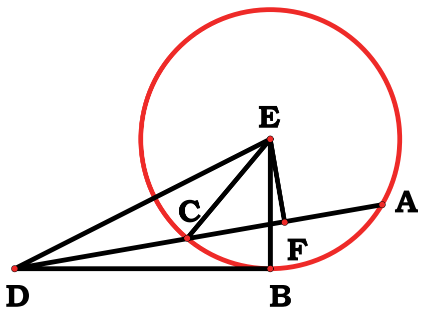
THEREFORE,

$AD \boxtimes DC = \boxdot DB$.

AGAIN, LET,

DCA NOT BE THROUGH

THE CENTRE OF $\odot ABC$;



LET,

THE CENTRE, E ,BE TAKEN,

AND LET,

FROM E, $EF \perp AC$;

LET,

EB, EC, ED, BE JOINED.

[III. 18] THEN,

$\llcorner EBD$, IS RIGHT.

[III. 3] AND, SINCE,

EF, THROUGH THE CENTRE, INTERSECTS AC,
NOT THROUGH THE CENTRE, AT RIGHT ANGLES,
IT, ALSO, BISECTS IT;

THEREFORE,

$AF = FC$.

NOW, SINCE,

AC, HAS BEEN BISECTED AT F,

[II. 6] AND,

CD IS ADDED TO IT.

$AD \boxtimes DC + \boxdot FC = \boxdot FD$.

LET,

$\boxdot FE$, BE ADDED TO EACH;

THEREFORE,

$$AD \boxtimes DC + \boxdot CF + \boxdot FE = \boxdot FD + \boxdot FE.$$

[I. 47] BUT,

$$\boxdot EC = \boxdot CF + \boxdot FE,$$

FOR,

$\llcorner EFC$, IS RIGHT; AND

$$\boxdot ED = \boxdot DF + \boxdot FE;$$

THEREFORE,

$$AD \boxtimes DC + \boxdot EC = \boxdot ED.$$ AND,

$$EC = EB;$$

THEREFORE,

$$AD \boxtimes DC + \boxdot EB = \boxdot ED.$$

[I. 47] BUT,

$\boxdot EB + \boxdot BD = \boxdot ED$, FOR,

$\llcorner EBD$, IS RIGHT;

THEREFORE,

$$AD \boxtimes DC + \boxdot EB = \boxdot EB + \boxdot BD.$$

LET,

$\boxdot EB$, BE SUBTRACTED FROM EACH;

THEREFORE, WHICH REMAINS

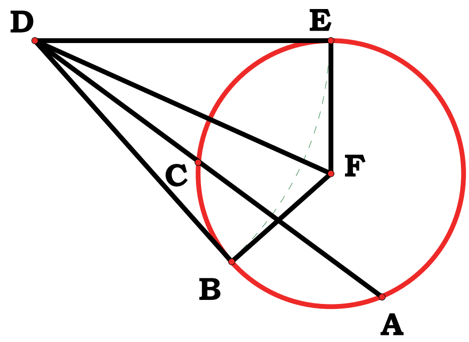
$$AD \boxtimes DC = \boxdot DB.$$

THEREFORE ETC.

Q. E. D.

PROPOSITION 37.

IF A POINT BE TAKEN OUTSIDE A CIRCLE AND FROM THE POINT THERE FALL ON THE CIRCLE TWO STRAIGHT LINES, IF ONE OF THEM CUT THE CIRCLE, AND THE OTHER FALL ON IT, AND IF FURTHER THE RECTANGLE CONTAINED BY THE WHOLE OF THE STRAIGHT LINE WHICH CUTS THE CIRCLE AND THE STRAIGHT LINE INTERCEPTED ON IT OUTSIDE BETWEEN THE POINT AND THE CONVEX CIRCUMFERENCE BE EQUAL, TO THE SQUARE, ON THE STRAIGHT LINE WHICH FALLS ON THE CIRCLE, THE STRAIGHT LINE WHICH FALLS ON IT WILL TOUCH THE CIRCLE.



FOR LET,

D, BE TAKEN OUTSIDE ⊙*ABC*;

LET,

FROM *D*, *DCA*, *DB*, FALL ON ⊙*ACB*;

LET,

DCA INTERSECT ⊙*ACB* AND *DB* FALL ON IT;

AND LET,

AD ⊠ *DC* = ⊡*DB*.

I SAY THAT;

DB TOUCHES ⊙*ABC*.

[III. 18] FOR LET,

DE BE DRAWN TOUCHING ⊙*ABC*;

LET,

THE CENTRE OF ⊙*ABC*, BE TAKEN,

AND LET,

IT BE *F*;

LET,

FE, *FB*, *FD*, BE JOINED.

THUS,

∟*FED*, IS RIGHT.

[III. 36] NOW, SINCE,

DE TOUCHES ⊙*ABC*, AND

DCA CUTS IT,

$AD \boxtimes DC = \boxdot DE$.

BUT,

$AD \boxtimes DC = \boxdot DB$;

THEREFORE,

$\boxdot DE = \boxdot DB$;

THEREFORE,

$DE = DB$. AND,
$FE = FB$;

THEREFORE,

THE TWO SIDES, DE, EF, ARE EQUAL, TO
THE TWO SIDES, DB, BF; AND
FD IS THE COMMON BASE OF THE TRIANGLES;

[I. 8] THEREFORE,

$\llcorner DEF = \llcorner DBF$. BUT,

$\llcorner DEF$, IS RIGHT;

THEREFORE,

$\llcorner DBF$, IS, ALSO, RIGHT. AND
FB PRODUCED IS A DIAMETER;

[III. 16, POR.] AND,

THE STRAIGHT LINE DRAWN AT RIGHT ANGLES TO
THE DIAMETER OF A CIRCLE, FROM ITS EXTREMITY,
TOUCHES THE CIRCLE;

THEREFORE,

DB TOUCHES THE CIRCLE.

SIMILARLY THIS CAN BE PROVED,
TO BE THE CASE EVEN IF THE CENTRE BE ON AC.

THEREFORE ETC.

Q. E. D.

BOOK IV.

OF

EUCLID'S ELEMENTS

TRANSLATED FROM THE TEXT OF HEIBERG

BY

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2013 *EDITION*

REVISED WITH SUBTRACTIONS

REFORMATTED AND ABRIDGED
FOR STUDY OF
***THE ELEMENTS*.**

BY JOHN CLARK.

BOOK IV.

Definitions.

1. A RECTILINEAL FIGURE IS SAID TO BE **INSCRIBED IN A RECTILINEAL FIGURE** WHEN THE RESPECTIVE ANGLES OF THE INSCRIBED FIGURE LIE ON THE RESPECTIVE SIDES OF THAT IN WHICH IT IS INSCRIBED.

2. SIMILARLY, A FIGURE IS SAID TO BE **CIRCUMSCRIBED ABOUT A FIGURE** WHEN THE RESPECTIVE SIDES OF THE CIRCUMSCRIBED FIGURE PASS THROUGH THE RESPECTIVE ANGLES OF THAT ABOUT WHICH IT IS CIRCUMSCRIBED.

3. A RECTILINEAL FIGURE IS SAID TO BE **INSCRIBED IN A CIRCLE** WHEN EACH ANGLE OF THE INSCRIBED FIGURE LIES ON THE CIRCUMFERENCE OF THE CIRCLE.

4. A RECTILINEAL FIGURE IS SAID TO BE **CIRCUMSCRIBED ABOUT A CIRCLE**, WHEN EACH SIDE OF THE CIRCUMSCRIBED FIGURE TOUCHES THE CIRCUMFERENCE OF THE CIRCLE.

5. SIMILARLY A CIRCLE IS SAID TO BE **INSCRIBED IN A FIGURE** WHEN THE CIRCUMFERENCE OF THE CIRCLE TOUCHES EACH SIDE OF THE FIGURE IN WHICH IT IS INSCRIBED.

6. A CIRCLE IS SAID TO BE **CIRCUMSCRIBED ABOUT A FIGURE** WHEN THE CIRCUMFERENCE OF THE CIRCLE PASSES THROUGH EACH ANGLE OF THE FIGURE ABOUT WHICH IT IS CIRCUMSCRIBED.

7. A STRAIGHT LINE IS SAID TO BE **FITTED INTO A CIRCLE** WHEN ITS EXTREMITIES ARE ON THE CIRCUMFERENCE OF THE CIRCLE.

NOTES.

Definition 1. *A rectilineal figure is said to be* INSCRIBED IN A RECTILINEAL FIGURE *when the respective angles of the inscribed figure lie on the respective sides of that in which it is inscribed*.

NOTES.

Definition 2. *SIMILARLY, A FIGURE IS SAID TO BE* CIRCUMSCRIBED ABOUT A FIGURE *WHEN THE RESPECTIVE SIDES OF THE CIRCUMSCRIBED FIGURE PASS THROUGH THE RESPECTIVE ANGLES OF THAT ABOUT WHICH IT IS CIRCUMSCRIBED*.

NOTES.

Definition 3. *A rectilineal figure is said to be* INSCRIBED IN A CIRCLE *when each angle of the inscribed figure lies on the circumference of the circle*.

Notes.

Definition 4. *A rectilineal figure is said to be* circumscribed about a circle, *when each side of the circumscribed figure touches the circumference of the circle*.

NOTES.

Definition 5. *SIMILARLY A CIRCLE IS SAID TO BE* INSCRIBED IN A FIGURE *WHEN THE CIRCUMFERENCE OF THE CIRCLE TOUCHES EACH SIDE OF THE FIGURE IN WHICH IT IS INSCRIBED*.

NOTES.

Definition 6. *A circle is said to be* CIRCUMSCRIBED ABOUT A FIGURE *when the circumference of the circle passes through each angle of the figure about which it is circumscribed*.

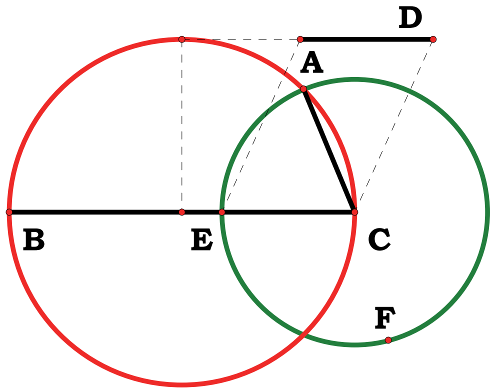
Notes.

Definition 7. *A straight line is said to be* fitted into a circle *when its extremities are on the circumference of the circle.*

BOOK IV.

PROPOSITIONS

Proposition 1.



Into a given circle to fit a straight line equal, to a given straight line which is not greater than the diameter of the circle.

Let,

$\odot ABC$ be given,

and

D, $\ngtr$ *the diameter* $\odot ABC$;

thus it is required,

to fit into $\odot ABC$, a line equal, to D.

Let,

a diameter, BC, of $\odot ABC$, be drawn.

Then, if,

$BC = D$,

that which was enjoined will have been done;

for,

BC has been fitted into $\odot ABC$, equal, to D.

But, if,

$BC > D$, let,

$CE = D$, and

with centre, C, and distance, CE; let,

$\odot EAF$, be described; let,

CA, be joined.

Then, since,

C, is the centre of $\odot EAF$,

$CA = CE$. But,

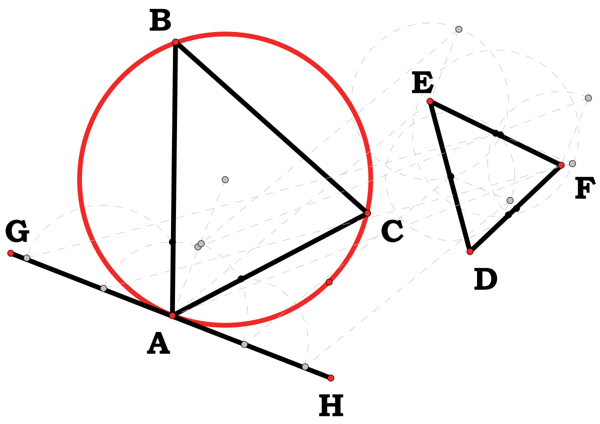
$CE = D$; therefore,

$D = CA$.

Therefore,

$\odot ABC$, there has been fitted $CA = D$.

PROPOSITION 2.



IN A GIVEN CIRCLE TO INSCRIBE A TRIANGLE EQUIANGULAR WITH A GIVEN TRIANGLE.

LET,

$\odot ABC$ AND ΔDEF,

BE GIVEN,

THUS IT IS REQUIRED,

TO INSCRIBE IN $\odot ABC$,

A TRIANGLE EQUIANGULAR WITH ΔDEF.

[III. 16, POR.] LET,

GH BE DRAWN, TOUCHING $\odot ABC$, AT A; ON AH,

AND LET,

AT A, ON IT,

$\angle HAC = \angle DEF$,

[I. 23] AND LET,

AG, AND AT A,

$\angle GAB = \angle DFE$;

LET,

BC BE JOINED.

THEN, SINCE,

AH, TOUCHES $\odot ABC$,

AND FROM,

THE POINT OF CONTACT, AT A,

AC, IS DRAWN ACROSS IN THE CIRCLE,

[III. 32] THEREFORE,

$\angle HAC = \angle ABC$,

IN THE ALTERNATE SEGMENT OF THE CIRCLE.

BUT,

$\angle HAC = \angle DEF$;

THEREFORE,

$\angle ABC = \angle DEF$.

FOR THE SAME REASON,

$\angle ACB = \angle DFE$;

[I. 32] THEREFORE, REMAINING

$\angle BAC = \angle EDF$.

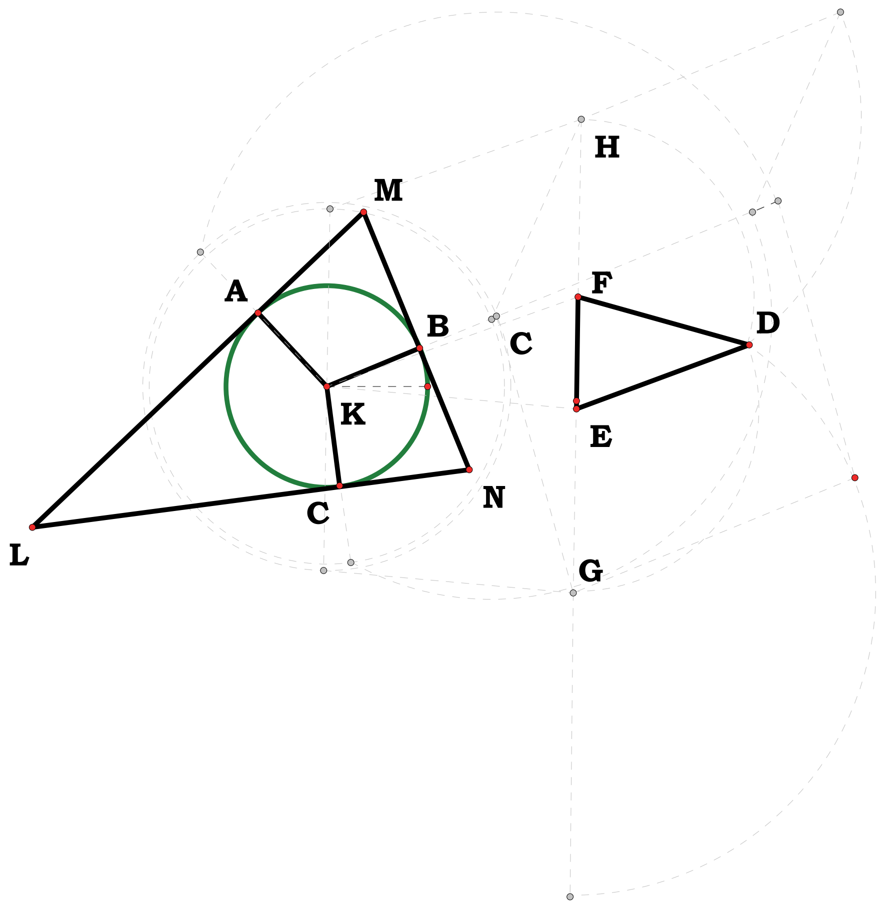
THEREFORE,

IN THE GIVEN CIRCLE,
THERE HAS BEEN INSCRIBED A TRIANGLE EQUIANGULAR WITH
THE GIVEN TRIANGLE.

Q. E. F.

PROPOSITION 3.

ABOUT A GIVEN CIRCLE TO CIRCUMSCRIBE A TRIANGLE EQUIANGULAR WITH A GIVEN TRIANGLE.



LET,

⊙ABC AND ΔDEF BE GIVEN,

THUS IT IS REQUIRED,

TO CIRCUMSCRIBE ABOUT

⊙ABC, A TRIANGLE EQUIANGULAR WITH ΔDEF.

LET,

EF BE PRODUCED, IN BOTH DIRECTIONS, TO THE POINTS, G, H,

[III. 1] LET,

THE CENTRE, K, OF ⊙ABC, BE TAKEN

AND LET,

KB, BE DRAWN ACROSS AT RANDOM;

ON KB, AND AT K,

[I. 23] LET,

$\angle BKA = \angle DEG$, AND

$\angle BKC = \angle DFH$;

[III. 16, Por.] and let,
through A, B, C,
LAM, MBN, NCL, be drawn touching $\odot ABC$.

[III. 18] Now, since,
LM, MN, NL, touch $\odot ABC$, at A, B, C, and
KA, KB, KC, have been joined from
the centre, K to A, B, C,

therefore,
the angles, at A, B, C, are right.

And, since,
the four angles of
the quadrilateral, $AMBK$, are equal, to
four right angles,

inasmuch as,
$AMBK$ is in fact divisible into two triangles, and
$\llcorner KAM$, $\llcorner KBM$, are right,

therefore,
the remainings, $\llcorner AKB$, $\llcorner AMB$, are equal, to
two right angles.

[I. 13] But,
$\angle DEG$, $\angle DEF$, are, also, equal, to
two right angles;

therefore,
$\angle AKB = \angle DEG$, and
$\angle AMB = \angle DEF$,

of which,
$\angle AKB = \angle DEG$;

therefore, which remains
$\angle AMB, = \angle DEF$.

Similarly it can be proved that,
$\angle LNB = \angle DFE$;

[I. 32] therefore,
the remaining angle, $MLN = \angle EDF$.

Therefore,

ΔLMN, IS EQUIANGULAR WITH ΔDEF;

AND,

IT HAS BEEN CIRCUMSCRIBED ABOUT $\odot ABC$.

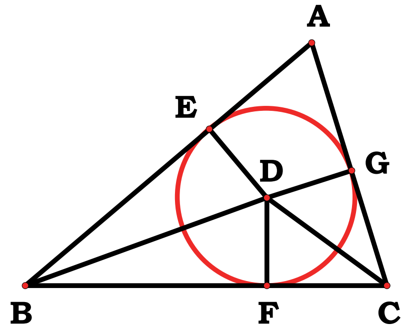
THEREFORE,

ABOUT A GIVEN CIRCLE,
THERE HAS BEEN CIRCUMSCRIBED A TRIANGLE,
EQUIANGULAR WITH THE GIVEN TRIANGLE.

Q. E. F.

PROPOSITION 4.

IN A GIVEN TRIANGLE TO INSCRIBE A CIRCLE.



LET,

ΔABC BE GIVEN;

THUS IT IS REQUIRED,

TO INSCRIBE A CIRCLE IN ΔABC.

[I. 9]

LET,

$\angle ABC$, $\angle ACB$, BE BISECTED BY BD, CD,

AND LET,

THESE MEET ONE ANOTHER AT THE POINT, D;

LET,

FROM D,

DE, DF, DG, BE DRAWN PERPENDICULAR TO THE STRAIGHT LINES, AB, BC, CA.

NOW, SINCE,

$\angle ABD = \angle CBD$,

AND,

$\llcorner BED = \llcorner BFD$, ΔEBD, ΔFBD, HAVE TWO ANGLES EQUAL, TO TWO ANGLES, AND ONE SIDE EQUAL, TO ONE SIDE,

NAMELY,

THAT SUBTENDING ONE OF THE EQUAL ANGLES, WHICH IS BD, COMMON TO THE TRIANGLES;

[I. 26] THEREFORE,

THEY WILL, ALSO, HAVE THE REMAINING SIDES EQUAL, TO THE REMAINING SIDES;

THEREFORE,

$DE = DF$.

FOR THE SAME REASON,

$DG = DF$.

THEREFORE,

DE, DF, DG, ARE EQUAL, TO ONE ANOTHER;

THEREFORE,

THE CIRCLE DESCRIBED WITH CENTRE, D, AND DISTANCE ONE DE, DF, DG,

WILL PASS, ALSO, THROUGH THE REMAINING POINTS, AND
WILL TOUCH AB, BC, CA,

BECAUSE,
THE ANGLES, AT E, F, G, ARE RIGHT.

[III. 16] FOR, IF,
IT CUTS THEM,
THE STRAIGHT LINE DRAWN AT RIGHT ANGLES TO
THE DIAMETER OF THE CIRCLE FROM ITS EXTREMITY
WILL BE FOUND TO FALL WITHIN THE CIRCLE:
WHICH WAS PROVED ABSURD;

THEREFORE,
THE CIRCLE DESCRIBED WITH CENTRE, D, AND
DISTANCE ONE OF DE, DF, DG, WILL NOT CUT
AB, BC, OR CA;

[IV. DEF. 5] THEREFORE,
IT WILL TOUCH THEM, AND
WILL BE THE CIRCLE INSCRIBED IN ΔABC,

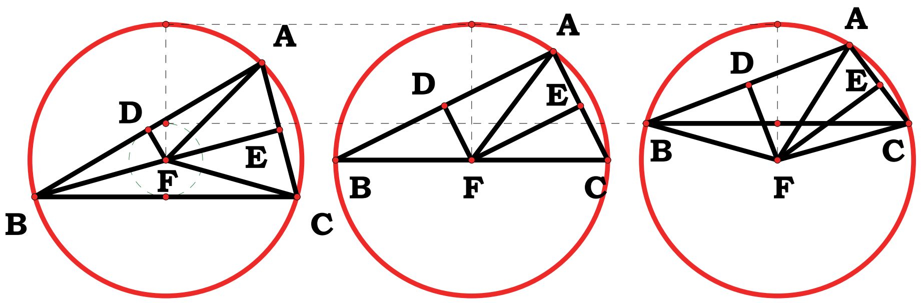
LET,
IT BE INSCRIBED, AS $\odot FGE$.

THEREFORE,
IN ΔABC, $\odot EFG$, HAS BEEN INSCRIBED.

Q. E. F.

Proposition 5.

About a given triangle to circumscribe a circle.



Let,

ΔABC, be given;

thus it is required,

to circumscribe a circle about ΔABC.

[I. 10]

Let,

AB, AC, be bisected at D, E,

and let,

from D, E, $DF \perp AB$, $EF \perp AC$;

they will then meet within ΔABC,

or,

on the straight line, BC, or
outside, BC.

First let,

them meet within, at F,

and let,

FB, FC, FA, be joined.

[I. 4] Then, since,

$AD = DB$, and DF is common and
at right angles,

therefore,

the bases, $AF = FB$.

Similarly, we can prove that;

$CF = AF$; so that,
$FB = FC$;

therefore,

FA, FB, FC, are equal, to one another.

Therefore,

THE CIRCLE DESCRIBED WITH CENTRE, F, AND
DISTANCE ONE OF FA, FB, FC,
WILL PASS, ALSO, THROUGH THE REMAINING POINTS, AND
THE CIRCLE WILL HAVE BEEN CIRCUMSCRIBED ABOUT
ΔABC.

LET IT,
BE CIRCUMSCRIBED, AS $\odot ABC$.

NEXT, LET,
DF, EF MEET ON BC, AT F,
AS IS THE CASE IN THE SECOND FIGURE;

AND LET,
AF BE JOINED.

THEN, SIMILARLY, WE SHALL PROVE THAT;
THE POINT, F, IS THE CENTRE OF THE CIRCLE
CIRCUMSCRIBED ABOUT ΔABC.

AGAIN, LET,
DF, BF MEET OUTSIDE ΔABC, AT F,
AS IS THE CASE IN THE THIRD FIGURE,

AND LET,
AF, BF, CF, BE JOINED.

[I. 4] THEN AGAIN, SINCE,
$AD = DB$, AND DF IS COMMON, AND
AT RIGHT ANGLES,

THEREFORE,
THE BASES, $AF = BF$.

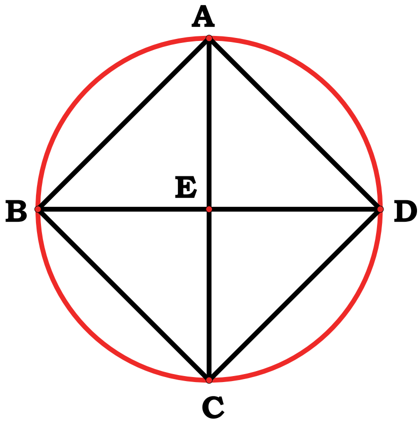
SIMILARLY WE CAN PROVE THAT,
$CF = AF$; SO THAT,
$BF = FC$;

THEREFORE,
THE CIRCLE DESCRIBED WITH CENTRE, F, AND
DISTANCE ONE FA, FB, FC,
WILL PASS, ALSO, THROUGH THE REMAINING POINTS, AND
WILL HAVE BEEN CIRCUMSCRIBED ABOUT
ΔABC.

THEREFORE,
ABOUT THE GIVEN TRIANGLE,
A CIRCLE HAS BEEN CIRCUMSCRIBED.

Q. E. F.

Proposition 6.



In a given circle to inscribe a square.

Let,

$\odot ABCD$, be given;

thus it is required,

to inscribe a square in

$\odot ABCD$.

Let,

two diameters, $AC \perp BD$, of $\odot ABCD$,

and let,

AB, BC, CD, DA, be joined.

Then, since,

$BE = ED$, for,

E is the centre, and

EA is common and at right angles,

[I. 4] therefore,

the bases, $AB = AD$.

For the same reason,

$BC = AB$, $CD = AD$;

therefore,

the quadrilateral, $ABCD$, is equilateral.

I say next that;

it is, also, right-angled.

For, since,

BD, is a diameter of $\odot ABCD$,

therefore,

BAD is a semicircle;

[III. 31] therefore,

$\llcorner BAD$, is right.

For the same reason,

each, of $\llcorner ABC$, $\llcorner BCD$, $\llcorner CDA$, is, also, right;

therefore,

the quadrilateral, $ABCD$, is right-angled.

[I. Def. 22] But,

IT WAS, ALSO, PROVED EQUILATERAL;

THEREFORE,

IT IS A SQUARE; AND,

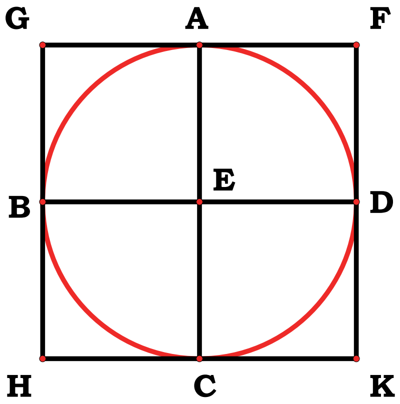
IT HAS BEEN INSCRIBED IN ⊙*ABCD*.

THEREFORE,

IN THE GIVEN CIRCLE, ⊡*ABCD*, HAS BEEN INSCRIBED.

Q. E. F.

PROPOSITION 7.



ABOUT A GIVEN CIRCLE TO CIRCUMSCRIBE A SQUARE.

LET,

$\odot ABCD$, BE GIVEN;

THUS IT IS REQUIRED,

TO CIRCUMSCRIBE

A SQUARE, ABOUT $\odot ABCD$.

LET,

DIAMETERS, $AC \perp BD$, OF $\odot ABCD$,

[III. 16, POR.] AND LET,

THROUGH A, B, C, D,

FG, GH, HK, KF, BE DRAWN TOUCHING $\odot ABCD$.

[III. 18] THEN, SINCE,

FG TOUCHES $\odot ABCD$, AND

EA HAS BEEN JOINED FROM

THE CENTRE, E, TO THE POINT OF CONTACT, AT A,

THEREFORE,

$\llcorner$AT A, ARE RIGHT.

FOR THE SAME REASON,

$\llcorner$AT B, C, D, ARE, ALSO, RIGHT.

[I. 28] NOW, SINCE,

$\llcorner AEB$, IS RIGHT, AND $\llcorner EBG$, IS, ALSO, RIGHT, THEREFORE,

$GH \parallel AC$.

[I. 30] FOR THE SAME REASON,

$AC \parallel FK$, SO THAT,

$GH \parallel FK$.

SIMILARLY WE CAN PROVE THAT,

EACH, OF GF, $HK \parallel BED$.

THEREFORE,

$\boxminus GK$, $\boxminus GC$, $\boxminus AK$, $\boxminus FB$, $\boxminus BK$;

[I. 34] THEREFORE,

$GF = HK$, AND $GH = FK$.

AND, SINCE,

$AC = BD$, AND

$AC = GH$, $AC = FK$,

[I. 34] WHILE,

$BD = GF$, $BD = HK$,

THEREFORE,

THE QUADRILATERAL, $FGHK$, IS EQUILATERAL.

I SAY NEXT THAT;

IT IS, ALSO, RIGHT-ANGLED.

[I. 34]

FOR, SINCE,

⊟$GBEA$, AND ∟AEB, IS RIGHT,

THEREFORE,

∟AGB, IS, ALSO, RIGHT.

SIMILARLY WE CAN PROVE THAT,

∠AT H, K, F, ARE, ALSO, RIGHT.

THEREFORE,

⊟$FGHK$ IS RIGHT-ANGLED.

BUT,

IT WAS, ALSO, PROVED EQUILATERAL;

THEREFORE,

IT IS A SQUARE; AND

IT HAS BEEN CIRCUMSCRIBED ABOUT ⊙$ABCD$.

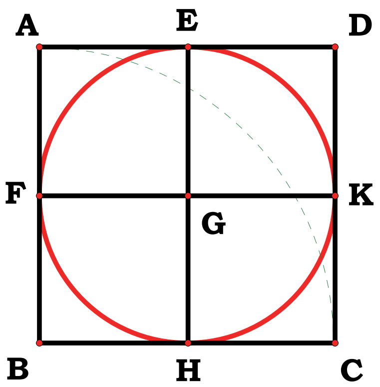
THEREFORE,

ABOUT THE GIVEN CIRCLE,

A SQUARE HAS BEEN CIRCUMSCRIBED.

Q. E. F.

Proposition 8.



In a given square to inscribe a circle.

Let,

⊡*ABCD* be given;

thus it is required,

to inscribe a circle in ⊡*ABCD*.

[I. 10] Let,

AD, *AB*, be bisected at *E*, *F*, respectively,

let,

through *E*,

EH ∥ to either, *AB* or *CD*,

[I. 31] and let,

through *F*,

FK ∥ to either, *AD* or *BC*;

[I. 34]

therefore,

each, of the figures,

⊟*AK*, ⊟*KB*, ⊟*AH*, ⊟*HD*, ⊟*AG*, ⊟*GC*, ⊟*BG*, ⊟*GD*,,

and,

their opposite sides are evidently equal.

Now, since,

AD = *AB*, and

2*AE* = *AD*, and

2*AF* = *AB*,

therefore,

AE = *AF*,

so that,

the opposite sides are, also, equal;

therefore,

FG = *GE*.

Similarly we can prove that,

each, of *GH*, *GK* = each, of , *FG*, *GE*;

therefore,

GE, *GF*, *GH*, *GK*, are equal, to one another.

THEREFORE,

THE CIRCLE DESCRIBED WITH CENTRE, *G*, AND
DISTANCE ONE OF *GE*, GF, *GH*, *GK*,
WILL PASS, ALSO, THROUGH THE REMAINING POINTS. AND
IT WILL TOUCH *AB*, *BC*, *CD*, *DA*,

BECAUSE,

THE ∠AT *E*, *F*, *H*, *K*, ARE RIGHT.

[III. 16] FOR,

IF THE CIRCLE CUTS, *AB*, *BC*, *CD*, *DA*,
THE STRAIGHT LINE DRAWN AT RIGHT ANGLES TO
THE DIAMETER OF THE CIRCLE FROM
ITS EXTREMITY, WILL FALL WITHIN THE CIRCLE:

WHICH,

WAS PROVED ABSURD;

THEREFORE,

THE CIRCLE DESCRIBED WITH CENTRE, *G*, AND
DISTANCE ONE OF *GE*, *GF*, *GH*, *GK*,
WILL NOT CUT *AB*, *BC*, *CD*, *DA*.

THEREFORE,

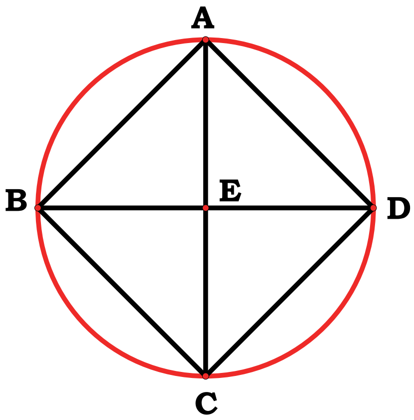
IT WILL TOUCH THEM, AND

WILL HAVE BEEN INSCRIBED IN ⊡*ABCD*.

THEREFORE,

IN THE GIVEN SQUARE, A CIRCLE HAS BEEN INSCRIBED.

PROPOSITION 9.



ABOUT A GIVEN SQUARE TO CIRCUMSCRIBE A CIRCLE.

LET,

⊡*ABCD*, BE GIVEN;

THUS IT IS REQUIRED,

TO CIRCUMSCRIBE A CIRCLE

ABOUT ⊡*ABCD*.

FOR LET,

AC, *BD*, BE JOINED,

AND LET,

THEM INTERSECT ONE ANOTHER, AT *E*.

THEN, SINCE,

$DA = AB$, AND *AC* IS COMMON,

THEREFORE,

THE TWO SIDES, *DA*, *AC*, ARE EQUAL, TO

THE TWO SIDES, *BA*, *AC*; AND

THE BASES, $DC = BC$;

[I. 8] THEREFORE,

$\angle DAC = \angle BAC$.

THEREFORE,

$\angle DAB$, IS BISECTED BY *AC*.

SIMILARLY WE CAN PROVE THAT,

EACH, OF $\angle ABC$, $\angle BCD$, $\angle CDA$,

IS BISECTED BY *AC*, *DB*.

[I. 6] NOW, SINCE,

$\angle DAB = \angle ABC$, AND

$2\angle EAB = \angle DAB$, AND

$2\angle EBA = \angle ABC$,

THEREFORE,

$\angle EAB = \angle EBA$;

SO THAT,

THE SIDES, $EA = EB$.

SIMILARLY WE CAN PROVE THAT,

EACH, OF EA, EB = EACH, OF EC, ED.

THEREFORE,

EA, EB, EC, ED, ARE EQUAL, TO ONE ANOTHER.

THEREFORE,

THE CIRCLE DESCRIBED WITH CENTRE, E, AND
DISTANCE ONE OF EA, EB, EC, ED,
WILL PASS, ALSO, THROUGH THE REMAINING POINTS; AND
IT WILL HAVE BEEN CIRCUMSCRIBED ABOUT $\boxdot ABCD$.

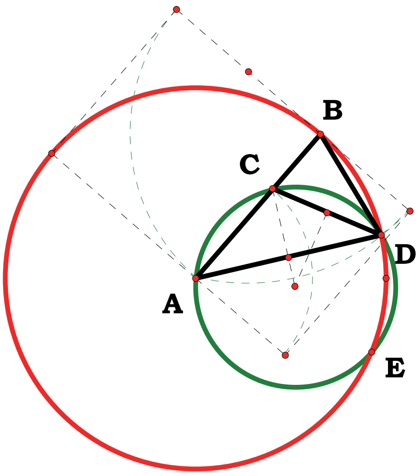
LET IT,

BE CIRCUMSCRIBED, $\odot ABCD$.

THEREFORE,

ABOUT THE GIVEN SQUARE,
A CIRCLE HAS BEEN CIRCUMSCRIBED.

PROPOSITION 10.



TO CONSTRUCT AN ISOSCELES TRIANGLE HAVING EACH, OF THE ANGLES AT THE BASE DOUBLE OF THE REMAINING ONE.

[II. 11] LET,

AB, BE SET OUT,

AND LET,

IT BE DIVIDED AT C,

SO THAT,

$AB \boxtimes BC = \boxdot CA$;

LET,

WITH CENTRE A, AND DISTANCE, AB, $\odot BDE$, BE DESCRIBED,

[IV. 1] AND LET,

THERE BE FITTED IN $\odot BDE$, $BD = AC$,

WHICH IS NOT GREATER THAN THE DIAMETER OF $\odot BDE$.

LET,

AD, DC BE JOINED,

[IV. 5] AND LET,

$\odot ACD$, BE CIRCUMSCRIBED ABOUT ΔACD.

THEN, SINCE,

$AB \boxtimes BC = \boxdot AC$, AND

$AC = BD$,

THEREFORE,

$AB \boxtimes BC = \boxdot BD$.

AND, SINCE,

B, HAS BEEN TAKEN OUTSIDE $\odot ACD$, AND FROM B,

BA, BD, HAVE FALLEN ON $\odot ACD$, AND

ONE OF THEM INTERSECTS IT,

WHILE,

THE OTHER TOUCHES IT, AND

$AB \boxtimes BC = \boxdot BD$,

[III. 37] THEREFORE,

BD TOUCHES $\odot ACD$.

SINCE, THEN,

BD TOUCHES IT, AND

DC IS DRAWN ACROSS FROM THE POINT OF CONTACT, AT D,

[III. 32] THEREFORE,

$\angle BDC = \angle DAC$,

IN THE ALTERNATE SEGMENT OF THE CIRCLE.

SINCE, THEN,

$\angle BDC = \angle DAC$, LET,

$\angle CDA$, BE ADDED TO EACH;

THEREFORE,

$\angle BDA = \angle CDA + \angle DAC$.

[I. 32] BUT,

THE EXTERIOR ANGLE, $BCD = \angle CDA + \angle DAC$;

THEREFORE,

$\angle BDA = \angle BCD$.

[I. 5] BUT,

$\angle BDA = \angle CBD$, SINCE,

THE SIDES, $AD = AB$; SO THAT,

$\angle DBA = \angle BCD$.

THEREFORE,

$\angle BDA$, $\angle DBA$, $\angle BCD$, ARE EQUAL, TO ONE ANOTHER.

[I. 6] AND, SINCE,

$\angle DBC = \angle BCD$,

THE SIDES, $BD = DC$.

BUT, BY HYPOTHESIS,

$BD = CA$;

[I. 5] THEREFORE,

$CA = CD$, SO THAT,

$\angle CDA = \angle DAC$;

THEREFORE,

$\angle CDA + \angle DAC = 2\angle DAC$.

BUT,

$\angle BCD = \angle CDA + \angle DAC$;

THEREFORE,

$\angle BCD = 2\angle CAD.$

BUT,

$\angle BCD = BDA,$

$\angle BCD = DBA;$

THEREFORE,

$\angle BDA + \angle DBA, = 2\angle DAB.$

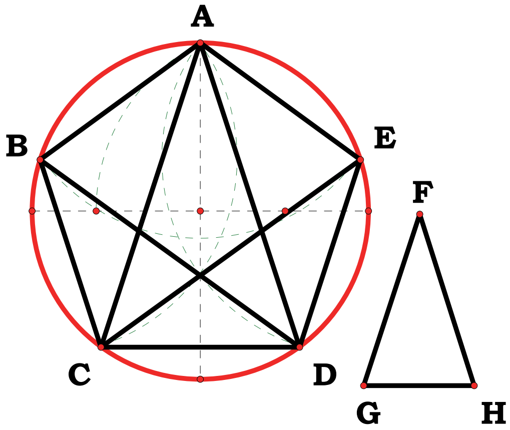
THEREFORE,

THE ISOSCELES TRIANGLE,
ABD, HAS BEEN CONSTRUCTED HAVING EACH, OF
THE ANGLES AT THE BASE, DB, DOUBLE OF
THE REMAINING ONE.

Q. E. F.

PROPOSITION 11.

IN A GIVEN CIRCLE TO INSCRIBE AN EQUILATERAL AND EQUIANGULAR PENTAGON.



LET,

$\odot ABCDE$ BE GIVEN;

THUS IT IS REQUIRED,

TO INSCRIBE IN $\odot ABCDE$, AN EQUILATERAL, AND

EQUIANGULAR PENTAGON.

[IV. 10]LET,

THE ISOSCELES TRIANGLE, FGH,

BE SET OUT HAVING EACH, OF

THE ANGLES, AT G, H, DOUBLE OF $\angle$AT F;

LET,

THERE BE INSCRIBED IN $\odot ABCDE$, AN

ΔACD, EQUIANGULAR WITH ΔFGH,

SO THAT,

$\angle CAD = \angle$AT F,

[IV. 2] AND,

$\angle$AT G, H, RESPECTIVELY, EQUAL, TO $\angle ACD$, $\angle CDA$;

THEREFORE,

$\angle ACD = 2\angle CAD$,

$\angle CDA = 2\angle CAD$.

[I. 9] NOW LET,

$\angle ACD$, $\angle CDA$, BE BISECTED, RESPECTIVELY, BY CE, DB,

AND LET,

AB, BC, DE, EA, BE JOINED.

THEN, SINCE,

EACH, OF $\angle ACD$, $\angle CDA$, IS DOUBLE OF $\angle CAD$, AND

THEY HAVE BEEN BISECTED BY CE, DB,

THEREFORE,

THE FIVE ANGLES,

$\angle DAC$, $\angle ACE$, $\angle ECD$, $\angle CDB$, $\angle BDA$,

ARE EQUAL, TO ONE ANOTHER.

[III. 26] BUT,

EQUAL ANGLES STAND ON EQUAL CIRCUMFERENCES;

THEREFORE,

THE FIVE CIRCUMFERENCES,

AB, *BC*, *CD*, *DE*, *EA*, ARE EQUAL, TO ONE ANOTHER.

[III. 29] BUT,

EQUAL CIRCUMFERENCES ARE SUBTENDED BY
EQUAL STRAIGHT LINES;

THEREFORE,

AB, *BC*, *CD*, *DE*, *EA*, ARE EQUAL, TO ONE ANOTHER;

THEREFORE,

THE PENTAGON, *ABCDE*, IS EQUILATERAL.

I SAY NEXT THAT;

IT IS, ALSO, EQUIANGULAR.

FOR, SINCE,

THE CIRCUMFERENCES, $AB = DE$,

LET,

BCD BE ADDED TO EACH;

THEREFORE,

THE WHOLE CIRCUMFERENCES, $ABCD = EDCB$. AND

$\angle AED$, STANDS ON THE CIRCUMFERENCE, *ABCD*, AND

$\angle BAE$, ON THE CIRCUMFERENCE, *EDCB*;

[III. 27] THEREFORE,

$\angle BAE = \angle AED$.

FOR THE SAME REASON,

EACH, OF $\angle ABC$, $\angle BCD$, $\angle CDE$ = EACH, OF $\angle BAE$, $\angle \mathrm{AED}$;

THEREFORE,

THE PENTAGON, *ABCDE*, IS EQUIANGULAR.

BUT,

IT WAS, ALSO, PROVED EQUILATERAL;

THEREFORE,

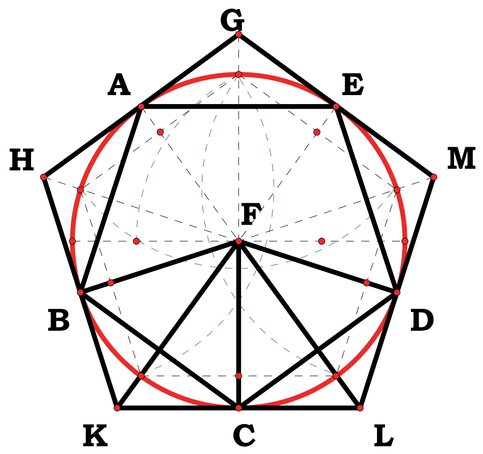
IN THE GIVEN CIRCLE, AN EQUILATERAL,

AND,

EQUIANGULAR, PENTAGON HAS BEEN INSCRIBED.

Q. E. F.

PROPOSITION 12.



ABOUT A GIVEN CIRCLE TO CIRCUMSCRIBE AN EQUILATERAL AND EQUIANGULAR PENTAGON.

LET,

⊙*ABCDE* BE GIVEN;

THUS IT IS REQUIRED,

TO CIRCUMSCRIBE AN EQUILATERAL

AND,

EQUIANGULAR PENTAGON ABOUT ⊙*ABCDE*.

[IV. 11] LET,

A, *B*, *C*, *D*, *E*, BE CONCEIVED TO BE THE ANGULAR POINTS OF THE INSCRIBED PENTAGON,

SO THAT,

THE CIRCUMFERENCES,

AB, *BC*, *CD* *DE*, *EA*, ARE EQUAL;

[III. 16, POR.] LET,

THROUGH *A*, *B*, *C*, *D*, *E*,

GH, *HK*, *KL*, *LM*, *MG*, BE DRAWN TOUCHING THE CIRCLE;

[III. 1] LET,

THE CENTRE, *F*, OF ⊙*ABCDE*, BE TAKEN,

AND LET,

FB, *FK*, *FC*, *FL*, *FD*, BE JOINED.

[III. 18] THEN, SINCE,

KL, TOUCHES ⊙*ABCDE*, AT *C*, AND

FC HAS BEEN JOINED FROM THE CENTRE, *F*, TO THE POINT OF CONTACT, AT *C*,

THEREFORE,

$FC \perp KL$;

THEREFORE,

EACH, OF ∟AT *C*, IS RIGHT.

FOR THE SAME REASON,

THE ANGLES, AT THE POINTS *B*, *D*, ARE, ALSO, RIGHT.

AND, SINCE,

$\llcorner FCK$, IS RIGHT,

[I. 47] THEREFORE,

$\boxdot FK = \boxdot FC + \boxdot CK$.

FOR THE SAME REASON,

$\boxdot FK = \boxdot FB + \boxdot BK$;

SO THAT,

$\boxdot FC + \boxdot CK, = \boxdot FB + \boxdot BK$,

OF WHICH,

$\boxdot FC = \boxdot FB$;

THEREFORE, REMAINS

$\boxdot CK = \boxdot BK$.

THEREFORE,
$BK = CK$.

AND, SINCE,
$FB = FC$, AND FK COMMON,
THE TWO SIDES, BF, FK, ARE EQUAL, TO
THE TWO SIDES, CF, FK; AND
THE BASES, $BK = CK$;

[I. 8]

THEREFORE,

$\angle BFK = \angle KFC$, AND

$\angle BKF = \angle FKC$.

THEREFORE,

$\angle BFC = 2\angle KFC$, AND

$\angle BKC = 2\angle FKC$.

FOR THE SAME REASON,

$\angle CFD = 2\angle CFL$, AND

$\angle DLC = 2\angle FLC$.

[III. 27] NOW, SINCE,
THE CIRCUMFERENCE, $BC = CD$,

$\angle BFC = \angle CFD$. AND,

$\angle BFC = 2\angle KFC$, AND

$\angle DFC = 2\angle LFC$;

THEREFORE,

$\angle KFC = \angle LFC$.

BUT,

$\angle FCK = \angle FCL$;

THEREFORE,

ΔFKC, ΔFLC ARE TWO TRIANGLES HAVING
TWO ANGLES EQUAL, TO TWO ANGLES AND
ONE SIDE EQUAL, TO ONE SIDE,

NAMELY,
FC WHICH IS COMMON TO THEM;

[I. 26] THEREFORE,
THEY WILL, ALSO, HAVE THE REMAINING SIDES EQUAL, TO
THE REMAINING SIDES, AND
THE REMAINING ANGLE TO THE REMAINING ANGLE;

THEREFORE,

$KC = CL$, AND $\angle FKC$, TO $\angle FLC$.

AND, SINCE,
$KC = CL$,

THEREFORE,
$KL = 2KC$.

FOR THE SAME REASON, IT CAN BE PROVED THAT;
$HK = 2BK$. AND, $BK = KC$;

THEREFORE,
$HK = KL$.

SIMILARLY EACH, OF,
HG, GM, ML, CAN, ALSO, BE PROVED EQUAL, TO
EACH, OF HK, KL;

THEREFORE,
THE PENTAGON, $GHKLM$, IS EQUILATERAL.

I SAY NEXT THAT;
IT IS, ALSO, EQUIANGULAR.

FOR, SINCE,

$\angle FKC = \angle FLC$, AND

$\angle HKL = 2\angle FKC$, AND

$\angle KLM = 2\angle FLC$,

THEREFORE,

$\angle HKL = \angle KLM$.

SIMILARLY,

EACH, OF

$\angle KHG$, $\angle HGM$, $\angle GML$, CAN, ALSO, BE PROVED EQUAL, TO

EACH, OF $\angle HKL$, $\angle KLM$

THEREFORE,

THE FIVE,

$\angle GHK$, $\angle HKL$, $\angle KLM$, $\angle LMG$, $\angle MGH$,

ARE EQUAL, TO ONE ANOTHER.

THEREFORE,

THE PENTAGON, $GHKLM$, IS EQUIANGULAR.

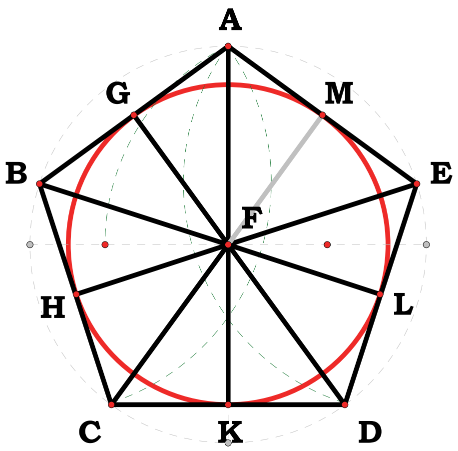
AND,

IT WAS, ALSO, PROVED EQUILATERAL; AND

IT HAS BEEN CIRCUMSCRIBED ABOUT THE CIRCLE $ABCDE$.

Q. E. F.

PROPOSITION 13.



IN A GIVEN PENTAGON, WHICH IS EQUILATERAL AND EQUIANGULAR, TO INSCRIBE A CIRCLE.

LET,

$ABCDE$, BE

THE GIVEN EQUILATERAL

AND,

EQUIANGULAR PENTAGON;

THUS IT IS REQUIRED,

TO INSCRIBE A CIRCLE, IN THE PENTAGON, $ABCDE$.

FOR LET,

$\angle BCD$, $\angle CDE$, BE BISECTED BY CF, DF, RESPECTIVELY; AND FROM F, AT WHICH CF, DF, MEET ONE ANOTHER,

LET,

FB, FA, FE, BE JOINED.

THEN, SINCE,

$BC = CD$, AND CF COMMON, THE TWO SIDES,

$BC = DC$, $CF = CF$; AND

$\angle BCF = \angle DCF$;

[I. 4] THEREFORE,

THE BASES, $BF = DF$, AND

$\Delta BCF = \Delta DCF$, AND

THE REMAINING ANGLES WILL BE EQUAL, TO THE REMAINING ANGLES,

NAMELY,

THOSE WHICH THE EQUAL SIDES SUBTEND.

THEREFORE,

$\angle CBF = \angle CDF$.

AND, SINCE,

$\angle CDE = 2\angle CDF$, AND

$\angle CDE = \angle ABC$, WHILE

$\angle CDF = \angle CBF$;

THEREFORE,

$\angle CBA = 2\angle CBF$;

THEREFORE,

$\angle ABF = \angle FBC$;

THEREFORE,

$\angle ABC$, HAS BEEN BISECTED BY BF.

SIMILARLY IT CAN BE PROVED THAT,

$\angle BAE$, $\angle AED$, HAVE, ALSO, BEEN BISECTED BY
FA, FE, RESPECTIVELY.

NOW LET,
FG, FH, FK, FL, FM, BE DRAWN FROM F,
PERPENDICULAR TO AB, BC, CD, DE, EA.

THEN, SINCE,

$\angle HCF = \angle KCF$, AND

$\llcorner FHC = \llcorner FKC$,

ΔFHC, ΔFKC, HAVE

TWO ANGLES EQUAL, TO TWO ANGLES, AND
ONE SIDE EQUAL, TO ONE SIDE,

NAMELY,
FC WHICH IS COMMON TO THEM, AND
SUBTENDS ONE OF THE EQUAL ANGLES;

[I. 26] THEREFORE,
THEY WILL, ALSO, HAVE,
THE REMAINING SIDES EQUAL, TO THE REMAINING SIDES;

THEREFORE,
$FH = FK$.

SIMILARLY IT CAN BE PROVED THAT,
EACH, OF FL, FM, FG = EACH, OF FH, FK;

THEREFORE,
FG, FH, FK, FL, FM, ARE EQUAL, TO ONE ANOTHER.

THEREFORE,
THE CIRCLE DESCRIBED WITH CENTRE, F, AND
DISTANCE ONE OF FG, FH, FK, FL, FM,
WILL PASS, ALSO, THROUGH THE REMAINING POINTS; AND
IT WILL TOUCH AB, BC, CD, DE, EA,

BECAUSE,

THE ANGLES AT G, H, K, L, M, ARE RIGHT.

For,

IF IT DOES NOT TOUCH THEM,

[III. 16] BUT,

DIVIDES THEM, IT WILL RESULT THAT
THE STRAIGHT LINE DRAWN AT RIGHT ANGLES TO
THE DIAMETER OF THE CIRCLE FROM ITS EXTREMITY,
FALLS WITHIN THE CIRCLE:

WHICH,

WAS PROVED ABSURD.

Therefore,

THE CIRCLE DESCRIBED WITH CENTRE, F, AND
DISTANCE ONE OF FG, FH, FK, FL, FM,
WILL NOT CUT AB, BC, CD, DE, EA;

THEREFORE,

IT WILL TOUCH THEM.

Let,

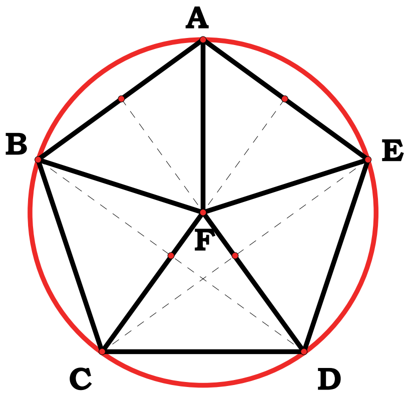
IT BE DESCRIBED, AS $GHKLM$.

Therefore,

IN THE GIVEN PENTAGON,
WHICH IS EQUILATERAL AND EQUIANGULAR,
A CIRCLE HAS BEEN INSCRIBED.

Q. E. F.

PROPOSITION 14.



ABOUT A GIVEN PENTAGON, WHICH IS EQUILATERAL AND EQUIANGULAR, TO CIRCUMSCRIBE A CIRCLE.

LET,
ABCDE, BE THE GIVEN PENTAGON,
WHICH,
IS EQUILATERAL AND
EQUIANGULAR;
THUS IT IS REQUIRED,
TO CIRCUMSCRIBE A CIRCLE ABOUT THE PENTAGON, *ABCDE*.

LET,
$\angle BCD$, $\angle CDE$, BE BISECTED BY *CF*, *DF*, RESPECTIVELY,

AND LET,
FROM *F*,
AT WHICH THE STRAIGHT LINES MEET,
FB, *FA*, *FE*, BE JOINED TO *B*, *A*, *E*.

THEN, IN MANNER SIMILAR TO THE PRECEDING,
IT CAN BE PROVED, THAT;
$\angle CBA$, $\angle BAE$, $\angle AED$, HAVE, ALSO, BEEN BISECTED BY
FB, *FA*, *FE*, RESPECTIVELY.

[I. 6] NOW, SINCE,
$\angle BCD = \angle CDE$, AND
$2\angle FCD = \angle BCD$, AND
$2\angle CDF = \angle CDE$,

THEREFORE,
$\angle FCD = \angle CDF$,

SO THAT,
THE SIDES, $FC = FD$.

SIMILARLY IT CAN BE PROVED THAT,
EACH, OF *FB*, *FA*, *FE* = EACH, OF *FC*, *FD*;

THEREFORE,
FA, *FB*, *FC*, *FD*, *FE*, ARE EQUAL, TO ONE ANOTHER.

THEREFORE,
THE CIRCLE DESCRIBED WITH CENTRE, *F*, AND
DISTANCE ONE OF *FA*, *FB*, *FC*, *FD*, *FE*,

WILL PASS, ALSO, THROUGH THE REMAINING POINTS, AND
WILL HAVE BEEN CIRCUMSCRIBED.

LET,

IT BE CIRCUMSCRIBED,

AND LET,

IT BE $\odot ABCDE$.

THEREFORE,

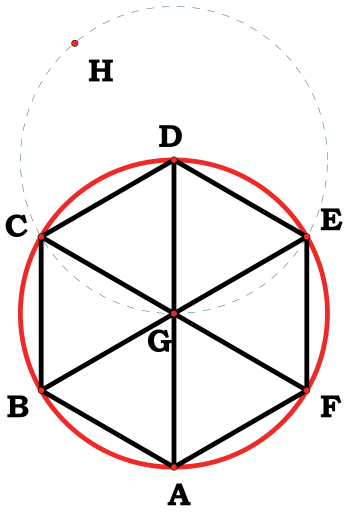
ABOUT THE GIVEN PENTAGON,

WHICH,

IS EQUILATERAL AND EQUIANGULAR,
A CIRCLE HAS BEEN CIRCUMSCRIBED.

Q. E. F.

PROPOSITION 15.



IN A GIVEN CIRCLE TO INSCRIBE AN EQUILATERAL AND EQUIANGULAR HEXAGON.

LET,

$\odot ABCDEF$, BE GIVEN;

THUS IT IS REQUIRED,

TO INSCRIBE AN EQUILATERAL

AND,

EQUIANGULAR HEXAGON IN

$\odot ABCDEF$.

LET,

THE DIAMETER, AD, OF $\odot ABCDEF$, BE DRAWN;

LET,

THE CENTRE, G, OF THE CIRCLE BE TAKEN, AND
WITH CENTRE, D, AND DISTANCE DG.

LET,

$\odot EGCH$, BE DESCRIBED;

LET,

EG, CG BE JOINED, AND CARRIED THROUGH TO B, F,

AND LET,

AB, BC, CD, DE, EF, FA BE JOINED.

I SAY THAT;

THE HEXAGON, $ABCDEF$, IS EQUILATERAL AND EQUIANGULAR.

FOR, SINCE,

G, IS THE CENTRE OF $\odot ABCDEF$,

$GE = GD$.

AGAIN, SINCE,

D, IS THE CENTRE OF $\odot GCH$,

$DE = DG$.

BUT,

$GE = GD$;

THEREFORE,

$GE = ED$;

THEREFORE,

ΔEGD, IS EQUILATERAL;

[I. 5] AND THEREFORE,

$\angle EGD$, $\angle GDE$, $\angle DEG$, ARE EQUAL, TO ONE ANOTHER,
INASMUCH AS, IN ISOSCELES TRIANGLES,
THE ANGLES AT THE BASE ARE EQUAL, TO ONE ANOTHER.

[I. 32] AND,
THE THREE ANGLES OF THE TRIANGLE ARE EQUAL, TO
TWO RIGHT ANGLES;

THEREFORE,

$\angle EGD$, IS ONE-THIRD OF TWO RIGHT ANGLES.

SIMILARLY,

$\angle DGC$, CAN, ALSO, BE PROVED TO BE ONE-THIRD OF
TWO RIGHT ANGLES.

AND, SINCE,
CG, STANDING, ON EB, MAKES THE ADJACENT,
$\angle EGC$, $\angle CGB$, EQUAL, TO TWO RIGHT ANGLES,

THEREFORE,

THE REMAINING, $\angle CGB$, IS, ALSO, ONE-THIRD OF
TWO RIGHT ANGLES.

[I. 15]

THEREFORE,

$\angle EGD$, $\angle DGC$, $\angle CGB$, ARE EQUAL, TO ONE ANOTHER;

SO THAT,
THE ANGLES VERTICAL TO THEM,
$\angle BGA$, $\angle AGE$, $\angle FGE$, ARE EQUAL.

THEREFORE,

$\angle EGD$, $\angle DGC$, $\angle CGB$, $\angle BGA$, $\angle AGE$, $\angle FGE$,
ARE EQUAL, TO ONE ANOTHER.

[III. 26] BUT,
EQUAL ANGLES STAND ON EQUAL CIRCUMFERENCES;

THEREFORE,
THE SIX CIRCUMFERENCES,
AB, BC, CD, DE, EF, FA, ARE EQUAL, TO ONE ANOTHER.

[III. 29] AND,
EQUAL CIRCUMFERENCES ARE SUBTENDED BY
EQUAL STRAIGHT LINES;

THEREFORE,

THE SIX STRAIGHT LINES ARE EQUAL, TO ONE ANOTHER;

THEREFORE,

THE HEXAGON, $ABCDEF$, IS EQUILATERAL.

I SAY NEXT THAT;

IT IS, ALSO, EQUIANGULAR.

FOR, SINCE,

THE CIRCUMFERENCES, $FA = ED$,

LET,

THE CIRCUMFERENCE, $ABCD$, BE ADDED TO EACH;

THEREFORE,

THE WHOLES, $FABCD = EDCBA$; AND,

$\angle FED$, STANDS ON THE CIRCUMFERENCE, $FABCD$, AND

$\angle AFE$, ON THE CIRCUMFERENCE, $EDCBA$;

[III. 27] THEREFORE,

$\angle AFE = \angle DEF$.

SIMILARLY IT CAN BE PROVED THAT,

THE REMAINING ANGLES OF
THE HEXAGON, $ABCDEF$, ARE, ALSO, SEVERALLY
EQUAL, TO EACH, OF $\angle AFE$, $\angle FED$;

THEREFORE,

THE HEXAGON, $ABCDEF$, IS EQUIANGULAR.

BUT,

IT WAS, ALSO, PROVED EQUILATERAL; AND

IT HAS BEEN INSCRIBED IN $\odot ABCDEF$.

THEREFORE,

IN THE GIVEN CIRCLE,
AN EQUILATERAL AND EQUIANGULAR HEXAGON,
HAS BEEN INSCRIBED.

Q. E. F.

PORISM.

FROM THIS IT IS MANIFEST THAT THE SIDE OF THE HEXAGON EQUALS THE RADIUS OF THE CIRCLE.

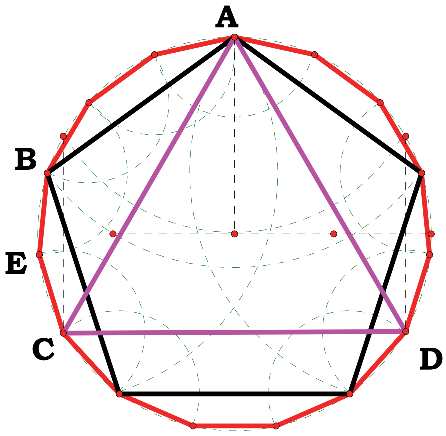
AND, IN LIKE MANNER AS IN THE CASE OF THE PENTAGON, IF THROUGH THE POINTS OF DIVISION ON THE CIRCLE WE DRAW TANGENTS TO $\odot$THERE WILL BE CIRCUMSCRIBED ABOUT THE CIRCLE AN EQUILATERAL AND EQUIANGULAR HEXAGON IN

conformity with what was explained in the case of the pentagon.

And further by means similar to those explained in the case of the pentagon we can both inscribe a circle in a given hexagon and circumscribe one about it.

Q. E. F.

Proposition 16.



In a given circle, to inscribe a fifteen-angled figure which shall be both equilateral and equiangular.

Let,

ABCD be
the given circle;

thus it is required,
to inscribe
in ⊙*ABCD*
a fifteen-angled figure which shall be both equilateral and equiangular.

let,

In ⊙*ABCD*,
there be inscribed a side, *AC*, of
the equilateral triangle inscribed in it, and
a side, *AB*, of an equilateral pentagon;

therefore,

of the equal segments of which
there are fifteen in ⊙*ABCD*,
there will be five in the circumference, *ABC*,
which is one-third of ⊙*ABCD* and
there will be three in the circumference, *AB*,
which is one-fifth of the circle;

therefore,

in the remainder, *BC*,
there will be two of the equal segments.

[III. 30] Let,

BC be bisected, at *E*;

therefore,

each, of the circumferences,
BE, *EC*, is a fifteenth of ⊙*ABCD*.

If therefore,

we join *BE*, *EC*, and fit into ⊙*ABCD*,
straight lines equal, to them, and
in contiguity,

A FIFTEEN-ANGLED FIGURE WHICH IS BOTH
EQUILATERAL AND EQUIANGULAR
WILL HAVE BEEN INSCRIBED IN IT.

Q. E. F.

AND, IN LIKE MANNER AS IN THE CASE OF THE PENTAGON, IF THROUGH THE POINTS OF DIVISION ON THE CIRCLE WE DRAW TANGENTS TO THE CIRCLE THERE WILL BE CIRCUMSCRIBED ABOUT THE CIRCLE A FIFTEEN-ANGLED FIGURE WHICH IS EQUILATERAL AND EQUIANGULAR.

AND FURTHER, BY PROOFS SIMILAR TO THOSE IN THE CASE OF THE PENTAGON, WE CAN BOTH INSCRIBE A CIRCLE IN THE GIVEN FIFTEEN-ANGLED FIGURE AND CIRCUMSCRIBE ONE ABOUT IT.

BOOK V.

OF

EUCLID'S ELEMENTS

TRANSLATED FROM THE TEXT OF HEIBERG

BY

SIR THOMAS L. HEATH,

K. C. B. K. C. V. O. F. R. S.

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HONORARY FELLOW (SOMETIME FELLOW) OF TRINITY COLLEGE CAMBRIDGE

2013 *EDITION*

REVISED WITH SUBTRACTIONS

REFORMATTED AND ABRIDGED FOR STUDY OF *THE ELEMENTS*.

BY JOHN CLARK.

BOOK V.

DEFINITIONS.

1. A MAGNITUDE IS A **PART** OF A MAGNITUDE, THE LESS OF THE GREATER, WHEN IT MEASURES THE GREATER.

2. THE GREATER IS A **MULTIPLE** OF THE LESS WHEN IT IS MEASURED BY THE LESS.

3. A **RATIO** IS A SORT OF RELATION IN RESPECT OF SIZE BETWEEN TWO MAGNITUDES OF THE SAME KIND.

4. MAGNITUDES ARE SAID TO **HAVE A RATIO** TO ONE ANOTHER WHICH ARE CAPABLE, WHEN MULTIPLIED, OF EXCEEDING ONE ANOTHER.

5. MAGNITUDES ARE SAID TO **BE IN THE SAME RATIO**, THE FIRST TO THE SECOND AND THE THIRD TO THE FOURTH, WHEN, IF ANY EQUIMULTIPLES WHATEVER BE TAKEN OF THE FIRST AND THIRD, AND ANY EQUIMULTIPLES WHATEVER OF THE SECOND AND FOURTH, THE FORMER EQUIMULTIPLES ALIKE EXCEED, ARE ALIKE EQUAL TO, OR ALIKE FALL SHORT OF, THE LATTER EQUIMULTIPLES RESPECTIVELY TAKEN IN CORRESPONDING ORDER.

6. LET MAGNITUDES WHICH HAVE THE SAME RATIO BE CALLED **PROPORTIONAL**.

7. WHEN, OF THE EQUIMULTIPLES, THE MULTIPLE OF THE FIRST MAGNITUDE EXCEEDS THE MULTIPLE OF THE SECOND, BUT THE MULTIPLE OF THE THIRD DOES NOT EXCEED THE MULTIPLE OF THE FOURTH, THEN THE FIRST IS SAID TO **HAVE A GREATER RATIO** TO THE SECOND THAN THE THIRD HAS TO THE FOURTH.

8. A PROPORTION IN THREE TERMS IS THE LEAST POSSIBLE.

9. WHEN THREE MAGNITUDES ARE PROPORTIONAL, THE FIRST IS SAID TO HAVE TO THE THIRD THE **DUPLICATE RATIO** OF THAT WHICH IT HAS TO THE SECOND.

10. WHEN FOUR MAGNITUDES ARE < CONTINUOUSLY > PROPORTIONAL, THE FIRST IS SAID TO HAVE TO THE FOURTH THE **TRIPLICATE RATIO** OF THAT WHICH IT HAS TO THE SECOND, AND SO ON CONTINUALLY, WHATEVER BE THE PROPORTION.

11. THE TERM **CORRESPONDING MAGNITUDES** IS USED OF ANTECEDENTS IN RELATION TO ANTECEDENTS, AND OF CONSEQUENTS IN RELATION TO CONSEQUENTS.

12. **ALTERNATE RATIO** MEANS TAKING THE ANTECEDENT IN RELATION TO THE ANTECEDENT AND THE CONSEQUENT IN RELATION TO THE CONSEQUENT.

13. **INVERSE RATIO** MEANS TAKING THE CONSEQUENT AS ANTECEDENT IN RELATION TO THE ANTECEDENT AS CONSEQUENT.

14. **COMPOSITION OF A RATIO** MEANS TAKING THE ANTECEDENT TOGETHER WITH THE CONSEQUENT AS ONE IN RELATION TO THE CONSEQUENT BY ITSELF.

15. **Separation of a ratio** means taking the excess by which the antecedent exceeds the consequent in relation to the consequent by itself.

16. **Conversion of a ratio** means taking the antecedent in relation to the excess by which the antecedent exceeds the consequent.

17. A ratio ***ex aequali*** arises when, there being several magnitudes and another set equal to them in multitude which taken two and two are in the same proportion, as the first is to the last among the first magnitudes, so is the first to the last among the second magnitudes;

Or, in other words, it means taking the extreme terms by virtue of the removal of the intermediate terms.

18. A **perturbed proportion** arises when, there being three magnitudes and another set equal to them in multitude, as antecedent is to consequent among the first magnitudes, so is antecedent to consequent among the second magnitudes, while, as the consequent is to a third among the first magnitudes, so is a third to the antecedent among the second magnitudes.

NOTES.

DEFINITION 1. *A MAGNITUDE IS A* PART *OF A MAGNITUDE, THE LESS OF THE GREATER, WHEN IT MEASURES THE GREATER.*

NOTES.

DEFINITION 2. *THE GREATER IS A* MULTIPLE *OF THE LESS WHEN IT IS MEASURED BY THE LESS*.

NOTES.

DEFINITION 3. *A* RATIO *is a sort of relation in respect of size between two magnitudes of the same kind.*

NOTES.

DEFINITION 4. *MAGNITUDES ARE SAID TO* HAVE A RATIO *TO ONE ANOTHER WHICH ARE CAPABLE, WHEN MULTIPLIED, OF EXCEEDING ONE ANOTHER.*

NOTES.

DEFINITION 5. *MAGNITUDES ARE SAID TO* BE IN THE SAME RATIO, *THE FIRST TO THE SECOND AND THE THIRD TO THE FOURTH, WHEN, IF ANY EQUIMULTIPLES WHATEVER BE TAKEN OF THE FIRST AND THIRD, AND ANY EQUIMULTIPLES WHATEVER OF THE SECOND AND FOURTH, THE FORMER EQUIMULTIPLES ALIKE EXCEED, ARE ALIKE EQUAL TO, OR ALIKE FALL SHORT OF, THE LATTER EQUIMULTIPLES RESPECTIVELY TAKEN IN CORRESPONDING ORDER.*

NOTES.

DEFINITION 6. *LET MAGNITUDES WHICH HAVE THE SAME RATIO BE CALLED* PROPORTIONAL.

Notes.

Definition 7. *When, of the equimultiples, the multiple of the first magnitude exceeds the multiple of the second, but the multiple of the third does not exceed the multiple of the fourth, then the first is said to* have a greater ratio *to the second than the third has to the fourth.*

Notes.

Definition 8. *A proportion in three terms is the least possible.*

NOTES.

DEFINITION 9. *WHEN THREE MAGNITUDES ARE PROPORTIONAL, THE FIRST IS SAID TO HAVE TO THE THIRD THE* DUPLICATE RATIO *OF THAT WHICH IT HAS TO THE SECOND.*

Notes.

Definition 10. *When four magnitudes are < continuously > proportional, the first is said to have to the fourth the* triplicate ratio *of that which it has to the second, and so on continually, whatever be the proportion.*

NOTES.

DEFINITION 11. *THE TERM* CORRESPONDING MAGNITUDES *IS USED OF ANTECEDENTS IN RELATION TO ANTECEDENTS, AND OF CONSEQUENTS IN RELATION TO CONSEQUENTS.*

NOTES.

DEFINITION 12. ALTERNATE RATIO *MEANS TAKING THE ANTECEDENT IN RELATION TO THE ANTECEDENT AND THE CONSEQUENT IN RELATION TO THE CONSEQUENT*.

NOTES.

DEFINITION 13. INVERSE RATIO *MEANS TAKING THE CONSEQUENT AS ANTECEDENT IN RELATION TO THE ANTECEDENT AS CONSEQUENT*.

NOTES.

DEFINITION 14. COMPOSITION OF A RATIO *means taking the antecedent together with the consequent as one in relation to the consequent by itself*.

NOTES.

DEFINITION 15. SEPARATION OF A RATIO *MEANS TAKING THE EXCESS BY WHICH THE ANTECEDENT EXCEEDS THE CONSEQUENT IN RELATION TO THE CONSEQUENT BY ITSELF*.

NOTES.

DEFINITION 16. CONVERSION OF A RATIO *MEANS TAKING THE ANTECEDENT IN RELATION TO THE EXCESS BY WHICH THE ANTECEDENT EXCEEDS THE CONSEQUENT*.

NOTES.

DEFINITION 17. *A RATIO* EX AEQUALI *ARISES WHEN, THERE BEING SEVERAL MAGNITUDES AND ANOTHER SET EQUAL TO THEM IN MULTITUDE WHICH TAKEN TWO AND TWO ARE IN THE SAME PROPORTION, AS THE FIRST IS TO THE LAST AMONG THE FIRST MAGNITUDES, SO IS THE FIRST TO THE LAST AMONG THE SECOND MAGNITUDES;*

OR, IN OTHER WORDS, IT MEANS TAKING THE EXTREME TERMS BY VIRTUE OF THE REMOVAL OF THE INTERMEDIATE TERMS.

Notes.

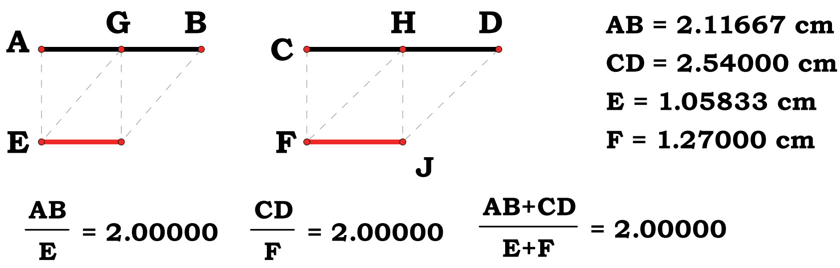
Definition 18. *A* perturbed proportion *arises when, there being three magnitudes and another set equal to them in multitude, as antecedent is to consequent among the first magnitudes, so is antecedent to consequent among the second magnitudes, while, as the consequent is to a third among the first magnitudes, so is a third to the antecedent among the second magnitudes.*

BOOK V.

PROPOSITIONS.

PROPOSITION 1.

IF THERE BE ANY NUMBER OF MAGNITUDES WHATEVER WHICH ARE, RESPECTIVELY, EQUIMULTIPLES OF ANY MAGNITUDES EQUAL IN MULTITUDE, THEN, WHATEVER MULTIPLE ONE OF THE MAGNITUDES IS OF ONE, THAT MULTIPLE, ALSO, *WILL ALL BE OF ALL.*



LET,

ANY NUMBER OF MAGNITUDES, WHATEVER,
AB, CD, BE RESPECTIVELY EQUIMULTIPLES OF
ANY MAGNITUDES, E, F, EQUAL IN MULTITUDE;

I SAY THAT;

WHATEVER MULTIPLE AB IS OF E,
THAT MULTIPLE WILL $AB + CD$, ALSO, BE OF $E + F$.

FOR, SINCE,

AB IS THE SAME MULTIPLE OF E,
THAT CD IS OF F,
AS MANY MAGNITUDES AS THERE ARE IN AB EQUAL TO E,
SO MANY, ALSO, ARE THERE IN CD EQUAL TO F.

LET,

AB BE DIVIDED INTO
THE MAGNITUDES, AG, GB, EQUAL TO E, AND
CD INTO CH, HD EQUAL TO F;

THEN,

THE MULTITUDE OF THE MAGNITUDES,
AG, GB, WILL BE EQUAL TO THE MULTITUDE OF
THE MAGNITUDES, CH, HD.

NOW, SINCE,

$AG = E$, AND $CH = F$,

THEREFORE,

$AG = E$, AND $AG + CH$ TO $E + F$.

FOR THE SAME REASON,

$GB = E$, AND $GB + HD = E + F$;

THEREFORE,

AS MANY MAGNITUDES AS THERE ARE IN AB EQUAL TO E,
SO MANY, ALSO, ARE THERE IN $AB + CD$ EQUAL TO $E + F$;

THEREFORE,

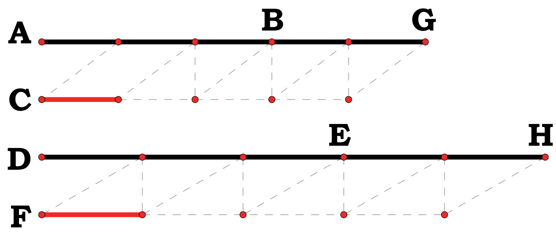
WHATEVER MULTIPLE AB IS OF E,
THAT MULTIPLE WILL $AB + CD$, ALSO, BE OF $E + F$.

THEREFORE ETC.

Q. E. D.

Proposition 2.

If a first magnitude be the same multiple of a second that a third is of a fourth, and a fifth, also, *be the same multiple of the second that a sixth is of the fourth, the sum of the first and fifth will,* also, *be the same multiple of the second that the sum of the third and sixth is of the fourth.*



AB = 3.04800 cm
C = 1.01600 cm
DE = 4.00050 cm
F = 1.33350 cm
BG = 2.03200 cm
EH = 2.66700 cm

$$\frac{AB}{C} = 3.00000 \qquad \frac{BG}{C} = 2.00000 \qquad \frac{AB+BG}{C} = 5.00000$$

$$\frac{DE}{F} = 3.00000 \qquad \frac{EH}{F} = 2.00000 \qquad \frac{DE+EH}{F} = 5.00000$$

$$\left(\frac{AB}{C}+\frac{BG}{C}\right)-\frac{AB+BG}{C} = 0.00000$$

Let,

a first magnitude,

AB, be the same multiple of a second, *C*,

that a third, *DE*, is of a fourth, *F*,

and let,

a fifth, *BG*, also, be the same multiple of the second, *C*,

that a sixth, *EH*, is of the fourth *F*;

I say that;

the sum of the first and fifth, *AG*,

will be the same multiple of the second, *C*,

that the sum of the third and sixth, *DH*, is of the fourth, *F*.

For, since,

AB is the same multiple of *C*,

that *DE* is of *F*,

therefore,

as many magnitudes as there are in *AB* equal to *C*,

so many, also, are there in *DE* equal to *F*.

For the same reason also,

as many as there are in *BG* equal to *C*,

so many are there, also, in *EH* equal to *F*;

therefore,

AS MANY AS THERE ARE IN THE WHOLE, AG, EQUAL TO C,
SO MANY, ALSO, ARE THERE IN THE WHOLE, DH, EQUAL TO F.

THEREFORE,
WHATEVER MULTIPLE AG IS OF C,
THAT MULTIPLE, ALSO, IS DH OF F.

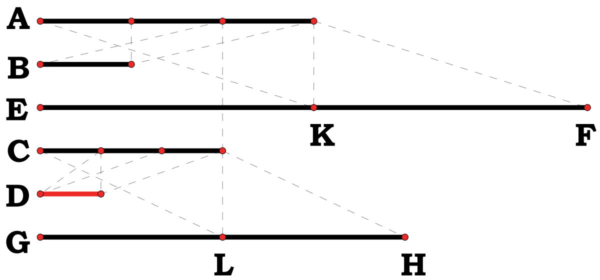
THEREFORE,
THE SUM OF THE FIRST AND FIFTH, AG,
IS THE SAME MULTIPLE OF THE SECOND, C,
THAT THE SUM OF THE THIRD AND SIXTH, DH,
IS OF THE FOURTH, F.

THEREFORE ETC.

Q. E. D.

Proposition 3.

If a first magnitude be the same multiple of a second that a third is of a fourth, and if equimultiples be taken of the first and third, then, also, ex aequali *the magnitudes taken will be equimultiples respectively, the one of the second and the other of the fourth.*



A = 3.61950 cm
B = 1.20650 cm
C = 2.41300 cm
D = 0.80433 cm
EF = 7.23900 cm
GH = 4.82600 cm

$\frac{A}{B} = 3.00000$ $\frac{EF}{A} = 2.00000$ $\frac{EF}{B} = 6.00000$ $\frac{C+GH}{D} = 9.00000$

$\frac{C}{D} = 3.00000$ $\frac{GH}{C} = 2.00000$ $\frac{GH}{D} = 6.00000$ $\frac{A+EF}{B} = 9.00000$

$$\frac{A+EF}{B} - \frac{C+GH}{D} = 0.00000$$

Let,

a first magnitude, *A*, be
the same multiple of a second, *B*,
that a third, *C*, is of a fourth, *D*,

and let,

equimultiples, *EF*, *GH*, be taken of *A*, *C*;

I say that;

EF is the same multiple of *B*, that
GH is of *D*.

For, since,

EF is the same multiple of *A*, that
GH is of *C*,

therefore,

as many magnitudes as there are in *EF* equal to *A*,
so many, also, are there in *GH* equal to *C*.

Let,

EF be divided into
the magnitudes, *EK*, *KF*, equal to *A*, and
GH into the magnitudes, *GL*, *LH*, equal to *C*;

then,

the multitude of the magnitudes,
EK, *KF*, will be equal to the multitude of

THE MAGNITUDES, GL, LH.

AND, SINCE,

A IS THE SAME MULTIPLE OF B, THAT C IS OF D,

WHILE,

$EK = A$, AND $GL = C$,

THEREFORE,

EK IS THE SAME MULTIPLE OF B, THAT GL IS OF D.

FOR THE SAME REASON,

KF IS THE SAME MULTIPLE OF B, THAT LH IS OF D.

SINCE, THEN,

A FIRST MAGNITUDE, EK, IS
THE SAME MULTIPLE OF A SECOND, B,
THAT A THIRD, GL, IS OF A FOURTH, D, AND
A FIFTH, KF, IS, ALSO, THE SAME MULTIPLE OF THE SECOND, B,
THAT A SIXTH, LH, IS OF THE FOURTH, D,

[V. 2] THEREFORE,

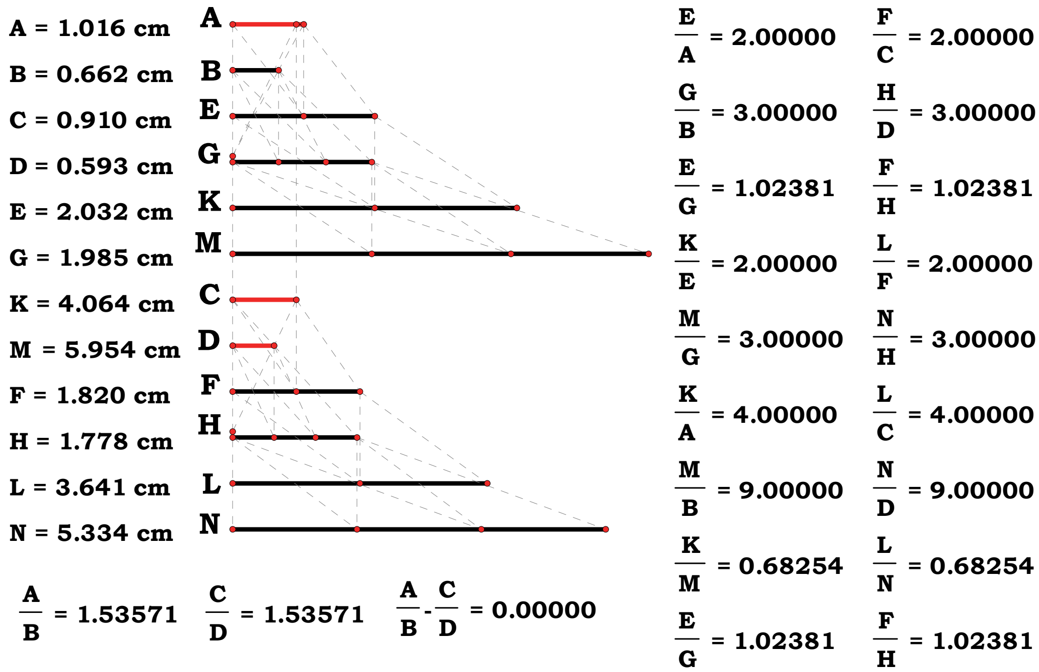
THE SUM OF THE FIRST AND FIFTH, EF, IS ALSO
THE SAME MULTIPLE OF THE SECOND, B, THAT THE SUM OF
THE THIRD AND SIXTH, GH, IS OF THE FOURTH, D.

THEREFORE ETC.

Q. E. D.

Proposition 4.

If a first magnitude have to a second the same ratio as a third to a fourth, any equimultiples whatever of the first and third will, also, have the same ratio to any equimultiples whatever of the second and fourth respectively, taken in corresponding order.



For let,

a first magnitude, *A*, have to a second, *B*,
the same ratio as a third, *C*, to a fourth, *D*;

and let,

equimultiples, *E*, *F*, be taken of *A*, *C*, and
G, *H* other, chance, equimultiples of *B*, *D*;

I say that;

as *E* is to *G*,
so is *F* to *H*.

For let,

equimultiples, *K*, *L*, be taken of *E*, *F*, and
other, chance, equimultiples *M*, *N* of *G*, *H*.

[v. 3] Since,

E is the same multiple of *A*,
that *F* is of *C*, and
equimultiples, *K*, *L*, of *E*, *F* have been taken,

therefore,

K is the same multiple of *A*, that *L* is of *C*.

For the same reason,

M is the same multiple of *B*, that *N* is of *D*.

AND, SINCE,

as A is to B,
so is C to D, and
of A, C, equimultiples, K, L, have been taken, and
of B, D, other, chance, equimultiples, M, N,

[v. Def. 5] therefore,

if K is in excess of M,
L, also, is in excess of N,
if it is equal, equal, and
if less, less.

[v. Def. 5] And,

K, L are equimultiples of E, F, and
M, N other, chance, equimultiples of G, H;

therefore,

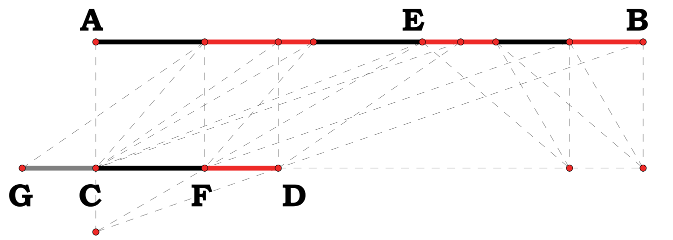
as E is to G,
so is F to H.

Therefore etc.

Q. E. D.

PROPOSITION 5.

IF A MAGNITUDE BE THE SAME MULTIPLE OF A MAGNITUDE THAT A PART SUBTRACTED IS OF A PART SUBTRACTED, THE REMAINDER WILL, ALSO, *BE THE SAME MULTIPLE OF THE REMAINDER THAT THE WHOLE IS OF THE WHOLE.*



AB = 7.23900 cm
CD = 2.41300 cm
CF = 1.44229 cm
AE = 4.32687 cm
EB = 2.91213 cm
FD = 0.97071 cm
CG = 0.97071 cm

$\frac{AB}{CD} = 3.00000$ $\frac{AE}{CF} = 3.00000$ $\frac{EB}{FD} = 3.00000$

$\frac{EB}{CG} = 3.00000$ $\frac{AB}{CG+CF} = 3.00000$

FOR LET,

THE MAGNITUDE, *AB*, BE
THE SAME MULTIPLE OF THE MAGNITUDE, *CD*,
THAT THE PART, *AE*, SUBTRACTED IS OF
THE PART, *CF*, SUBTRACTED;

I SAY THAT;

THE REMAINDER, *EB*, IS ALSO
THE SAME MULTIPLE OF THE REMAINDER, *FD*, THAT
THE WHOLE, *AB*, IS OF THE WHOLE, *CD*.

FOR,

WHATEVER MULTIPLE OF *AE* IS OF *CF*,

LET,

EB, BE MADE THAT MULTIPLE OF *CG*.

[V. 1] THEN, SINCE,

AE IS THE SAME MULTIPLE OF *CF*, THAT
EB IS OF *GC*,

THEREFORE,

AE IS THE SAME MULTIPLE OF *CF*, THAT
AB IS OF *GF*.

BUT, BY THE ASSUMPTION,

AE IS THE SAME MULTIPLE OF *CF*, THAT
AB IS OF *CD*.

THEREFORE,

AB IS THE SAME MULTIPLE OF EACH, OF
THE MAGNITUDES *GF*, *CD*;

THEREFORE,

$GF = CD$.

LET,

CF BE SUBTRACTED FROM EACH;

THEREFORE,

THE REMAINDERS, $GC = FD$.

AND, SINCE,

AE IS THE SAME MULTIPLE OF CF, THAT

EB IS OF GC, AND

$GC = DF$,

THEREFORE,

AE IS THE SAME MULTIPLE OF CF, THAT

EB IS OF FD.

BUT, BY HYPOTHESIS,

AE IS THE SAME MULTIPLE OF CF, THAT

AB IS OF CD;

THEREFORE,

EB IS THE SAME MULTIPLE OF FD, THAT

AB IS OF CD.

THAT IS,

THE REMAINDER, EB, WILL BE

THE SAME MULTIPLE OF THE REMAINDER, FD, THAT

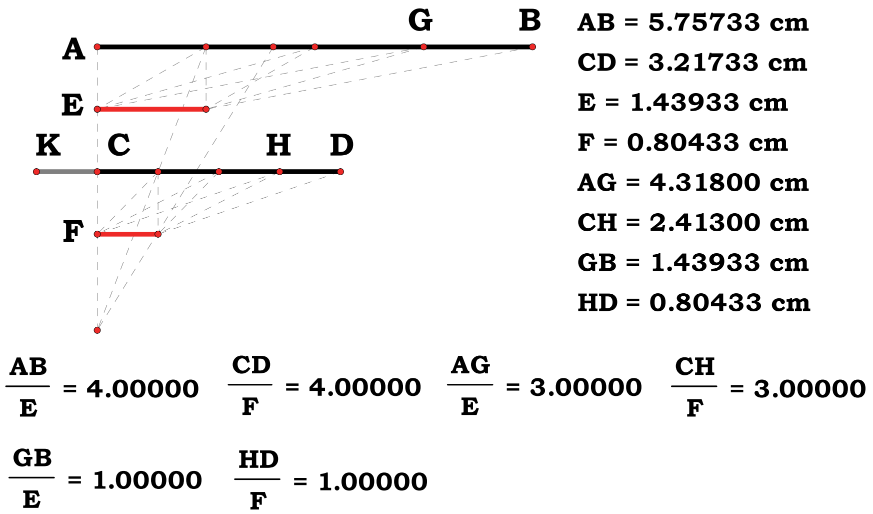
THE WHOLE, AB, IS OF THE WHOLE, CD.

THEREFORE ETC.

Q. E. D.

Proposition 6.

If two magnitudes be equimultiples of two magnitudes, and any magnitudes subtracted from them be equimultiples of the same, the remainders, also, *are either equal to the same or equimultiples of them.*



For let,

two magnitudes, AB, CD, be equimultiples of two magnitudes, E, F,

and let,

AG, CH, subtracted from them, be equimultiples of the same two, E, F;

I say that;

the remainders also, GB, HD, are either equal to E, F, or equimultiples of them.

For, first, let,

GB be equal to E;

I say that;

$HD = F$.

For let,

$CK = F$.

[v. 2] Since,

AG is the same multiple of E, that CH is of F,

while,

$GB = E$, and $KC = F$,

therefore,

AB is the same multiple of E, that

KH IS OF F.

BUT, BY HYPOTHESIS,

AB IS THE SAME MULTIPLE OF E, THAT
CD IS OF F;

THEREFORE,

KH IS THE SAME MULTIPLE OF F, THAT
CD IS OF F.

SINCE THEN,

EACH, OF THE MAGNITUDES, KH, CD, IS
THE SAME MULTIPLE OF F,

THEREFORE,

$KH = CD$.

LET,

CH BE SUBTRACTED FROM EACH;

THEREFORE,

THE REMAINDERS, $KC = HD$.

BUT,

$F = KC$; THEREFORE,
$HD = F$.

HENCE,

IF, $GB = E$,
$HD = F$.

SIMILARLY WE CAN PROVE THAT; EVEN,

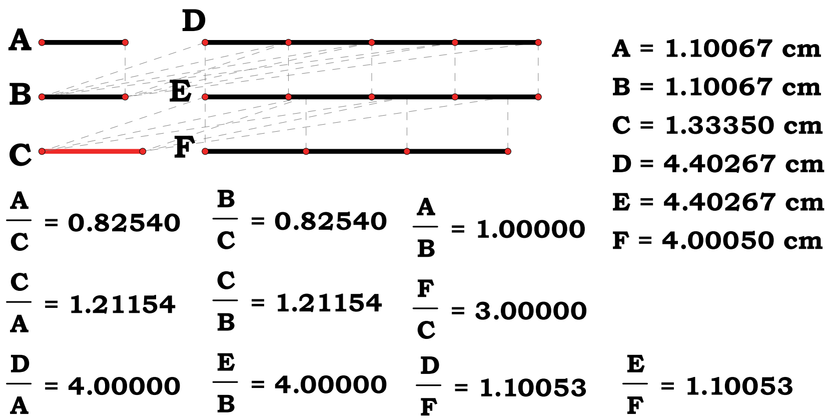
IF, GB BE A MULTIPLE OF E,
HD IS, ALSO, THE SAME MULTIPLE OF F.

THEREFORE ETC.

Q. E. D.

PROPOSITION 7.

EQUAL MAGNITUDES HAVE TO THE SAME THE SAME RATIO, AS, ALSO, *HAS THE SAME TO EQUAL MAGNITUDES.*



LET,

A, *B* BE EQUAL MAGNITUDES, AND
C ANY OTHER, CHANCE, MAGNITUDE;

I SAY THAT;

EACH, OF THE MAGNITUDES, *A*, *B*, HAS
THE SAME RATIO TO *C*, AND
C HAS THE SAME RATIO TO EACH, OF THE MAGNITUDES, *A*, *B*.

FOR LET,

EQUIMULTIPLES, *D*, *E* OF *A*, *B*, BE TAKEN, AND
OF *C*, ANOTHER, CHANCE, MULTIPLE, *F*.

THEN, SINCE,

D IS THE SAME MULTIPLE OF *A*, THAT
E IS OF *B*, WHILE
$A = B$, THEREFORE,
$D = E$.

BUT,

F IS ANOTHER, CHANCE, MAGNITUDE.

IF THEREFORE,

D IS IN EXCESS OF *F*,
E IS, ALSO, IN EXCESS OF *F*,
IF EQUAL TO IT, EQUAL; AND
IF LESS, LESS. AND,
D, *E* ARE EQUIMULTIPLES OF *A*, *B*, WHILE
F IS ANOTHER, CHANCE, MULTIPLE OF *C*;

[V. DEF. 5] THEREFORE,

AS *A* IS TO *C*,
SO IS *B* TO *C*.

I SAY NEXT THAT;

C, also, has the same ratio to each, of the magnitudes, A, B.

For, with the same construction,

we can prove, similarly, that;
$D = E$; and
F is some other magnitude.

If therefore,
F is in excess of D,
it is, also, in excess of E,
if equal, equal; and
if less, less.

[v. Def. 5] And,
F is a multiple of C, while
D, E are other, chance, equimultiples of A, B;

therefore,
as C is to A,
so is C to B.

Therefore etc.

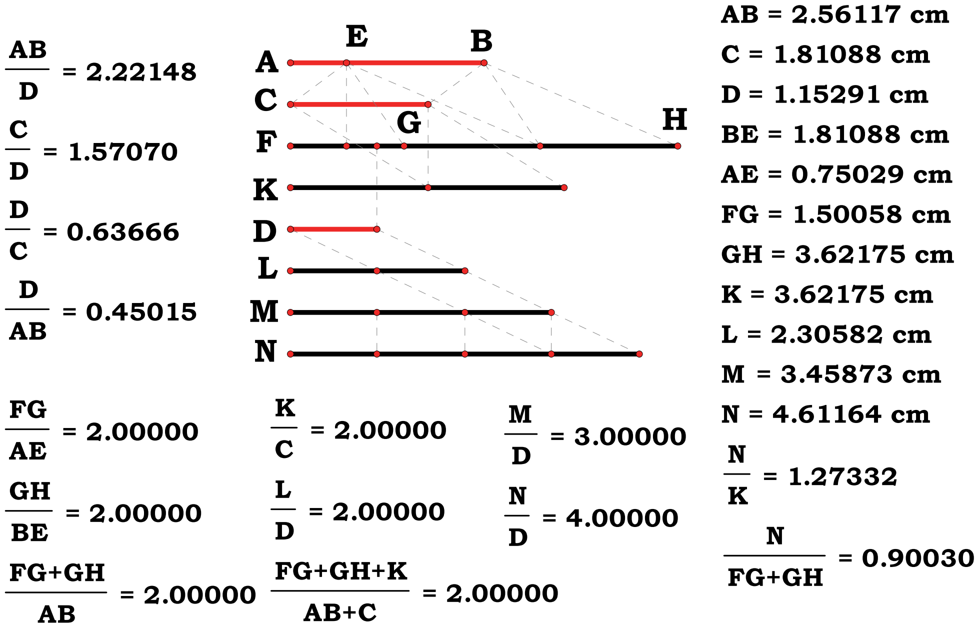
Porism.

From this it is manifest that, if any magnitudes are proportional, they will, also, be proportional inversely.

Q. E. D.

Proposition 8.

Of unequal magnitudes, the greater has to the same a greater ratio than the less has; and the same has to the less a greater ratio than it has to the greater.



Let,

AB, *C*

be unequal magnitudes,

and let,

AB be greater;

let,

D be another, chance, magnitude;

I say that;

AB has to *D*, a greater ratio than *C* has to *D*, and
D has to *C*, a greater ratio than it has to *AB*.

For, since,

AB is greater than *C*,

let,

BE = *C*;

[v. Def. 4] then,

the less of the magnitudes *AE*, *EB*,
if multiplied, will sometime be greater than *D*.

[Case 1.]

First, let,

AE be less than *EB*;

let,

AE be multiplied,

AND LET,

FG, BE A MULTIPLE OF IT WHICH IS GREATER THAN D;

THEN,

WHATEVER MULTIPLE FG IS OF AE,

LET,

GH, BE MADE THE SAME MULTIPLE OF EB, AND
K OF C;

AND LET,

L BE TAKEN DOUBLE OF D,
M TRIPLE OF IT, AND
SUCCESSIVE MULTIPLES INCREASING BY ONE,

UNTIL,

WHAT IS TAKEN IS A MULTIPLE OF D, AND
THE FIRST THAT IS GREATER THAN K.

LET,

IT BE TAKEN,

AND LET IT,

BE N, WHICH IS QUADRUPLE OF D, AND
THE FIRST MULTIPLE OF IT THAT IS GREATER THAN K.

THEN, SINCE,

K IS LESS THAN N FIRST,

THEREFORE,

K IS NOT LESS THAN M.

AND, SINCE,

FG IS THE SAME MULTIPLE OF AE, THAT
GH IS OF EB,

[V. I] THEREFORE,

FG IS THE SAME MULTIPLE OF AE, THAT
FH IS OF AB.

BUT,

FG IS THE SAME MULTIPLE OF AE, THAT
K IS OF C;

THEREFORE,

FH IS THE SAME MULTIPLE OF AB, THAT
K IS OF C;

THEREFORE,

FH, K ARE EQUIMULTIPLES OF AB, C.

AGAIN, SINCE,

GH IS THE SAME MULTIPLE OF EB, THAT

K is of C, and
$EB = C$,

therefore,
$GH = K$.

But,
K is not less than M;

therefore,
neither is GH less than M.

And,
FG is greater than D;

therefore,
the whole, FH, is greater than D, M, together.

But,
D, M together are equal to N,

inasmuch as,
M is triple of D, and
M, D together are quadruple of D, while
N is, also, quadruple of D; whence
M, D together are equal to N.

But,
FH is greater than M, D;

therefore,
FH is in excess of N, while
K is not in excess of N. And,
FH, K are equimultiples of AB, C, while
N is another, chance, multiple of D;

[v. Def. 7] therefore,
AB has to D a greater ratio than C has to D.

I say next, that;
D, also, has to C, a greater ratio than D has to AB.

For, with,
the same construction,

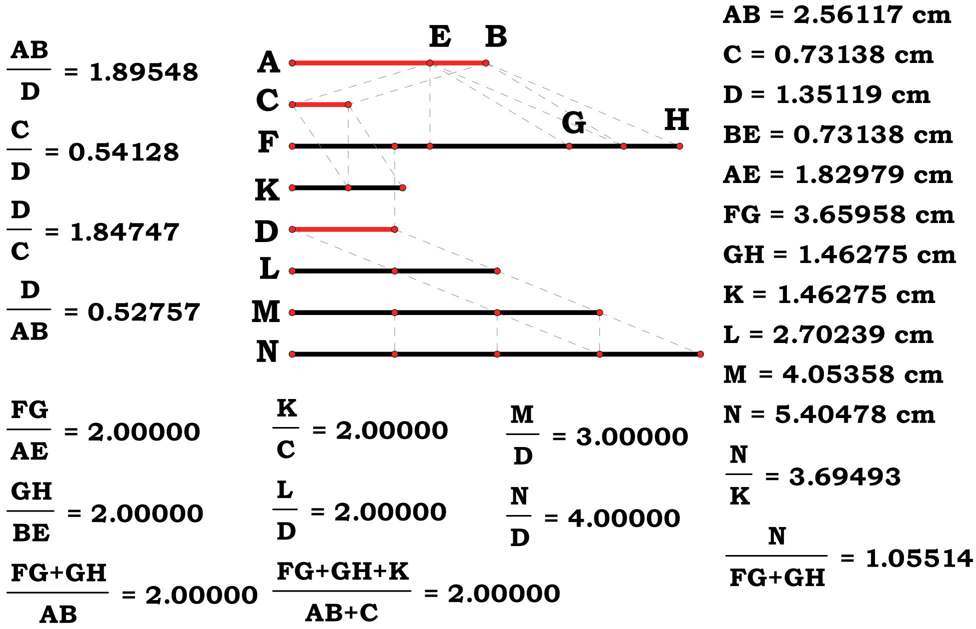
we can prove similarly, that;
N is in excess of K, while
N is not in excess of FH. And
N is a multiple of D, while
FH, K are other, chance, equimultiples of AB, C;

[v. Def. 7] therefore,
D has to C, a greater ratio than D has to AB.

[CASE 2.]

AGAIN, LET,

AE BE GREATER THAN *EB*.



[V. DEF. 4] THEN,

THE LESS, *EB*,
IF MULTIPLIED,
WILL SOMETIME
BE GREATER THAN *D*.

LET IT,

BE MULTIPLIED,

AND LET,

GH BE A MULTIPLE OF *EB*

AND,

GREATER THAN *D*; AND
WHATEVER MULTIPLE *GH* IS OF *EB*,

LET,

FG BE MADE THE SAME MULTIPLE OF *AE*, AND
K OF *C*.

THEN WE CAN PROVE SIMILARLY THAT;

FH, *K* ARE EQUIMULTIPLES OF *AB*, *C*,

AND, SIMILARLY, LET,

N BE TAKEN A MULTIPLE OF *D*,

BUT,

THE FIRST THAT IS GREATER THAN *FG*,

SO THAT,

FG IS AGAIN NOT LESS THAN *M*.

BUT,

GH IS GREATER THAN *D*;

THEREFORE,

THE WHOLE, *FH*, IS IN EXCESS OF *D*, *M*,
THAT IS, OF *N*.

NOW,

K IS NOT IN EXCESS OF *N*,
INASMUCH AS *FG*, ALSO,
WHICH IS GREATER THAN *GH*,
THAT IS, THAN *K*, IS NOT IN EXCESS OF *N*.

AND,

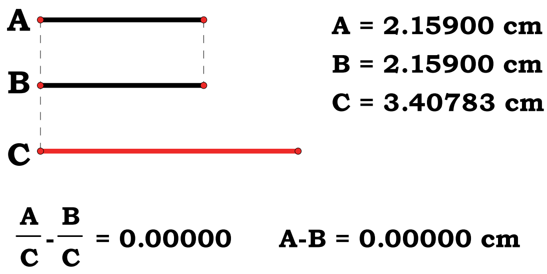
IN THE SAME MANNER,
BY FOLLOWING THE ABOVE ARGUMENT,
WE COMPLETE THE DEMONSTRATION.

THEREFORE ETC.

Q. E. D.

Proposition 9.

Magnitudes which have the same ratio to the same are equal to one another; and magnitudes to which the same has the same ratio are equal.



For let,

each, of the magnitudes, A, B, have
the same ratio to C;

I say that;

$A = B$.

[v. 8] For, otherwise,

each, of the magnitudes A, B,
would not have had the same ratio, to C,

but,

it has;

therefore,

$A = B$.

Again, let,

C have the same ratio to each, of
the magnitudes, A, B;

I say that;

$A = B$.

[v. 8] For, otherwise,

C would not have had
the same ratio to each, of the magnitudes A, B;

but,

it has;

therefore,

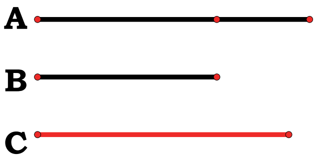
$A = B$.

Therefore etc.

Q. E. D.

Proposition 10.

Of magnitudes which have a ratio to the same, that which has a greater ratio is greater; and that to which the same has a greater ratio is less.



A = 3.59833 cm
B = 2.36538 cm
C = 3.32317 cm

$\frac{A}{C} = 1.08280 \quad \frac{B}{C} = 0.71178 \quad \frac{A}{B} = 1.52125$

For let,

A have to C, a greater ratio than B has to C;

I say that;

A is greater than B.

For, if not,

A is either equal to B or less.

[v. 7] Now,

A is not equal to B;

for in that case,

each, of the magnitudes, A, B, would have had
the same ratio, to C;

but,

they have not;

therefore,

A is not equal to B.

again,

Nor is A less than B;

[v. 8] for,

in that case
A would have had to C, a less ratio than B has to C;

but,

it has not;

therefore,

A is not less than B.

But,

it was proved not to be equal either;

therefore,

A is greater than B.

Again, let,

C have to B, a greater ratio than C has to A;

I say that;

B is less than A.

For,

if not, it is either equal or greater.

Now,

B is not equal to A;

[v. 7] for, in that case,

C would have had the same ratio to each, of the magnitudes, A, B;

but,

it has not;

therefore,

A is not equal to B.

Nor again,

is B greater than A;

[v. 8] for, in that case

C would have had to B, a less ratio than it has to A;

but,

it has not;

therefore,

B is not greater than A.

But,

it was proved that it is not equal either;

therefore,

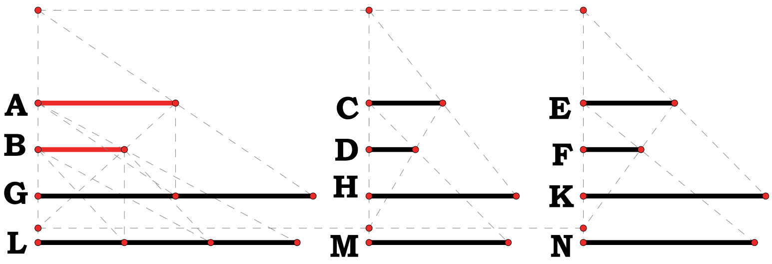
B is less than A.

Therefore etc.

Q. E. D.

PROPOSITION 11.

RATIOS WHICH ARE THE SAME WITH THE SAME RATIO ARE, ALSO, *THE SAME WITH ONE ANOTHER*.



A = 1.82033 cm
B = 1.14300 cm
G = 3.64067 cm
L = 3.42900 cm

C = 0.97367 cm
D = 0.61137 cm
H = 1.94733 cm
M = 1.83412 cm

E = 1.20650 cm
F = 0.75757 cm
K = 2.41300 cm
N = 2.27271 cm

$\frac{A}{B} = 1.59259$ $\frac{C}{D} = 1.59259$ $\frac{E}{F} = 1.59259$

$\frac{G}{A} = 2.00000$ $\frac{H}{C} = 2.00000$ $\frac{K}{E} = 2.00000$

$\frac{L}{B} = 3.00000$ $\frac{M}{D} = 3.00000$ $\frac{N}{F} = 3.00000$

$\frac{G}{L} = 1.06173$ $\frac{H}{M} = 1.06173$ $\frac{K}{N} = 1.06173$

FOR,
as *A* is to *B*,

so let,
C be to *D*, and,
as *C* is to *D*,

so let,
E be to *F*,

I say that;
as *A* is to *B*,
so is *E* to *F*.

FOR,
of *A*, *C*, *E*,

let,
equimultiples, *G*, *H*, *K*, be taken, and
of *B*, *D*, *F*, other, chance, equimultiples, *L*, *M*, *N*.

THEN SINCE,
as *A* is to *B*,
so is *C* to *D*, and
of *A*, *C* equimultiples *G*, *H* have been taken, and
of *B*, *D*, other, chance, equimultiples, *L*, *M*,

THEREFORE,

IF G IS IN EXCESS OF L,
H IS, ALSO, IN EXCESS OF M,
IF EQUAL, EQUAL, AND
IF LESS, LESS.

AGAIN, SINCE,

AS C IS TO D,
SO IS E TO F, AND
OF C, E, EQUIMULTIPLES, H, K, HAVE BEEN TAKEN, AND
OF D, F OTHER, CHANCE, EQUIMULTIPLES M, N

THEREFORE,

IF H IS IN EXCESS OF M,
K IS, ALSO, IN EXCESS OF N,
IF EQUAL, EQUAL, AND
IF LESS, LESS.

BUT WE SAW THAT,

IF H WAS IN EXCESS OF M,
G WAS, ALSO, IN EXCESS OF L;
IF EQUAL, EQUAL; AND
IF LESS, LESS;

SO THAT, IN ADDITION,

IF G IS IN EXCESS OF L,
K IS, ALSO, IN EXCESS OF N,
IF EQUAL, EQUAL, AND
IF LESS, LESS.

AND,

G, K ARE EQUIMULTIPLES OF A, E,

WHILE,

L, N ARE OTHER, CHANCE, EQUIMULTIPLES OF B, F;

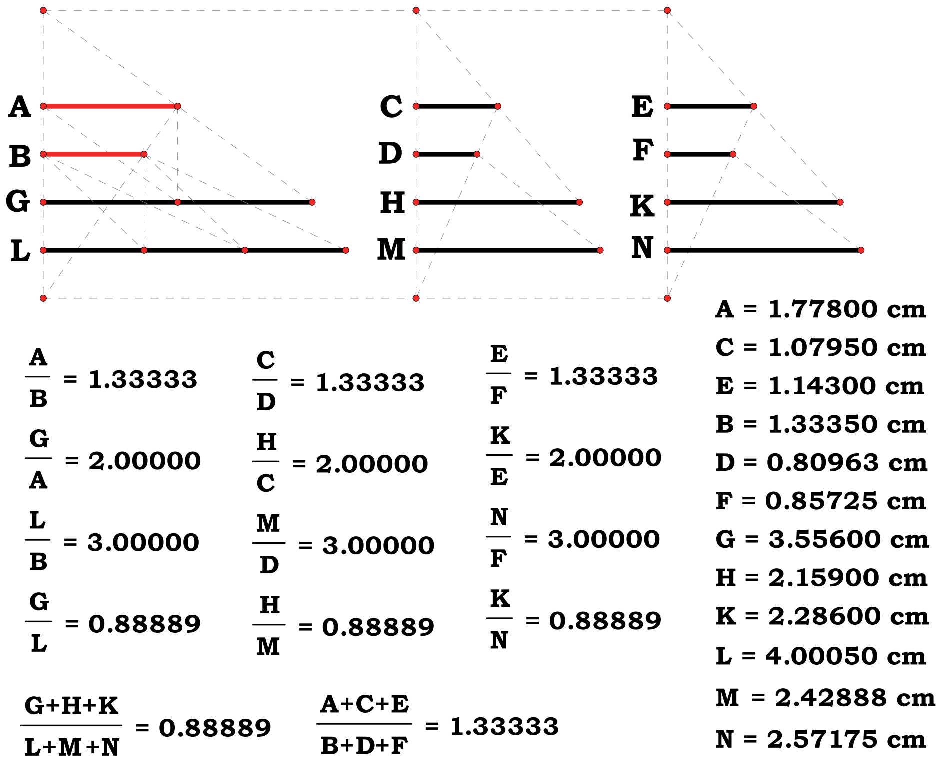
THEREFORE,

AS A IS TO B,
SO IS E TO F.

THEREFORE ETC.

Proposition 12.

If any number of magnitudes be proportional as one of the antecedents is to one of the consequents, so will all the antecedents be to all the consequents.



LET,

ANY NUMBER OF MAGNITUDES,
A, *B*, *C*, *D* , *E*, *F*, BE PROPORTIONAL,

SO THAT,

AS *A* IS TO *B*,
SO IS *C* TO *D*, AND
E TO *F*

I SAY THAT;

AS *A* IS TO *B*,
SO ARE *A*, *C*, *E*, TO *B*, *D*, *F*.

FOR,

OF *A*, *C*, *E*,

LET,

EQUIMULTIPLES, *G*, *H*, *K*, BE TAKEN, AND
OF *B*, *D*, *F*, OTHER, CHANCE, EQUIMULTIPLES, *L*, *M*, *N*.

THEN SINCE,

AS *A* IS TO *B*,
SO IS *C* TO *D*, AND
E TO F, AND,
OF *A*, *C*, *E*, EQUIMULTIPLES, *G*, *H*, *K*, HAVE BEEN TAKEN, AND

of B, D, F, other, chance, equimultiples, L, M, N,

therefore,

if G is in excess of L,
H is, also, in excess of M, and
K of N,
if equal, equal, and
if less, less;

so that, in addition,

if G is in excess of L,

then,

G, H, K, are in excess of L, M, N,
if equal, equal, and
if less, less.

Now,

G and G, H, K, are equimultiples of A and A, C, E,

[v. 1] since,

if any number of magnitudes
whatever are respectively equimultiples of
any magnitudes, equal in multitude, whatever
multiple one of the magnitudes is of one,
that multiple, also, will all be of all.

For the same reason,

L and L, M, N, are, also, equimultiples of B and B, D, F;

[v. Def. 5] therefore,

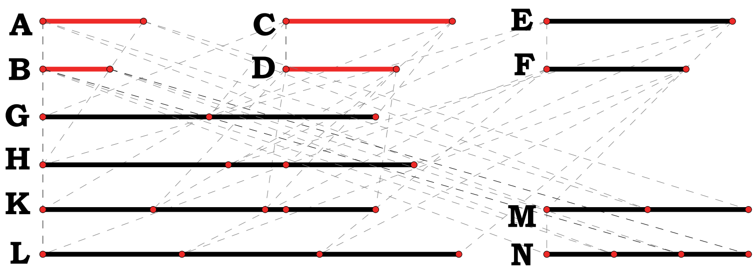
as A is to B,
so are A, C, E, to B, D, F.

Therefore etc.

Q. E. D.

Proposition 13.

If a first magnitude have to a second the same ratio as a third to a fourth, and the third have to the fourth a greater ratio than a fifth has to a sixth, the first will, also, *have to the second a greater ratio than the fifth to the sixth.*



A = 1.33350 cm
B = 0.88900 cm
C = 2.20133 cm
D = 1.46756 cm
E = 2.45533 cm
F = 1.83110 cm
G = 4.40267 cm
H = 4.91067 cm
K = 4.40267 cm
L = 5.49329 cm
M = 2.66700 cm
N = 2.66700 cm

$\frac{A}{B} = 1.50000$ $\frac{C}{D} = 1.50000$ $\frac{E}{F} = 1.34091$

$\frac{G}{C} = 2.00000$ $\frac{H}{E} = 2.00000$ $\frac{M}{A} = 2.00000$

$\frac{K}{D} = 3.00000$ $\frac{L}{F} = 3.00000$ $\frac{N}{B} = 3.00000$

$\frac{M}{N} = 1.00000$ $\frac{G}{K} = 1.00000$ $\frac{H}{L} = 0.89394$

$\frac{M+H}{A+E} = 2.00000$ $\frac{N+L}{B+F} = 3.00000$

For let,

a first magnitude, *A*, have to a second, *B*,
the same ratio as a third, *C*, has to a fourth, *D*,

and let,

the third, *C*, have to the fourth, *D*,
a greater ratio than a fifth, *E*, has to a sixth, *F*;

I say that;

the first, *A*, will, also, have to the second, *B*,
a greater ratio than the fifth, *E*, to the sixth, *F*.

For, since,

there are some equimultiples of *C*, *E*, and
of *D*, *F*, other, chance, equimultiples,

such that,

the multiple of *C* is in excess of the multiple of *D*,

[v. Def. 7] while,

the multiple of *E* is not in excess of the multiple of *F*,

let,

them be taken,

and let,

G, H, BE EQUIMULTIPLES OF C, E, AND
K, L OTHER, CHANCE, EQUIMULTIPLES OF D, F,

SO THAT;
G IS IN EXCESS OF K,

BUT,
H IS NOT IN EXCESS OF L;

AND, LET,
WHATEVER MULTIPLE G IS OF C,
M BE, ALSO, THAT MULTIPLE OF A,

AND, LET,
WHATEVER MULTIPLE K IS OF D,
N BE, ALSO, THAT MULTIPLE OF B.

NOW, SINCE,
AS A IS TO B,
SO IS C TO D, AND
OF A, C, EQUIMULTIPLES, M, G, HAVE BEEN TAKEN, AND
OF B, D, OTHER, CHANCE, EQUIMULTIPLES, N, K,

[V. DEF. 5] THEREFORE,
IF M IS IN EXCESS OF N,
G IS, ALSO, IN EXCESS OF K,
IF EQUAL, EQUAL, AND
IF LESS, LESS.

BUT,
G IS IN EXCESS OF K;

THEREFORE,
M IS, ALSO, IN EXCESS OF N.

BUT,
H IS NOT IN EXCESS OF L; AND
M, H ARE EQUIMULTIPLES OF A, E, AND
N, L, OTHER, CHANCE, EQUIMULTIPLES OF B, F;

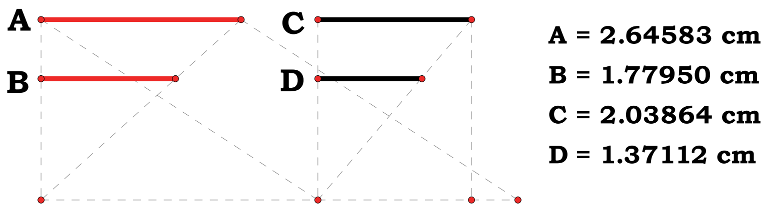
[V. DEF. 7] THEREFORE,
A HAS TO B, A GREATER RATIO THAN E HAS TO F.

THEREFORE ETC.

Q. E. D.

Proposition 14.

If a first magnitude have to a second the same ratio as a third has to a fourth, and the first be greater than the third, the second will, also, *be greater than the fourth; if equal, equal; and if less, less.*



$\frac{A}{B} = 1.48684 \quad \frac{C}{D} = 1.48684 \quad \frac{A}{C} = 1.29784 \quad \frac{B}{D} = 1.29784$

For let,

a first magnitude, *A*, have the same ratio
to a second, *B*, as a third, *C*, has to a fourth, *D*;

and let,

A be greater than *C*;

I say that;

B is, also, greater than *D*.

For, since,

A is greater than *C*, and
B is another, chance, magnitude,

[v. 8] therefore,

A has to *B*, a greater ratio than *C* has to *B*.

But,

as *A* is to *B*,
so is *C* to *D*;

[v. 13] therefore,

C has, also, to *D*, a greater ratio than *C* has to *B*.

[v. 10] But,

that to which the same has a greater ratio is less;

therefore,

D is less than *B*;

so that;

B is greater than *D*.

Similarly we can prove that,

if *A* be equal to *C*,
B will, also, be equal to *D*;

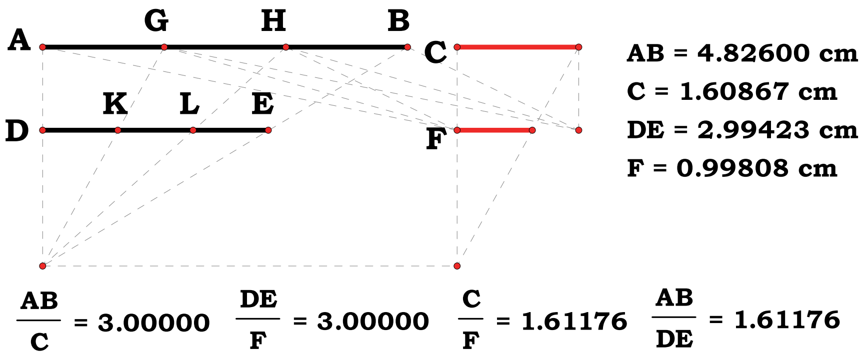
and,

if A be less than *C*,
B will, also, be less than *D*.

Therefore etc.

Proposition 15.

Parts have the same ratio as the same multiples of them taken in corresponding order.



For let,

AB be the same multiple of *C*,

that,

DE is of *F*;

I say that;

as *C* is to *F*,
so is *AB* to *DE*.

For, since,

AB is the same multiple of *C*, that
DE is of *F*,
as many magnitudes as there are in *AB* equal to *C*,
so many are there, also, in *DE* equal to *F*.

Let,

AB be divided into the magnitudes,
AG, *GH*, *HB*, equal to *C*, and
DE into the magnitudes,
DK, *KL*, *LE*, equal to *F*;

then,

the multitude of the magnitudes,
AG, *GH*, *HB*, will be equal to
the multitude of the magnitudes, *DK*, *KL*, *LE*.

And, since,

AG, *GH*, *HB*, are equal to one another, and
DK, *KL*, *LE*, are, also, equal to one another,

[v. 7] therefore,

as *AG* is to *DK*,
so is *GH* to *KL*, and
HB to *LE*.

[v. 12] Therefore,

AS ONE OF THE ANTECEDENTS IS TO
ONE OF THE CONSEQUENTS,
SO WILL ALL THE ANTECEDENTS BE TO
ALL THE CONSEQUENTS;

THEREFORE,
AS AG IS TO DK,
SO IS AB TO DE.

BUT,
$AG = C$, AND $DK = F$;

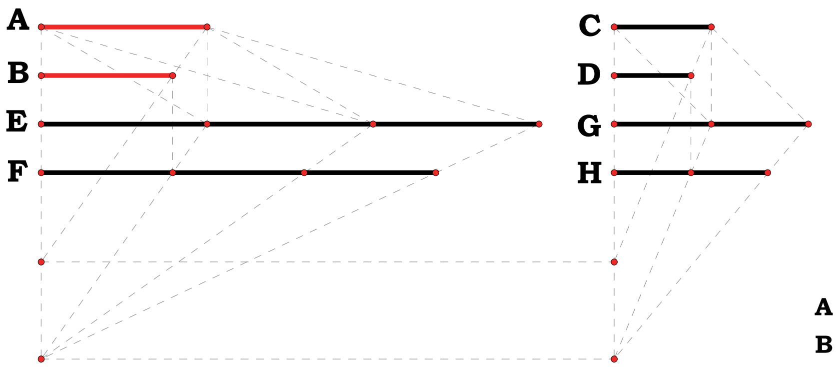
THEREFORE,
AS C IS TO F,
SO IS AB TO DE.

THEREFORE ETC.

Q. E. D.

Proposition 16.

If four magnitudes be proportional, they will, also, *be proportional alternately.*



$\frac{A}{B} = 1.26050$	$\frac{C}{D} = 1.26050$	$\frac{A}{C} = 1.70968$	$\frac{B}{D} = 1.70968$
$\frac{E}{A} = 3.00000$	$\frac{F}{B} = 3.00000$	$\frac{G}{C} = 2.00000$	$\frac{H}{D} = 2.00000$
$\frac{E}{F} = 1.26050$	$\frac{G}{H} = 1.26050$	$\frac{E}{G} = 2.56452$	$\frac{F}{H} = 2.56452$

A = 2.24367 cm
B = 1.77998 cm
C = 1.31233 cm
D = 1.04112 cm
E = 6.73100 cm
F = 5.33993 cm
G = 2.62467 cm
H = 2.08224 cm

Let,
A, *B*, *C*, *D*, be four proportional magnitudes,

so that,
as *A* is to *B*,
so is *C* to *D*;

I say that;
they will, also, be so alternately,

that is,
as *A* is to *C*,
so is *B* to *D*.

For let,
of *A*, *B*,
equimultiples, *E*, *F*, be taken, and
of *C*, *D*, other, chance, equimultiples, *G*, *H*.

[v. 15] Then, since,
E is the same multiple of *A*, that
F is of *B*, and
parts have the same ratio as
the same multiples of them,

therefore,
as *A* is to *B*,
so is *E* to *F*.

But,
as *A* is to *B*,

so is C to D;

[v. 11] therefore also,
as C is to D,
so is E to F.

Again, since,
G, H are equimultiples of C, D,

[v. 15] therefore,
as C is to D,
so is G to H.

But,
as C is to D
so is E to F;

[v. 11] therefore, also,
as E is to F,
so is G to H.

[v. 14] But,
if four magnitudes be proportional, and
the first be greater than the third,
the second will, also, be greater than the fourth;
if equal, equal; and
if less, less.

Therefore,
if E is in excess of G,
F is, also, in excess of H,
if equal, equal, and
if less, less.

Now,
E, F are equimultiples of A, B, and
G, H, other, chance, equimultiples of C, D;

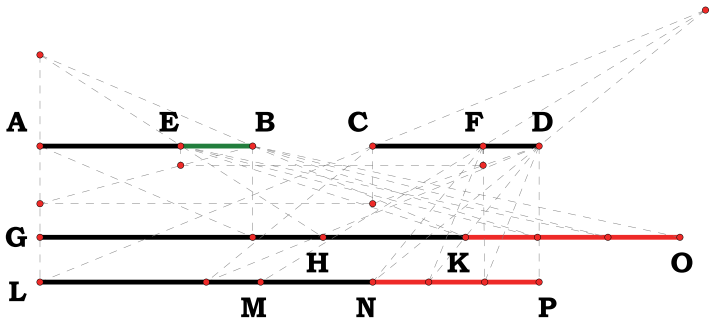
[v. Def. 5] therefore,
as A is to C,
so is B to D.

Therefore etc.

Q. E. D.

Proposition 17.

If magnitudes be proportional componendo, *they will*, also, *be proportional* separando.



AB = 2.81517 cm
BE = 0.94221 cm
CD = 2.20133 cm
DF = 0.73677 cm
AE = 1.87296 cm
CF = 1.46457 cm
GH = 3.74591 cm
HK = 1.88442 cm
LM = 2.92913 cm
MN = 1.47353 cm
KO = 2.82663 cm
NP = 2.21030 cm
HO = 4.71106 cm
MP = 3.68383 cm
GK = 5.63033 cm
LN = 4.40267 cm

$\frac{AB}{BE} = 2.98783$ $\frac{CD}{DF} = 2.98783$ $\frac{AE}{BE} = 1.98783$

$\frac{CF}{DF} = 1.98783$ $\frac{GH}{AE} = 2.00000$ $\frac{HK}{BE} = 2.00000$

$\frac{LM}{CF} = 2.00000$ $\frac{MN}{DF} = 2.00000$ $\frac{GK}{AB} = 2.00000$

$\frac{KO}{BE} = 3.00000$ $\frac{NP}{DF} = 3.00000$ $\frac{LN}{CD} = 2.00000$

$\frac{HO}{BE} = 5.00000$ $\frac{MP}{DF} = 5.00000$ $\frac{GK+LN}{AB+CD} = 2.00000$

$\frac{GK}{HO} = 1.19513$ $\frac{LN}{MP} = 1.19513$ $\frac{GH}{KO} = 1.32522$ $\frac{LM}{NP} = 1.32522$

Let,

AB, *BE*, *CD*, *DF*,

be magnitudes proportional *componendo*,

so that,

as *AB* is to *BE*,

so is *CD* to *DF*;

I say that;

they will, also, be proportional *separando*,

that is,

as *AE* is to *BE*,

so is *CF* to *DF*.

For let,

of *AE*, *BE*, *CF*, *DF*, equimultiples,

GH, *HK*, *LM*, *MN*, be taken, and

of *EB*, *FD*, other, chance, equimultiples, *KO*, *NP*,

Then, since,

GH is the same multiple of *AE*, that

HK is of *BE*,

[v. 1] therefore,

GH is the same multiple of *AE*, that
GK is of *AB*.

But,
GH is the same multiple of *AE*, that
LM is of *CF*;

therefore,
GK is the same multiple of *AB*, that
LM is of *CF*.

Again, since,
LM is the same multiple of *CF*, that
MN is of *DF*,

[v. 1] therefore,
LM is the same multiple of *CF*, that
LN is of *CD*.

But,
LM was the same multiple of *CF*, that
GK is of *AB*;

therefore,
GK is the same multiple of *AB*, that
LN is of *CD*.

Therefore,
GK, *LN* are equimultiples of *AB*, *CD*.

Again, since,
HK is the same multiple of *BE*, that
MN is of *DF*, and
KO is, also, the same multiple of *BE*, that
NP is of *DF*,

[v. 2] therefore,
the sum *HO* is, also, the same multiple of *BE*, that
MP is of *DF*,

And, since,
as *AB* is to *BE*,
so is *CD* to *DF*, and
of *AB*, *CD*, equimultiples, *GK*, *LN*, have been taken, and
of *BE*, *DF*, equimultiples, *HO*, *MP*,

therefore,
if *GK* is in excess of *HO*,
LN is, also, in excess of *MP*,
if equal, equal, and
if less, less.

LET,

GK BE IN EXCESS OF *HO*;

THEN,

IF *HK* BE SUBTRACTED FROM EACH,
GH IS, ALSO, IN EXCESS OF *KO*.

BUT WE SAW THAT,

IF *GK* WAS IN EXCESS OF *HO*,
LN WAS, ALSO, IN EXCESS OF *MP*;

THEREFORE,

LN IS, ALSO, IN EXCESS OF *MP*,

AND,

IF *MN* BE SUBTRACTED FROM EACH,
LM IS, ALSO, IN EXCESS OF *NP*;

SO THAT,

IF *GH* IS IN EXCESS OF *KO*,
LM IS, ALSO, IN EXCESS OF *NP*.

SIMILARLY WE CAN PROVE THAT,

IF *GH* BE EQUAL TO *KO*,
LM WILL, ALSO, BE EQUAL TO *NP*, AND
IF LESS, LESS.

AND,

GH, *LM* ARE EQUIMULTIPLES OF *AE*, *CF*, WHILE
KO, *NP* ARE OTHER, CHANCE, EQUIMULTIPLES OF *EB*, *FD*;

THEREFORE,

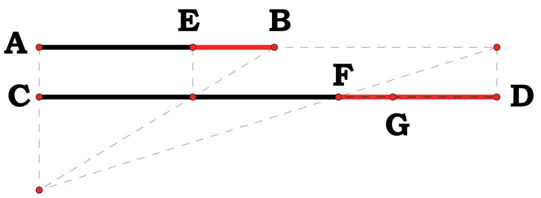
AS *AE* IS TO *BE*,
SO IS *CF* TO *DF*.

THEREFORE ETC.

Q. E. D.

Proposition 18.

If magnitudes be proportional separando, *they will*, also, *be proportional* componendo.



AE = 2.03200 cm
BE = 1.07950 cm
CF = 3.95817 cm
DF = 2.10278 cm

$$\frac{AE}{BE} = 1.88235 \quad \frac{CF}{DF} = 1.88235 \quad \frac{AE+BE}{BE} = 2.88235 \quad \frac{CF+DF}{DF} = 2.88235$$

Let,

AE, *EB*, *CF*, *FD*, be magnitudes proportional *separando*,

so that,

as *AE* is to *EB*,
so is *CF* to *FD*;

I say that;

they will, also, be proportional *componendo*,

that is,

as *AB* is to *BE*,
so is *CD* to *FD*.

For,

if *CD* be not to *DF* as *AB* to *BE*,

then,

as *AB* is to *BE*,
so will *CD* be

either,

to some magnitude less than *DF*, or
to a greater.

First, let,

it be in that ratio to a less magnitude *DG*.

[v. 17] Then, since,

as *AB* is to *BE*,
so is *CD* to *DG*,
they are magnitudes proportional *componendo*;

so that;

they will, also, be proportional *separando*.

Therefore,

as *AE* is to *EB*,
so is *CG* to *GD*.

BUT ALSO,

BY HYPOTHESIS,
AS *AE* IS TO *EB*,
SO IS *CF* TO *FD*.

[V. 11] THEREFORE ALSO,
AS *CG* IS TO *GD*,
SO IS *CF* TO *FD*.

BUT,
THE FIRST, *CG*, IS GREATER THAN THE THIRD, *CF*;

[V. 14] THEREFORE,
THE SECOND, *GD*, IS, ALSO, GREATER THAN THE FOURTH, *FD*.

BUT,
IT IS, ALSO, LESS:

WHICH,
IS IMPOSSIBLE.

THEREFORE,
AS *AB* IS TO *BE*,
SO IS NOT *CD* TO A LESS MAGNITUDE THAN *FD*.

SIMILARLY WE CAN PROVE,
THAT NEITHER IS IT IN THAT RATIO TO A GREATER;

THEREFORE,
IT IS IN THAT RATIO TO *FD* ITSELF.

THEREFORE ETC.

Q. E. D.

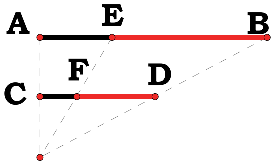
Proposition 19.

If, as a whole is to a whole, so is a part subtracted to a part subtracted, the remainder will, also, *be to the remainder as whole to whole.*

$\frac{AB}{CD} = 1.97368$

$\frac{AE}{CF} = 1.97368$

$\frac{BE}{DF} = 1.97368$



AB = 3.00567 cm
CD = 1.52287 cm
AE = 0.94283 cm
CF = 0.47770 cm
BE = 2.06283 cm
DF = 1.04517 cm

$\frac{AB}{AE} = 3.18791$ $\frac{CD}{CF} = 3.18791$ $\frac{BE}{AE} = 2.18791$ $\frac{DF}{CF} = 2.18791$

For, let,
as the whole, *AB*, is to the whole, *CD*,
the part, *AE*, subtracted, be to
the part, *CF*, subtracted;

I say that;
the remainder, *BE*, will, also, be
to the remainder, *DF*,
as the whole, *AB*, to the whole, *CD*.

[v. 16] For since,
as *AB* is to *CD*,
so is *AE* to *CF*,

alternately also,
as *AB* is to *AE*,
so is *CD* to *CF*.

[v. 17] And, since,
the magnitudes are proportional *componendo*,
they will, also, be proportional *separando*,

that is,
as *BE* is to *AE*,
so is *DF* to *CF*

[v. 16] and, alternately,
as *BE* is to *DF*,
so is *AE* to *CF*.

But,

as *AE* is to *CF*,
so by hypothesis, is the whole, *AB*, to the whole, *CD*.

[v. 11] Therefore also,
the remainder, *EB*, will be to the remainder, *DF*,
as the whole, *AB*, is to the whole, *CD*.

THEREFORE ETC.

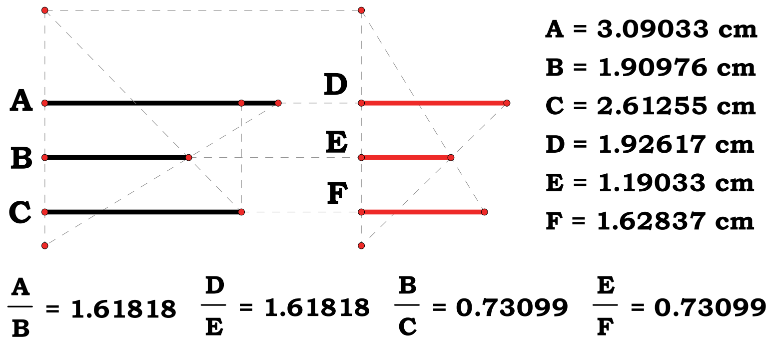
[PORISM.

FROM THIS IT IS MANIFEST THAT, IF MAGNITUDES BE PROPORTIONAL *COMPONENDO*, THEY WILL, ALSO, BE PROPORTIONAL *CONVERTENDO*.]

Q. *E. D.*

Proposition 20.

If there be three magnitudes, and others equal to them in multitude, which taken two and two are in the same ratio, and if ex aequali *the first be greater than the third, the fourth will,* also, *be greater than the sixth; if equal, equal; and, if less, less.*



Let there be,

three magnitudes, A, B, C, and

others, D, E, F, equal to them in multitude, which taken two and two are in the same ratio,

so that,

as A is to B,

so is D to E, and

as B is to C,

so is E to F

and let,

A be greater than C, *ex aequali*;

I say that;

D will, also, be greater than F;

if $A = C$, equal; and,

if less, less.

[v. 8] For, since,

A is greater than C, and

B is some other magnitude, and

the greater has to the same, a greater ratio than the less has,

therefore,

A has to B, a greater ratio than C has to B.

But,

as A is to B,

so is D to E,

and inversely,

as C is to B,

so is F to E;

[v. 13] therefore,

D has, also, to E, a greater ratio than F has to E.

[v. 10] But,

of magnitudes which have a ratio to the same,
that which has a greater ratio is greater;

therefore,

D is greater than F.

Similarly we can prove that;

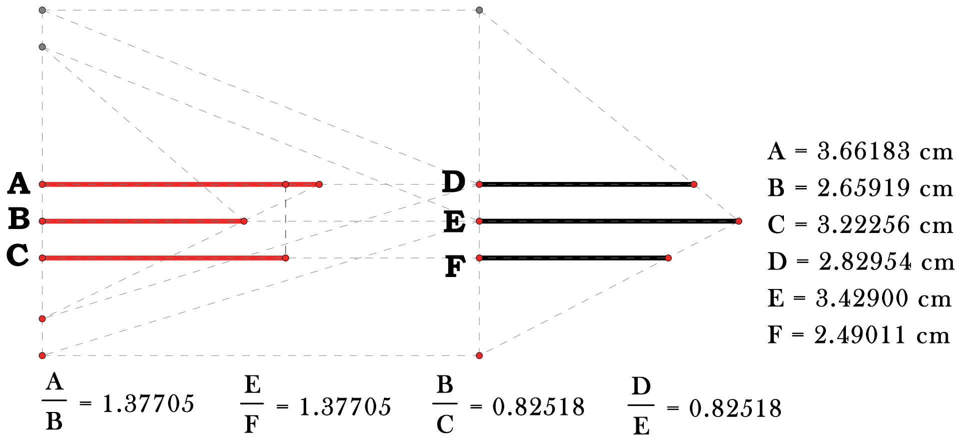
if A be equal to C,
D will, also, be equal to F; and,
if less, less.

Therefore etc.

Q. E. D.

PROPOSITION 21.

IF THERE BE THREE MAGNITUDES, AND OTHERS EQUAL TO THEM IN MULTITUDE, WHICH TAKEN TWO AND TWO TOGETHER ARE IN THE SAME RATIO, AND THE PROPORTION OF THEM BE PERTURBED, THEN, IF EX AEQUALI *THE FIRST MAGNITUDE IS GREATER THAN THE THIRD, THE FOURTH WILL,* ALSO, *BE GREATER THAN THE SIXTH; IF EQUAL, EQUAL; AND IF LESS, LESS.*



LET,

THERE BE THREE MAGNITUDES, A, B, C, AND
OTHERS, D, E, F, EQUAL TO THEM IN MULTITUDE, WHICH
TAKEN TWO AND TWO ARE IN THE SAME RATIO,

AND LET,

THE PROPORTION OF THEM BE PERTURBED,

SO THAT,

AS A IS TO B,
SO IS E TO F,

AND,

AS B IS TO C,
SO IS D TO E,

AND LET,

A BE GREATER THAN C, *EX AEQUALI*;

I SAY THAT;

D WILL, ALSO, BE GREATER THAN E;
IF $A = C$, EQUAL; AND
IF LESS, LESS.

FOR, SINCE,

A IS GREATER THAN C, AND
B IS SOME OTHER MAGNITUDE,

[V. 8] THEREFORE;

A HAS TO B, A GREATER RATIO THAN C HAS TO B.

BUT,

AS A IS TO B,
SO IS E TO F,

AND INVERSELY,

AS C IS TO B,
SO IS E TO D.

[V. 13] THEREFORE ALSO,

E HAS TO F, A GREATER RATIO THAN E HAS TO D.

[V. 10] BUT,

THAT TO WHICH THE SAME HAS A GREATER RATIO IS LESS;

THEREFORE,

F IS LESS THAN D;

THEREFORE,

D IS GREATER THAN F.

SIMILARLY WE CAN PROVE THAT,

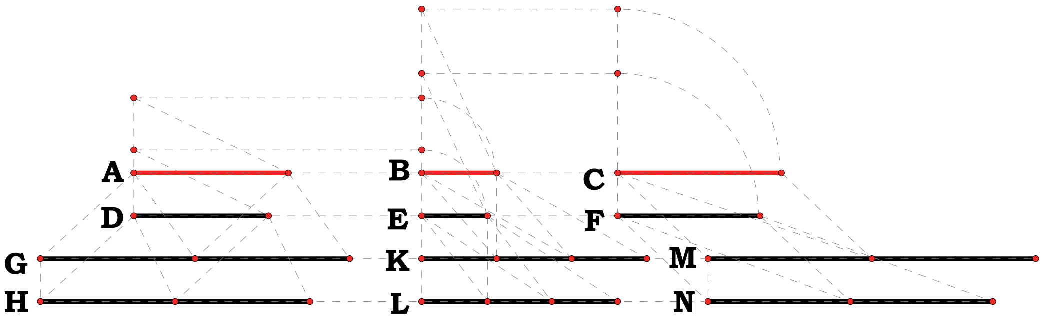
IF A BE EQUAL TO C,
D WILL, ALSO, BE EQUAL TO E;
AND IF LESS, LESS.

THEREFORE ETC.

Q. E. D.

PROPOSITION 22.

IF THERE BE ANY NUMBER OF MAGNITUDES WHATEVER, AND OTHERS EQUAL TO THEM IN MULTITUDE, WHICH TAKEN TWO AND TWO TOGETHER ARE IN THE SAME RATIO, THEY WILL, ALSO, *BE IN THE SAME RATIO* EX AEQUALI.



A = 2.13783 cm	D = 1.86267 cm	G = 4.27567 cm	H = 3.72533 cm
B = 1.03717 cm	E = 0.90367 cm	K = 3.11150 cm	L = 2.71101 cm
C = 2.26483 cm	F = 1.97332 cm	M = 4.52967 cm	N = 3.94664 cm

$\frac{A}{B} = 2.06122$	$\frac{D}{E} = 2.06122$	$\frac{B}{C} = 0.45794$	$\frac{E}{F} = 0.45794$	
$\frac{A}{C} = 0.94393$	$\frac{D}{F} = 0.94393$	$\frac{G}{A} = 2.00000$	$\frac{H}{D} = 2.00000$	
$\frac{K}{B} = 3.00000$	$\frac{L}{E} = 3.00000$	$\frac{M}{C} = 2.00000$	$\frac{N}{F} = 2.00000$	$\frac{G}{M} = 0.94393$
$\frac{G}{K} = 1.37415$	$\frac{H}{L} = 1.37415$	$\frac{K}{M} = 0.68692$	$\frac{L}{N} = 0.68692$	$\frac{H}{N} = 0.94393$

LET,

THERE BE ANY NUMBER OF MAGNITUDES *A*, *B*, *C*, AND OTHERS, *D*, *E*, *F*, EQUAL TO THEM IN MULTITUDE, WHICH TAKEN TWO AND TWO TOGETHER ARE IN THE SAME RATIO,

SO THAT,

AS *A* IS TO *B*,
SO IS *D* TO *E*, AND
AS *B* IS TO *C*,
SO IS *E* TO *F*;

I SAY THAT;

THEY WILL, ALSO, BE IN THE SAME RATIO *EX AEQUALI*,

< THAT IS,

AS *A* IS TO *C*,
SO IS *D* TO *F*. >

FOR LET ,

OF *A*, *D*,
EQUIMULTIPLES, *G*, *H*, BE TAKEN, AND
OF *B*, *E*, OTHER, CHANCE, EQUIMULTIPLES, *K*, *L*;

AND, FURTHER,

of C, F, other, chance, equimultiples, M, N.

Then, since,

as A is to B,
so is D to E, and
of A, D, equimultiples, G, H, have been taken, and
of B, E, other, chance, equimultiples, K, L,

[v. 4] therefore,

as G is to K,
so is H to L.

For the same reason also,

as K is to M,
so is L to N.

Since, then,

there are three magnitudes, G, K, M, and
others, H, L, N, equal to them in multitude, which
taken two and two together are in the same ratio,

[v. 20] therefore,

ex aequali, if G is in excess of M,
H is, also, in excess of N;
if equal, equal; and,
if less, less.

And,

G, H are equimultiples of A, D, and
M, N other, chance, equimultiples of C, F.

[v. Def. 5] Therefore,

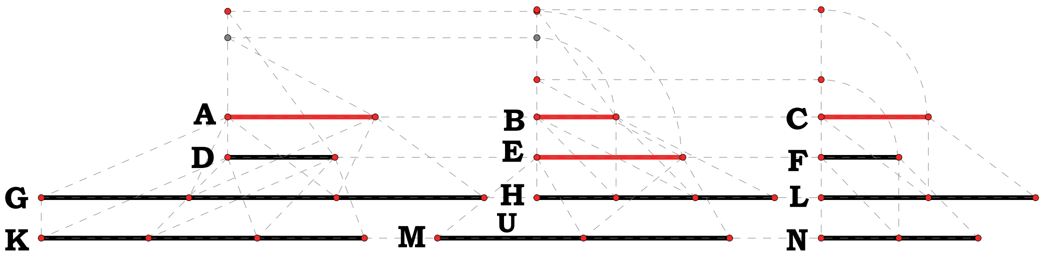
as A is to C,
so is D to F.

Therefore etc.

Q. E. D.

PROPOSITION 23.

IF THERE BE THREE MAGNITUDES, AND OTHERS EQUAL TO THEM IN MULTITUDE, WHICH TAKEN TWO AND TWO TOGETHER ARE IN THE SAME RATIO, AND THE PROPORTION OF THEM BE PERTURBED, THEY WILL, ALSO, *BE IN THE SAME RATIO* EX AEQUALI.



A = 2.01083 cm
B = 1.07950 cm
C = 1.46050 cm
D = 1.47062 cm
E = 1.98967 cm
F = 1.06814 cm
G = 6.03250 cm
H = 3.23850 cm
K = 4.41187 cm
L = 2.92100 cm
M = 3.97933 cm
N = 2.13627 cm

$\frac{A}{B} = 1.86275$
$\frac{E}{F} = 1.86275$
$\frac{B}{C} = 0.73913$
$\frac{D}{E} = 0.73913$
$\frac{A}{C} = 1.37681$
$\frac{D}{F} = 1.37681$

$\frac{G}{A} = 3.00000$
$\frac{H}{B} = 3.00000$
$\frac{K}{D} = 3.00000$
$\frac{L}{C} = 2.00000$
$\frac{M}{E} = 2.00000$
$\frac{N}{F} = 2.00000$

$\frac{G}{H} = 1.86275$
$\frac{M}{N} = 1.86275$
$\frac{D}{E} = 0.73913$
$\frac{B}{D} = 0.73404$
$\frac{C}{E} = 0.73404$
$\frac{H}{K} = 0.73404$

$\frac{L}{M} = 0.73404$
$\frac{H}{L} = 1.10870$
$\frac{K}{M} = 1.10870$
$\frac{G}{L} = 2.06522$
$\frac{K}{N} = 2.06522$

LET,

THERE BE THREE MAGNITUDES, *A*, *B*, *C*, AND
OTHERS EQUAL TO THEM IN MULTITUDE, WHICH
TAKEN TWO AND TWO TOGETHER,
ARE IN THE SAME PROPORTION,

NAMELY,

D, *E*, *F*,

AND LET,

THE PROPORTION OF THEM BE PERTURBED,

SO THAT,

AS *A* IS TO *B*,
SO IS *E* TO *F*, AND
AS *B* IS TO *C*,
SO IS *D* TO *E*;

I SAY THAT;

AS *A* IS TO *C*,
SO IS *D* TO *F*.

LET,

OF *A*, *B*, *D*, EQUIMULTIPLES, *G*, *H*, *K*, BE TAKEN, AND
OF *C*, *E*, *F*, OTHER, CHANCE, EQUIMULTIPLES, *L*, *M* *N*.

[V. 15] THEN, SINCE,
G, H ARE EQUIMULTIPLES OF A, B, AND
PARTS HAVE THE SAME RATIO AS
THE SAME MULTIPLES OF THEM,

THEREFORE,
AS A IS TO B,
SO IS G TO H.

FOR THE SAME REASON ALSO,
AS E IS TO F,
SO IS M TO N. AND
AS A IS TO B,
SO IS E TO F;

[V. 11] THEREFORE ALSO,
AS G IS TO H,
SO IS M TO N.

NEXT, SINCE,
AS B IS TO C,
SO IS D TO E,

[V. 16] ALTERNATELY, ALSO,
AS B IS TO D,
SO IS C TO E.

AND, SINCE,
H, K ARE EQUIMULTIPLES OF B, D, AND
PARTS HAVE THE SAME RATIO AS THEIR EQUIMULTIPLES,

[V. 15] THEREFORE,
AS B IS TO D,
SO IS H TO K.

BUT,
AS B IS TO D,
SO IS C TO E;

[V. 11] THEREFORE ALSO,
AS H IS TO K,
SO IS C TO E.

AGAIN, SINCE,
L, M ARE EQUIMULTIPLES OF C, E,

[V. 15] THEREFORE,
AS C IS TO E,
SO IS L TO M.

BUT,
AS C IS TO E,

so is H to K;

[v. 11] therefore also,
as H is to K,
so is L to M,

[v. 16] and, alternately,
as H is to L,
so is K to M.

But it was, also, proved that,
as G is to H,
so is M to N.

Since, then,
there are three magnitudes, G, H, L, and,
others equal to them in multitude, K, M, N, which taken two and two together are in the same ratio, and the proportion of them is perturbed,

[v. 21] therefore,
ex aequali, if G is in excess of L,
K is, also, in excess of N;
if equal, equal; and,
if less, less. And
G, K are equimultiples of A, D, and
L, N of C, F.

Therefore,
as A is to C,
so is D to F.

Therefore etc.

Q. E. D.

Proposition 24:

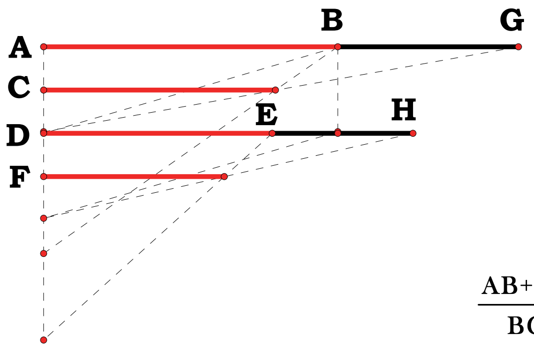
If a first magnitude have to a second the same ratio as a third has to a fourth, and, also, a fifth have to the second the same ratio as a sixth to the fourth, the first and fifth added together will have to the second the same ratio as the third and sixth have to the fourth:

$\frac{AB}{C} = 1.26471$

$\frac{DE}{F} = 1.26471$

$\frac{BG}{C} = 0.77828$

$\frac{EH}{F} = 0.77828$



AB = 3.89467 cm
C = 3.07950 cm
DE = 3.02683 cm
F = 2.39331 cm
BG = 2.39672 cm
EH = 1.86267 cm

$\frac{AB+BG}{BG} = 2.62500$

$\frac{AB+BG}{C} = 2.04299$ $\quad \frac{AB}{BG} = 1.62500$ $\quad \frac{DE}{EH} = 1.62500$ $\quad \frac{DE+EH}{EH} = 2.62500$

$\frac{DE+EH}{F} = 2.04299$

Let,
 a first magnitude,
 AB, have to a second, *C*, the same ratio as
 a third, *DE*, has to a fourth, *F*;

and let,
 also a fifth, *BG*, have to the second, *C*,
 the same ratio as a sixth, *EH*, has to the fourth, *F*;

I say that;
 the first and fifth added together, *AG*, will have to
 the second, *C*, the same ratio as
 the third and sixth, *DH*, has to the fourth, *F*:

For since,
 as *BG* is to *C*,
 so is *EH* to *F*,

inversely,
 as *C* is to *BG*,
 so is *F* to *EH*:

Since, then,
 as *AB* is to *C*,
 so is *DE* to *F*, and
 as *C* is to *BG*,
 so is *F* to *EH*,

[v. 22] therefore,
 ex aequali,

as AB is to BG,
so is DE to EH.

[v. 18] And, since,
the magnitudes are proportional *separando*,
they will, also, be proportional *componendo*;

therefore,
as AG is to GB,
so is DH to HE.

But also,
as BG is to C,
so is EH to F;

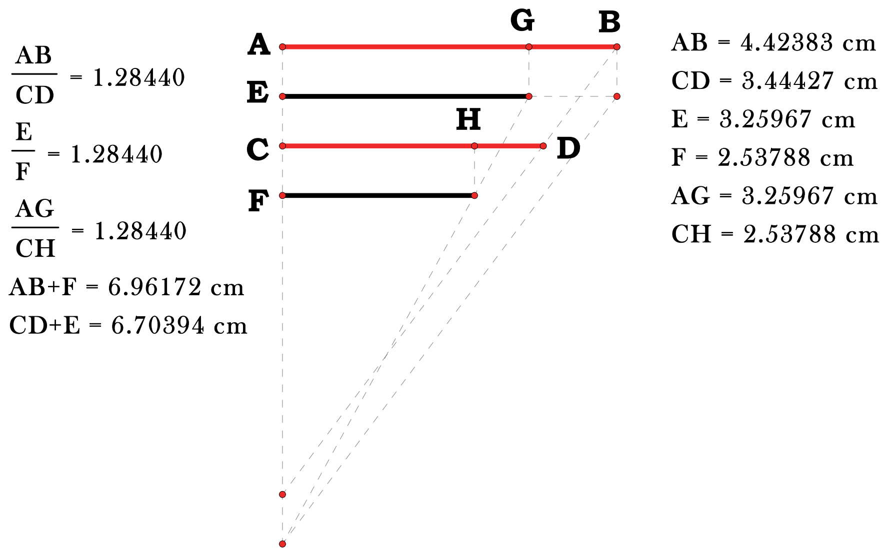
[v. 22] therefore,
ex aequali,
as AG is to C,
so is DH to F:

Therefore etc:,

Q. E. D.

Proposition 25:

If four magnitudes be proportional, the greatest and the least are greater than the remaining two:



Let,

the four magnitudes,

AB, *CD*, *E*, *F*, be proportional

so that,

as *AB* is to *CD*,

so is *E* to *F*,

and let,

AB be the greatest of them, and

F the least;

I say that;

AB, *F* are greater than *CD*, *E*.

For let,

AG be made equal to *E*, and

CH equal to *F*.

Since, as,

AB is to *CD*,

so is *E* to *F*, and

E = *AG*, and

F to *CH*,

therefore,

as *AB* is to *CD*,

so is *AG* to *CH*.

[v. 19] And since,

as the whole, *AB*, is to the whole, *CD*,

so is the part, AG, subtracted to
the part, CH, subtracted,
the remainder, GB, will, also, be to the remainder, HD,
as the whole, AB, is to the whole, CD.

But,
AB is greater than CD;

therefore,
GB is, also, greater than HD.

And, since,
$AG = E$, and
CH to F,

therefore,
AG, F are equal to CH, E.

And,
if, GB, HD being unequal, and
GB greater,
AG, F be added to GB and CH,
E be added to HD,

it follows that,
AB, F are greater than CD, E.

Therefore etc:,

Q. E. D.

BOOK VI.

OF

EUCLID'S ELEMENTS

TRANSLATED FROM THE TEXT OF HEIBERG

BY

SIR THOMAS L. HEATH,

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2013 *EDITION*

REVISED WITH SUBTRACTIONS

REFORMATTED AND ABRIDGED FOR STUDY OF *THE ELEMENTS*.

BY JOHN CLARK.

BOOK VI.

DEFINITIONS.

1. **SIMILAR RECTILINEAL FIGURES** ARE SUCH AS HAVE THEIR ANGLES SEVERALLY EQUAL AND THE SIDES ABOUT THE EQUAL ANGLES PROPORTIONAL.

[2. **RECIPROCALLY RELATED** FIGURES. SEE NOTE.]

3. A STRAIGHT LINE IS SAID TO HAVE BEEN **CUT IN EXTREME AND MEAN RATIO** WHEN, AS THE WHOLE LINE IS TO THE GREATER SEGMENT, SO IS THE GREATER TO THE LESS.

4. THE **HEIGHT** OF ANY FIGURE IS THE PERPENDICULAR DRAWN FROM THE VERTEX TO THE BASE.

Note.

Definition 1. Similar rectilineal figures *are such as have their angles severally equal and the sides about the equal angles proportional.*

Note.

Definition 2. Simson proposes in his note to substitute the following definition. "Two magnitudes are said to be reciprocally proportional to two others when one of the first is to one of the other magnitudes as the remaining one of the last two is to the remaining one of the first."

This definition requires that the magnitudes shall be all of the same kind.

NOTE.

DEFINITION 3. *A STRAIGHT LINE IS SAID TO HAVE BEEN* CUT IN EXTREME AND MEAN RATIO *WHEN*, *AS THE WHOLE LINE IS TO THE GREATER SEGMENT*, *SO IS THE GREATER TO THE LESS*.

NOTE.

DEFINITION 4. *The* HEIGHT *of any figure is the perpendicular drawn from the vertex to the base.*

NOTE.

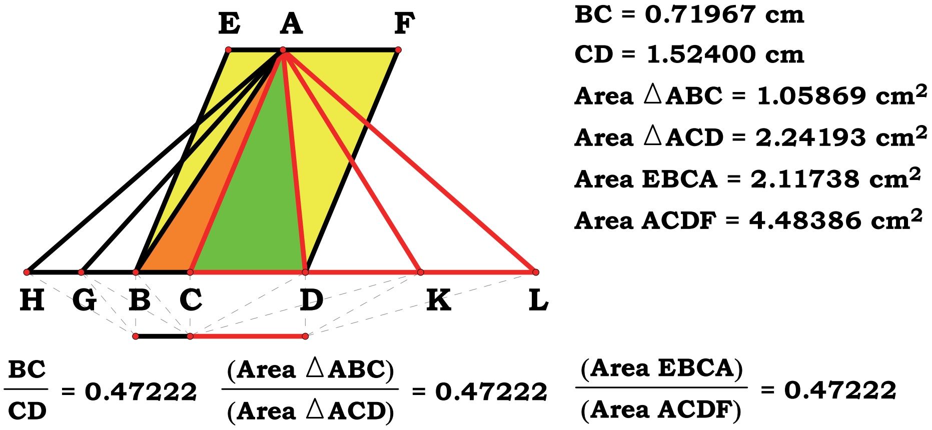
[**DEFINITION 5.** "*A RATIO IS SAID TO BE* COMPOUNDED *OF RATIOS WHEN THE SIZES OF THE RATIOS MULTIPLIED TOGETHER MAKE SOME* (? *RATIO*, *OR SIZE*)."]

BOOK VI.

PROPOSITIONS.

PROPOSITION 1.

TRIANGLES AND PARALLELOGRAMS WHICH ARE UNDER THE SAME HEIGHT ARE TO ONE ANOTHER AS THEIR BASES.



LET,

ΔABC, ΔACD,

AND,

⊟EC, ⊟CF, BE UNDER THE SAME HEIGHT;

I SAY THAT;

BC, IS TO *CD*,

SO IS Δ*ABC*, TO Δ*ACD*, AND ⊟ *EC*, TO ⊟ *CF*.

FOR LET,

BD BE PRODUCED IN BOTH DIRECTIONS TO *H*, *L*,

AND LET,

[ANY NUMBER OF] *BG*, *GH* BE MADE EQUAL TO *BC*, AND ANY NUMBER OF *DK*, *KL*, EQUAL TO *CD*;

LET,

AG, *AH*, *AK*, *AL*, BE JOINED.

[I. 38] THEN, SINCE,

CB, *BG*, *GH* ARE EQUAL TO ONE ANOTHER,

Δ*ABC*, Δ*AGB*, Δ*AHG*, ARE, ALSO, EQUAL TO ONE ANOTHER.

THEREFORE,

WHATEVER MULTIPLE *HC*, IS OF *BC*,

THAT MULTIPLE, ALSO, IS Δ*AHC*, OF Δ*ABC*.

FOR THE SAME REASON,

WHATEVER MULTIPLE *LC*, IS OF *CD*,

THAT MULTIPLE, ALSO, IS ΔALC, OF ΔACD;

[I. 38] AND,

IF THE BASES, $HC = CL$,

$\Delta AHC = \Delta ACL$,

IF THE BASE, HC, IS IN EXCESS OF THE BASE, CL,

ΔAHC, IS, ALSO, IN EXCESS OF ΔACL, AND IF LESS, LESS.

THUS,

THERE BEING FOUR MAGNITUDES,

BC, CD, AND ΔABC, ΔACD,

EQUIMULTIPLES HAVE BEEN TAKEN OF BC, AND ΔABC,

NAMELY,

HC, AND ΔAHC, AND CD, AND ΔADC,

OTHER, CHANCE, EQUIMULTIPLES,

NAMELY,

LC, AND ΔALC; AND,

IT HAS BEEN PROVED THAT,

IF HC, IS IN EXCESS OF CL,

ΔAHC, IS, ALSO, IN EXCESS OF ΔALC,

IF EQUAL, EQUAL; AND, IF LESS, LESS.

[V. DEF. 5] THEREFORE,

AS BC, IS TO CD,

SO IS ΔABC, TO ΔACD.

[I. 41] NEXT, SINCE,

$\boxminus EC = 2\Delta ABC$, AND $\boxminus FC = 2\Delta ACD$,

[V. 15] WHILE,

PARTS HAVE THE SAME RATIO AS

THE SAME MULTIPLES OF THEM,

THEREFORE,

AS ΔABC, IS TO ΔACD,

SO IS $\boxminus EC$, TO $\boxminus FC$.

SINCE, THEN, IT WAS PROVED THAT;

AS BC IS TO CD,

SO IS ΔABC, TO ΔACD, AND

AS ΔABC IS TO ΔACD,

so is ⊟EC, to ⊟CF,

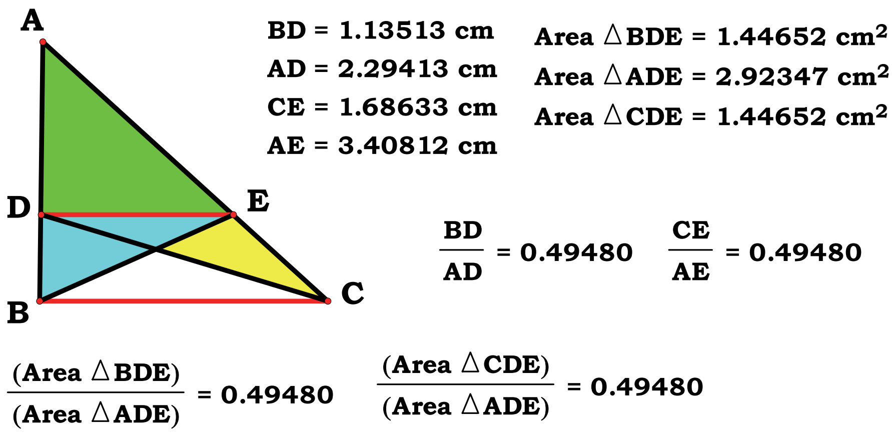
[v. 11] therefore also,
as BC, is to CD,
so is ⊟ EC, to ⊟FC.

Therefore etc.

Q. E. D.

Proposition 2.

If a straight line be drawn parallel to one of the sides of a triangle, it will cut the sides of the triangle proportionally; and, if the sides of the triangle be cut proportionally, the line joining the points of section will be parallel to the remaining side of the triangle.



For let,

DE be drawn parallel to *BC*,

one of the sides of ΔABC;

I say that;

as *BD* is to *DA*,
so is *CE* to *EA*.

For let,

BE, *CD* be joined.

Therefore,

$\Delta BDE = \Delta CDE$;

[I. 38] for,

they are on the same base, *DE*, and
in the same parallels, *DE*, *BC*. And,

ΔADE, is another area.

[V. 7] But,

equals have the same ratio to the same;

therefore,

as ΔBDE, is to ΔADE,

so is ΔCDE, to ΔADE.

But,

as ΔBDE, is to *ADE*,
so is *BD* to *DA*;

[VI. 1] FOR,

BEING UNDER THE SAME HEIGHT,
THE PERPENDICULAR DRAWN FROM E TO AB,
THEY ARE TO ONE ANOTHER AS THEIR BASES,

FOR THE SAME REASON ALSO,

AS ΔCDE, IS TO ADE,

SO IS CE TO EA.

[V. 11]

THEREFORE ALSO,

AS BD IS TO DA,
SO IS CE TO EA.

AGAIN, LET,

THE SIDES, AB, AC OF

ΔABC, BE CUT PROPORTIONALLY,

SO THAT,

AS BD IS TO DA,
SO IS CE TO EA;

AND LET,

DE, BE JOINED.

I SAY THAT;

$DE \parallel BC$.

FOR,

WITH THE SAME CONSTRUCTION,

SINCE, AS,

BD IS TO DA,
SO IS CE TO EA,

[VI. 1] BUT,

AS BD IS TO DA,

SO IS ΔBDE, TO ΔADE, AND

AS CE IS TO EA,

SO IS ΔCDE, TO ΔADE,

[V. 11] THEREFORE ALSO,

AS ΔBDE, IS TO ΔADE,

SO IS ΔCDE, TO ΔADE.

THEREFORE,

EACH, OF ΔBDE, ΔCDE, HAS THE SAME RATIO TO ΔADE.

[V. 9] THEREFORE,

$\Delta BDE = \Delta CDE$; AND

THEY ARE ON THE SAME BASE, DE.

[I. 39] BUT,

EQUAL TRIANGLES, WHICH ARE ON THE SAME BASE,
ARE, ALSO, IN THE SAME PARALLELS.

THEREFORE,

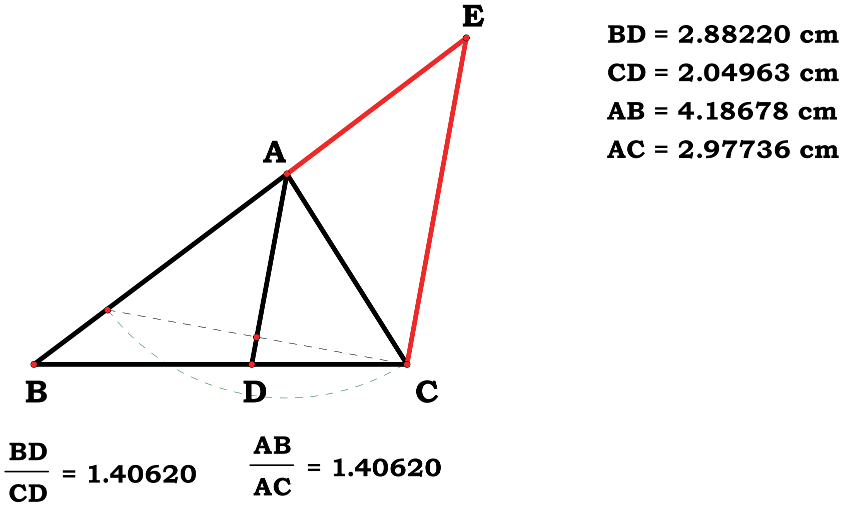
$DE \parallel BC$.

THEREFORE ETC.

Q. E. D.

Proposition 3.

If an angle of a triangle be bisected and the straight line cutting the angle cut the base also, the segments of the base will have the same ratio as the remaining sides of the triangle; and, if the segments of the base have the same ratio as the remaining sides of the triangle, the straight line joined from the vertex to the point of section will bisect the angle of the triangle.



Let,

ABC, be a triangle,

and let,

$\angle BAC$, be bisected by the straight line, AD;

I say that;

as BD is to CD,

so is BA to AC.

For let,

CE be drawn, through C, parallel to DA,

and let,

BA be carried through and meet it at E.

[I. 29] Then, since,

AC, falls upon the parallels, AD, EC,

$\angle ACE = \angle CAD$.

But, by hypothesis,

$\angle CAD = \angle BAD$;

therefore,

$\angle BAD = \angle ACE$.

[I. 29] Again, since,

THE BAE, FALLS UPON THE PARALLELS, AD, EC,
THE EXTERIOR $\angle BAD = \angle AEC$, THE INTERIOR .

BUT,

$\angle ACE = \angle BAD$;

THEREFORE,

$\angle ACE = \angle AEC$,

[I. 6] SO THAT,
THE SIDES, $AE = AC$.

AND, SINCE,

$AD \parallel EC$, ONE OF THE SIDES OF ΔBCE,

THEREFORE, PROPORTIONALLY,
AS BD IS TO DC,
SO IS BA TO AE.

[VI. 2] BUT,
$AE = AC$;

THEREFORE,
AS BD IS TO DC,
SO IS BA TO AC.

AGAIN, LET,
BA BE TO AC,
AS BD TO DC,

AND LET,
AD BE JOINED;

I SAY THAT;

$\angle BAC$, HAS BEEN BISECTED BY AD.

FOR,
WITH THE SAME CONSTRUCTION, SINCE,
AS BD IS TO DC,
SO IS BA TO AC, AND ALSO
AS BD IS TO DC,
SO IS BA TO AE

[VI. 2] FOR,

$AD \parallel EC$, ONE OF THE SIDES OF ΔBCE:

[V. 11] THEREFORE ALSO,
AS BA IS TO AC,
SO IS BA TO AE.

[V. 9] THEREFORE,

$AC = AE$,

[I. 5]

SO THAT,

$\angle AEC = \angle ACE$.

[I. 29] BUT,

$\angle AEC = \angle BAD$, THE EXTERIOR ANGLE,

[*ID.*] AND,

$\angle ACE = \angle CAD$, THE ALTERNATE ANGLE;

THEREFORE,

$\angle BAD = \angle CAD$.

THEREFORE,

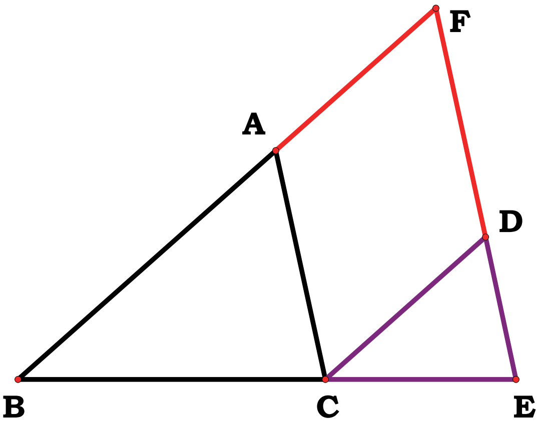
$\angle BAC$, HAS BEEN BISECTED BY AD.

THEREFORE ETC.

Q. E. D.

Proposition 4.

In equiangular triangles the sides about the equal angles are proportional, and those are corresponding sides which subtend the equal angles.



AB = 4.55797 cm
AF = 2.82499 cm
BC = 4.06400 cm
CE = 2.51883 cm
CD = 2.82499 cm
DF = 3.09714 cm
DE = 1.91958 cm
AC = 3.09714 cm

$\frac{AB}{AF} = 1.61345$ $\quad \frac{BC}{CE} = 1.61345$ $\quad \frac{AB}{CD} = 1.61345$ $\quad \frac{DF}{DE} = 1.61345$

$\frac{AC}{DE} = 1.61345$ $\quad \frac{AB}{BC} = 1.12155$ $\quad \frac{CD}{CE} = 1.12155$

$\frac{AB}{AC} = 1.47167$ $\quad \frac{CD}{DE} = 1.47167$

Let,

ΔABC, ΔDCE be equiangular, having

$\angle ABC = \angle DCE$, $\angle BAC = \angle CDE$, and $\angle ACB$, to $\angle CED$;

I say that;

in ΔABC, ΔDCE, the sides about
the equal angles are proportional, and
those are corresponding sides which subtend
the equal angles.

For let,

BC be collinear with CE.

[I. 17] Then, since,

$\angle ABC$, $\angle ACB$,

are less than two right angles, and

$\angle ACB = \angle DEC$,

therefore,

$\angle ABC$, $\angle DEC$, are less than two right angles;

[I. Post. 5] therefore,

BA, ED, when produced, will meet.

LET,

THEM BE PRODUCED AND MEET AT F.

[I. 28] NOW, SINCE,

$\angle DCE = \angle ABC$, $BF \parallel CD$.

[I. 28] AGAIN, SINCE,

$\angle ACB = \angle DEC$, $AC \parallel FE$.

THEREFORE,

$FACD$ IS A PARALLELOGRAM;

[I. 34] THEREFORE,

$FA = DC$, AND AC TO FD.

AND, SINCE,

$AC \parallel FE$, ONE SIDE OF ΔFBE,

[VI. 2] THEREFORE,

AS BA IS TO AF,
SO IS BC TO CE.

BUT,

$AF = CD$;

THEREFORE,

AS BA IS TO CD,
SO IS BC TO CE,

[V. 16]

AND ALTERNATELY,

AS AB IS TO BC,
SO IS DC TO CE.

AGAIN, SINCE,

$CD \parallel BF$,

[VI. 2] THEREFORE,

AS BC IS TO CE,
SO IS FD TO DE.

BUT,

$FD = AC$;

THEREFORE,

AS BC IS TO CE,
SO IS AC TO DE,

[V. 16]

AND ALTERNATELY,

AS BC IS TO CA,

so is *CE* to *ED*.

Since then,

it was proved that,
as *AB* is to *BC*,
so is *DC* to *CE*, and
as *BC* is to *CA*,
so is *CE* to *ED*;

[v. 22]

therefore,

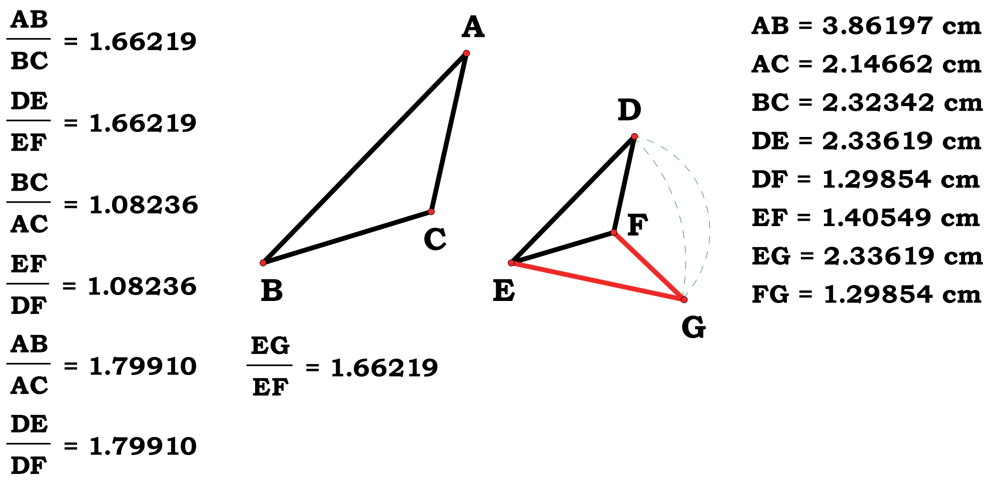
ex aequali,
as *BA* is to *AC*,
so is *CD* to *DE*.

Therefore etc.

Q. E. D.

Proposition 5.

If two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend.



Let,

ΔABC, ΔDEF, have their sides proportional,

so that,

as AB is to BC,
so is DE to EF,
as BC is to CA,
so is EF to FD, and further
as BA is to AC,
so is ED to DF;

I say that;

ΔABC is equiangular with ΔDEF, and

they will have those angles equal which
the corresponding sides subtend,

namely,

$\angle ABC = \angle DEF$, $\angle BCA = \angle EFD$, and $\angle BAC = \angle EDF$.

For,

on EF,

[I. 23] and let,

at E, F, on it, there be constructed

$\angle FEG = \angle ABC$, and $\angle EFG = \angle ACB$;

[I. 32] therefore, the remaining

$\angle$at $A = \measuredangle$at G.

Therefore,

ΔABC, IS EQUIANGULAR WITH ΔGEF.

[VI. 4] THEREFORE,

ΔABC, ΔGEF, THE SIDES ABOUT
THE EQUAL ANGLES ARE PROPORTIONAL, AND
THOSE ARE CORRESPONDING SIDES WHICH SUBTEND
THE EQUAL ANGLES;

THEREFORE,
AS AB IS TO BC,
SO IS GE TO EF.

BUT,
AS AB IS TO BC,
SO, BY HYPOTHESIS, IS DE TO EF;

[V. 11] THEREFORE,
AS DE IS TO EF,
SO IS GE TO EF.

THEREFORE,
EACH, DE, GE, HAS THE SAME RATIO TO EF;

[V. 9] THEREFORE,
$DE = GE$.

FOR THE SAME REASON,
$DF = GF$.

SINCE THEN,
$DE = EG$, AND EF IS COMMON,
THE TWO SIDES, DE, EF, ARE EQUAL TO
THE TWO SIDES, GE, EF; AND
THE BASES, $DF = FG$;

[I. 8] THEREFORE,

$\angle DEF = \angle GEF$, AND $\Delta DEF = \Delta GEF$, AND
THE REMAINING ANGLES ARE EQUAL TO
THE REMAINING ANGLES,

[I. 4] NAMELY,
THOSE WHICH THE EQUAL SIDES SUBTEND.

THEREFORE,

$\angle DFE = \angle GFE$, AND $\angle EDF = \angle EGF$.

AND, SINCE,

$\angle FED = \angle GEF$, WHILE,

$\angle GEF = \angle ABC$,

THEREFORE,

$\angle ABC = \angle DEF$.

FOR THE SAME REASON,

$\angle ACB = \angle DFE$,

AND FURTHER,

$\angle$AT A, TO $\angle$AT D;

THEREFORE,

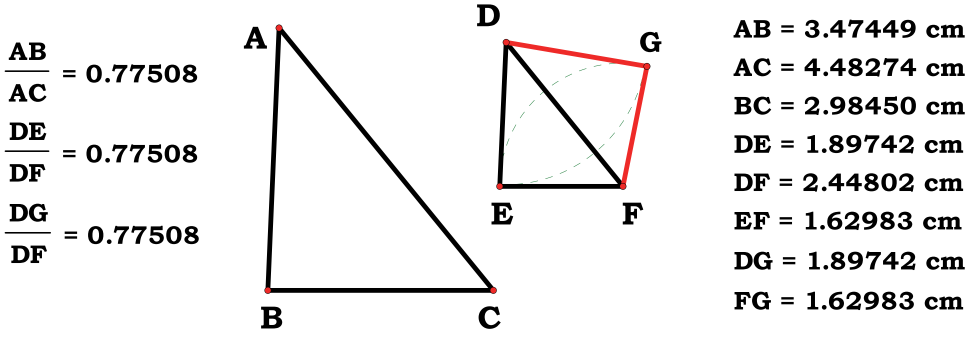
ΔABC, IS EQUIANGULAR WITH ΔDEF.

THEREFORE ETC.

Q. E. D.

Proposition 6.

If two triangles have one angle equal to one angle and the sides about the equal angles proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend.



Let,

ΔABC, ΔDEF, have $\angle BAC = \angle EDF$, and,

the sides about the equal angles proportional,

so that,

as BA is to AC,
so is ED to DF;

I say that;

ΔABC, is equiangular with

ΔDEF, and will have

$\angle ABC = \angle DEF$, and $\angle ACB = \angle DFE$.

[I. 23] For let,

on DF, and at D, F, on it, there be constructed

$\angle FDG = \angle BAC$, or $\angle EDF$, and $\angle DFG = \angle ACB$;

[I. 32] therefore, the remaining

$\angle$at $B = \angle$at G.

Therefore,

ΔABC, is equiangular with ΔDGF.

[VI. 4] Therefore, proportionally,

as BA is to AC,
so is GD to DF.

But, by hypothesis, also,

as BA is to AC,
so, is ED to DF;

[V. 11] therefore also,

as ED is to DF,
so is GD to DF.

[v. 9] Therefore,
$ED = DG$; and DF is common;

therefore,

ED, DF, are equal to GD, DF; and $\angle EDF = \angle GDF$;

[i. 4] therefore,

the bases, $EF = GF$, and $\Delta DEF = \Delta DGF$, and
the remaining angles will be equal to
the remaining angles,

namely,
those which the equal sides subtend.

Therefore,

$\angle DFG = \angle DFE$, and $\angle DGF = \angle DEF$.

But,

$\angle DFG = \angle ACB$; therefore,

$\angle ACB = \angle DFE$.

And, by hypothesis,

$\angle BAC = \angle EDF$;

[i. 32] therefore, the remaining

$\angle$at $B = \angle$at E;

therefore,

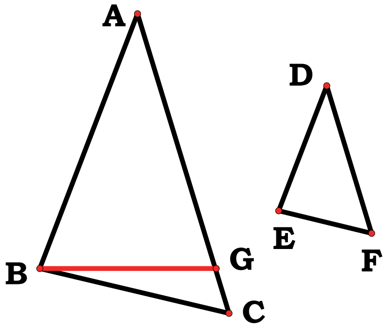
ΔABC, is equiangular with ΔDEF.

Therefore etc.

Q. E. D.

Proposition 7.

If two triangles have one angle equal to one angle, the sides about other angles proportional, and the remaining angles either both less or both not less than a right angle, the triangles will be equiangular and will have those angles equal, the sides about which are proportional.



Let,

ΔABC, $\measuredangle DEF$, have one angle equal to one angle,

$\angle BAC$, to $\angle EDF$,

the sides about other $\angle ABC$, $\angle DEF$, proportional,

so that,

as AB is to BC,

so is DE to EF,

and, first,

each, of the remaining angles, at C, F,

less than a right angle;

I say that;

ΔABC, is equiangular with ΔDEF,

$\angle ABC = \angle DEF$, and the remaining angles, namely,

$\angle$at $C = \angle$at F.

For,

if $\angle ABC \neq \angle DEF$, one of them is greater.

Let,

$\angle ABC$, be greater; and on AB, and at B, on it,

[I. 23] let,

$\angle ABG$ be constructed equal to $\angle DEF$.

Then, since,

$\angle A = D$, and

$\angle ABG$, to $\angle DEF$,

[I. 32] therefore, the remaining

$\angle AGB = \angle DFE$.

THEREFORE,

ΔABG, IS EQUIANGULAR WITH ΔDEF.

[VI. 4] THEREFORE,
AS AB IS TO BG,
SO IS DE TO EF.

BUT,
AS DE IS TO EF, SO BY HYPOTHESIS IS
AB TO BC;

[V. 11] THEREFORE,
AB HAS THE SAME RATIO TO EACH, OF BC, BG;

[V. 9] THEREFORE,
$BC = BG$,

[I. 5] SO THAT,

$\angle$AT $C = \angle BGC$.

BUT, BY HYPOTHESIS,

$\angle$AT C, IS LESS THAN A RIGHT ANGLE;

THEREFORE,

$\angle BGC$, IS, ALSO, LESS THAN A RIGHT ANGLE;

[I. 13] SO THAT,

$\angle AGB$, ADJACENT TO IT, IS GREATER THAN

A RIGHT ANGLE. AND,

$\angle AGB = \angle$AT F;

THEREFORE,

$\angle$AT F, IS, ALSO, GREATER THAN A RIGHT ANGLE.

BUT, BY HYPOTHESIS,
IT IS LESS THAN A RIGHT ANGLE: WHICH,
IS ABSURD.

THEREFORE,

$\angle ABC = \angle DEF$;

BUT,

$\angle$AT $A = \angle$AT D;

[I. 32] THEREFORE, THE REMAINING

$\angle$AT $C = \angle$AT F.

THEREFORE,

ΔABC, IS EQUIANGULAR WITH ΔDBF.

BUT, AGAIN, LET,

EACH, OF THE ANGLES, AT C, F,
BE SUPPOSED NOT LESS THAN A RIGHT ANGLE;

I SAY, AGAIN, THAT;

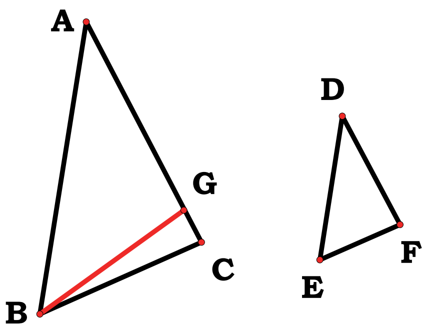
IN THIS CASE TOO,

ΔABC, IS EQUIANGULAR WITH

ΔDBF.

FOR,

WITH THE SAME CONSTRUCTION,
WE CAN PROVE SIMILARLY THAT;
$BC = BG$;



[I. 5] SO THAT,

$\angle$AT $C = \angle BGC$.

BUT,

$\angle$AT C, IS NOT LESS

THAN A RIGHT ANGLE;

THEREFORE,

NEITHER IS $\angle BGC$, LESS THAN A RIGHT ANGLE.

[I. 17] THUS,

IN ΔBGC,

THE TWO ANGLES ARE NOT LESS THAN TWO RIGHT ANGLES:

WHICH,

IS IMPOSSIBLE.

THEREFORE, ONCE MORE,

$\angle ABC = \angle DEF$;

BUT,

$\angle$AT $A = \angle$AT D;

[I. 32] THEREFORE, THE REMAINING

$\angle$AT $C = \angle$AT F.

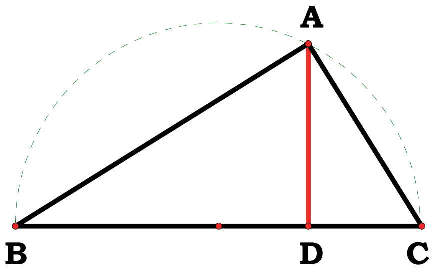
THEREFORE,

ΔABC, IS EQUIANGULAR WITH ΔDEF.

THEREFORE ETC.

Q. E. D.

PROPOSITION 8.



IF, IN A RIGHT-ANGLED TRIANGLE, A PERPENDICULAR BE DRAWN FROM THE RIGHT ANGLE TO THE BASE, THE TRIANGLES ADJOINING THE PERPENDICULAR ARE SIMILAR BOTH TO THE WHOLE AND TO ONE ANOTHER.

LET,

⊿ABC, BE RIGHT-ANGLED HAVING ∟BAC, RIGHT,

AND LET,

$AD \perp BC$;

I SAY THAT;

EACH, OF ⊿ABD, ⊿ADC, IS SIMILAR TO ⊿ABC,

AND, FURTHER,

THEY ARE SIMILAR TO ONE ANOTHER.

FOR, SINCE,

∟BAC = ∟ADB, FOR EACH IS RIGHT, AND

∠AT B, IS COMMON TO ⊿ABC AND ⊿ABD,

[I. 32] THEREFORE, THE REMAINING

∠ACB = ∠BAD;

THEREFORE,

ΔABC, IS EQUIANGULAR WITH ΔABD.

THEREFORE,

AS BC,

WHICH SUBTENDS THE RIGHT ANGLE IN ⊿ ABC,

IS TO BA,

WHICH SUBTENDS THE RIGHT ANGLE IN ∠ ABD,

SO IS AB, ITSELF, WHICH SUBTENDS ∠AT C, IN ⊿ ABC,

TO BD, WHICH SUBTENDS ∠ BAD IN ⊿ABD,

[VI. 4] AND SO ALSO,

IS AC TO AD,

WHICH SUBTENDS ∠AT B, COMMON TO THE TWO TRIANGLES.

THEREFORE,

ΔABC, IS BOTH EQUIANGULAR TO ΔABD,

AND,

HAS THE SIDES ABOUT THE EQUAL ANGLES PROPORTIONAL.

[VI. DEF. 1] THEREFORE,

ΔABC, IS SIMILAR TO ΔABD.

SIMILARLY WE CAN PROVE THAT,

ΔABC, IS, ALSO, SIMILAR TO ΔADC;

THEREFORE,

EACH, OF $\triangle ABD$, $\triangle ADC$, IS SIMILAR TO $\triangle ABC$.

I SAY NEXT THAT;

$\triangle ABD$, $\triangle ADC$, ARE, ALSO, SIMILAR TO ONE ANOTHER.

FOR, SINCE,

$\llcorner BDA = \llcorner ADC$,

AND, MOREOVER,

$\angle BAD = \angle$AT C,

[I. 32] THEREFORE, THE REMAINING

$\angle$AT $B = \angle DAC$;

THEREFORE,

ΔABD, IS EQUIANGULAR WITH ΔADC.

[VI. 4]

THEREFORE,

AS BD,

WHICH SUBTENDS $\angle BAD$, IN ΔABD,

IS TO DA,

WHICH SUBTENDS $\angle$AT C, IN ΔADC, $= \angle BAD$,

SO IS AD, ITSELF, WHICH SUBTENDS $\angle$AT B, IN ΔABD,

TO DC, WHICH SUBTENDS $\angle DAC$, IN ΔADC, $= \angle$AT B,

AND SO ALSO,

IS BA TO AC,

THESE SIDES SUBTENDING THE RIGHT ANGLES;

[VI. DEF. 1] THEREFORE,

ΔABD, IS SIMILAR TO ΔADC.

THEREFORE ETC.

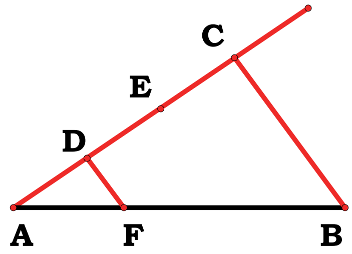
PORISM.

FROM THIS IT IS CLEAR THAT, IF IN A RIGHT-ANGLED TRIANGLE A PERPENDICULAR BE DRAWN FROM THE RIGHT ANGLE TO THE BASE, THE STRAIGHT LINE SO DRAWN IS A MEAN PROPORTIONAL BETWEEN THE SEGMENTS OF THE BASE.

Q. E. D.

PROPOSITION 9.

FROM A GIVEN STRAIGHT LINE TO CUT OFF A PRESCRIBED PART.



LET,
AB BE GIVEN;

THUS IT IS REQUIRED, TO CUT OFF,
FROM AB, A PRESCRIBED PART.

LET,
THE THIRD PART BE THAT PRESCRIBED.

LET,
AC, BE DRAWN THROUGH FROM A,
CONTAINING, WITH AB, ANY ANGLE;

LET, AT RANDOM
D, BE TAKEN, ON AC,

[I. 3] AND LET,
DE, EC BE MADE EQUAL TO AD.

LET,
BC BE JOINED,

[I. 31] AND LET,
THROUGH D,
DF, BE DRAWN PARALLEL TO IT.

THEN, SINCE,
$FD \parallel BC$, ONE OF THE SIDES OF ΔABC,

[VI. 2] THEREFORE, PROPORTIONALLY,
AS CD IS TO DA,
SO IS BF TO FA.

BUT,
CD IS DOUBLE OF DA;

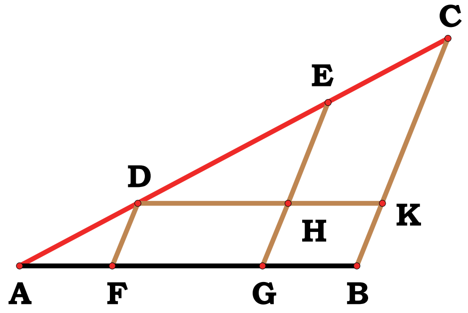
THEREFORE,
BF IS, ALSO, DOUBLE OF FA;

THEREFORE,
BA IS TRIPLE OF AF.

THEREFORE,
FROM AB,
THE PRESCRIBED THIRD PART, AF, HAS BEEN CUT OFF.

Q. E. F.

PROPOSITION 10.



TO CUT A GIVEN UNCUT STRAIGHT LINE SIMILARLY TO A GIVEN CUT STRAIGHT LINE.

LET,

AB BE GIVEN UNCUT,

AND

AC, CUT AT D, E;

AND LET,

THEM BE SO PLACED AS TO CONTAIN ANY ANGLE;

LET,

CB, BE JOINED,

AND LET,

THROUGH D, E, DF, EG, $\parallel$ BC,

[I. 31] AND LET,

THROUGH D, DHK, $\parallel$ AB.

THEREFORE,

EACH, OF THE FIGURES, ⊟FH, ⊟HB, IS A PARALLELOGRAM;

[I. 34] THEREFORE,

$DH = FG$ AND $HK = GB$.

NOW, SINCE,

HE, $\parallel$ KC, ONE OF THE SIDES OF ΔDKC,

[VI. 2] THEREFORE, PROPORTIONALLY,

AS CE IS TO ED,

SO IS KH TO HD.

BUT,

$KH = BG$, AND, $HD = GF$;

THEREFORE,

AS CE IS TO ED,

SO IS BG TO GF.

AGAIN, SINCE,

$FD \parallel GE$, ONE OF THE SIDES OF ΔAGE,

[VI. 2] THEREFORE, PROPORTIONALLY,

AS ED IS TO DA,

SO IS GF TO FA.

BUT,

IT WAS, ALSO, PROVED THAT,
AS CE IS TO ED,
SO IS BG TO GF;

THEREFORE,
AS CE IS TO ED,
SO IS BG TO GF,

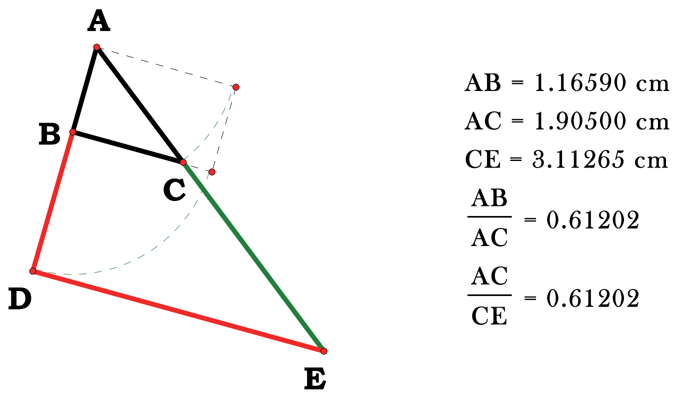
AND,
AS ED IS TO DA,
SO IS GF TO FA.

THEREFORE,
AB, GIVEN UNCUT, HAS BEEN CUT SIMILARLY TO
THE GIVEN CUT AC.

Q. E. F.

PROPOSITION 11.

TO TWO GIVEN STRAIGHT LINES TO FIND A THIRD PROPORTIONAL.



LET,

BA, *AC* BE
THE TWO GIVEN STRAIGHT LINES,

AND LET,

THEM BE PLACED SO AS TO CONTAIN ANY ANGLE;

THUS IT IS REQUIRED,

TO FIND A THIRD PROPORTIONAL, TO *BA*, *AC*.

FOR LET,

THEM BE PRODUCED TO THE POINTS, *D*, *E*,

[I. 3] AND LET,

BD BE MADE EQUAL TO *AC*;

LET,

BC BE JOINED,

[I. 3]

AND LET,

THROUGH *D*,
DE, BE DRAWN PARALLEL TO IT.

SINCE, THEN,

$BC \parallel DE$, ONE OF THE SIDES OF ΔADE,

[VI. 2] PROPORTIONALLY,

AS *AB* IS TO *BD*,
SO IS *AC* TO *CE*. BUT,
$BD = AC$;

THEREFORE,

AS *AB* IS TO *AC*,
SO IS *AC* TO *CE*.

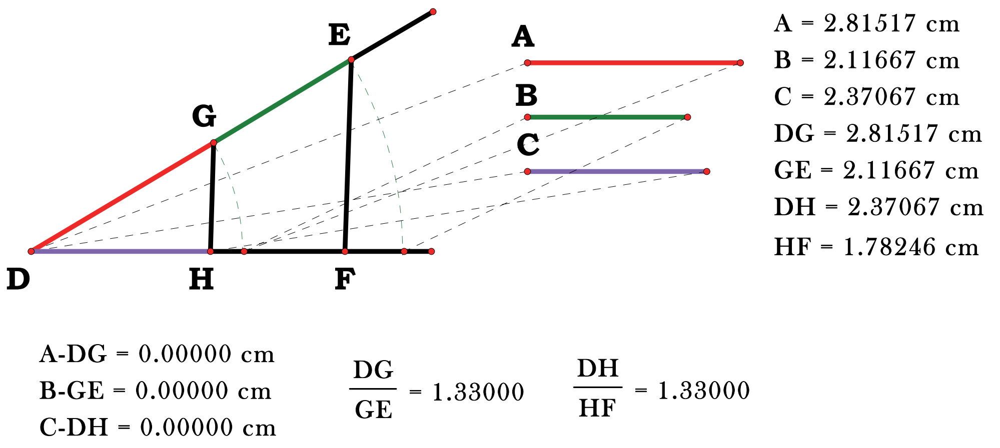
THEREFORE,

TO TWO GIVEN STRAIGHT LINES, AB, AC,
A THIRD PROPORTIONAL TO THEM, CE, HAS BEEN FOUND.

Q. E. F.

Proposition 12.

To three given straight lines to find a fourth proportional.



LET,

A, *B*, *C*, BE THE THREE GIVEN STRAIGHT LINES;

THUS IT IS REQUIRED,

TO FIND A FOURTH PROPORTIONAL, TO *A*, *B*, *C*.

LET,

TWO STRAIGHT LINES, *DE*, *DF*,

BE SET OUT CONTAINING ANY $\angle EDF$;

LET,

$DG = A$,

$GE = B$, AND FURTHER,

$DH = C$;

LET,

GH BE JOINED,

[I. 31] AND LET,

EF, BE DRAWN, THROUGH *E*, PARALLEL TO IT.

SINCE, THEN,

$GH \parallel EF$, ONE OF THE SIDES OF ΔDEF,

[VI. 2]

THEREFORE,

AS *DG* IS TO *GE*,

SO IS *DH* TO *HF*.

BUT,

$DG = A$,

GE TO *B*, AND,

DH TO *C*;

THEREFORE,

AS A IS TO B,
SO IS C TO HF.

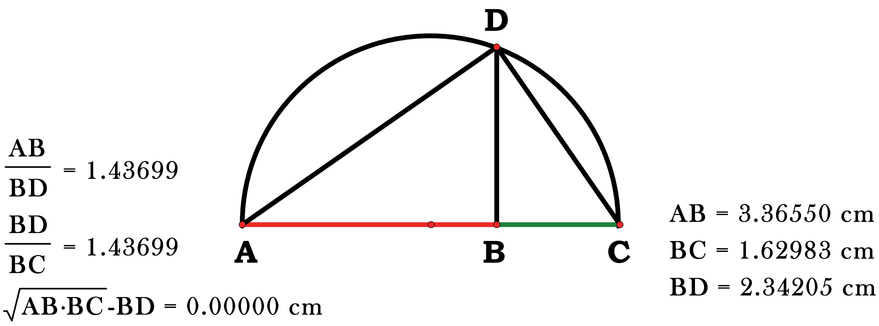
THEREFORE,

GIVEN A, B, C,
A FOURTH PROPORTIONAL, HF, HAS BEEN FOUND.

Q. E. F

Proposition 13.

To two given straight lines to find a mean proportional.



Let,

AB, *BC* be the two given straight lines;

thus it is required,

to find a mean proportional, to *AB*, *BC*.

Let,

them be placed in a straight line,

and let,

the semicircle, *ADC*, be described, on *AC*.

let,

BD, be drawn from the point, *B*,
at right angles to the straight line, *AC*,

and let,

AD, *DC*, be joined.

[iii. 31] Since,

∟*ADC*, is an angle in a semicircle,

it is right.

And, since,

in the right-angled ⊿*ADC*,

DB has been drawn from the right angle,
perpendicular to the base,

[vi. 8, Por.]

therefore,

DB is a mean proportional between
the segments of the base, *AB*, *BC*,

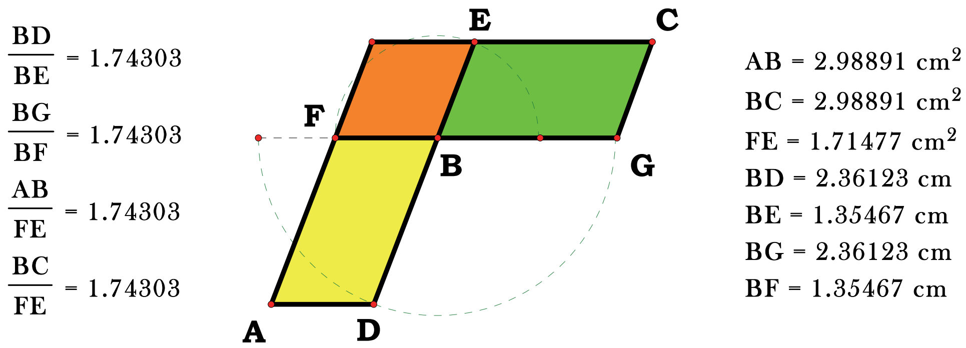
Therefore,

to the two given straight lines, *AB*, *BC*,
a mean proportional, *DB*, has been found.

Q. E. F.

Proposition 14.

In equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional; and equiangular parallelograms in which the sides about the equal angles are reciprocally proportional are equal.



Let,

AB, *BC* be equal, and
equiangular parallelograms
having the angles, at *B*, equal,

and let,

DB, *BE*, be placed in a straight line;

[I. 14]

therefore,

FB, *BG* are, also, in a straight line.

I say that;

in *AB*, *BC*, the sides about
the equal angles are reciprocally proportional,

that is to say, that;

as *DB* is to *BE*,
so is *GB* to *BF*.

For let,

the parallelogram, *FE*, be completed.

Since, then,

⊟*AB* = ⊟*BC*, and *FE* is another area,

[V. 7] therefore,

as *AB* is to *FE*,
so is *BC* to *FE*.

[VI. 1] But,

as *AB* is to *FE*,
so is *DB* to *BE*,

[*Id.*] and,

as BC is to FE,
so is GB to BF.

[v. 11] therefore also,
as DB is to BE,
so is GB to BF.

Therefore,

in $\boxminus AB$, $\boxminus BC$, the sides about

the equal angles are reciprocally proportional.

Next, let,
GB be to BF,
as DB to BE;

I say that;

$\boxminus AB = \boxminus BC$.

For since,
as DB is to BE,
so GB is to BF,

[vi. 1] while,
as DB is to BE,

so is $\boxminus AB$, to $\boxminus FE$,

[vi. 1] and,
as GB is to BF,

so is $\boxminus BC$, to $\boxminus FE$,

[v. 11]

therefore also,
as AB is to FE,
so is BC to FE;

[v. 9]

therefore,

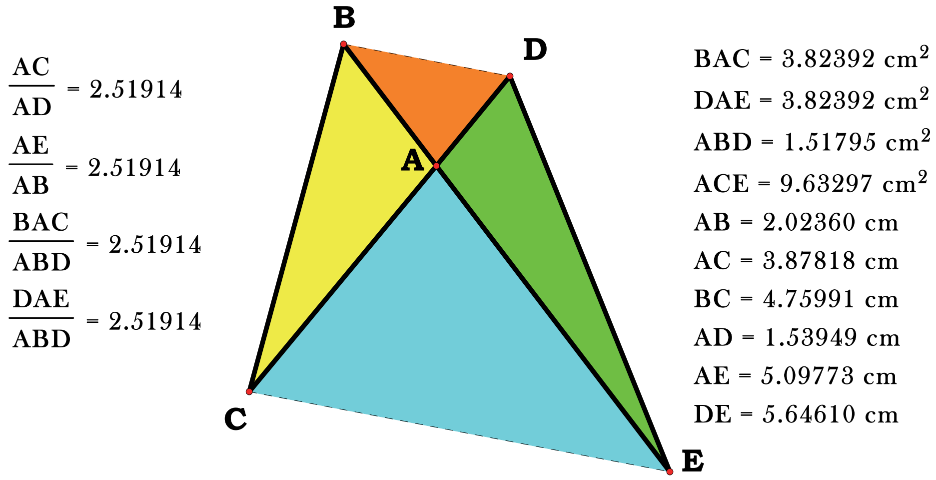
$\boxminus AB = \boxminus BC$.

Therefore etc.

Q. E. D.

Proposition 15.

In equal triangles which have one angle equal to one angle the sides about the equal angles are reciprocally proportional; and those triangles which have one angle equal to one angle, and in which the sides about the equal angles are reciprocally proportional, are equal.



Let,

ΔABC, ΔADE be equal having

one angle equal to one angle,

namely,

$\angle BAC$, to $\angle DAE$;

I say that;

in ΔABC, ΔADE, the sides about

the equal angles are reciprocally proportional,

that is to say, that;

as CA is to AD,

so is EA to AB.

For let,

them be placed so that CA is collinear with AD;

[I. 14] therefore,

EA is, also, collinear with AB.

Let,

BD be joined.

Since then,

$\Delta ABC = \Delta ADE$, and BAD is another area,

[V. 7] therefore,

as ΔCAB, is to ΔBAD,

so is ΔEAD, to ΔBAD.

[vi. 1] But,
as CAB is to BAD,
so is CA to AD,

[*id.*] and,
as EAD is to BAD,
so is EA to AB.

[v. 11] Therefore also,
as CA is to AD,
so is EA to AB.

Therefore,

in ΔABC, ΔADE, the sides about

the equal angles are reciprocally proportional.

Next, let,
the sides of

ΔABC, ΔADE, be reciprocally proportional,

that is to say, let,
EA be to AB,
as CA to AD;

I say that;

$\Delta ABC = \Delta ADE$.

For,
if BD be again joined,

since,
as CA is to AD,
so is EA to AB,

while,
as CA is to AD,

so is ΔABC, to ΔBAD,

[vi. 1] and,
as EA is to AB,

so is ΔEAD, to ΔBAD,

[v. 11] therefore,

as ΔABC, is to ΔBAD,

so is ΔEAD, to ΔBAD.

THEREFORE,

EACH, OF ΔABC, ΔEAD, HAS THE SAME RATIO, TO BAD.

[V. 9]

THEREFORE,

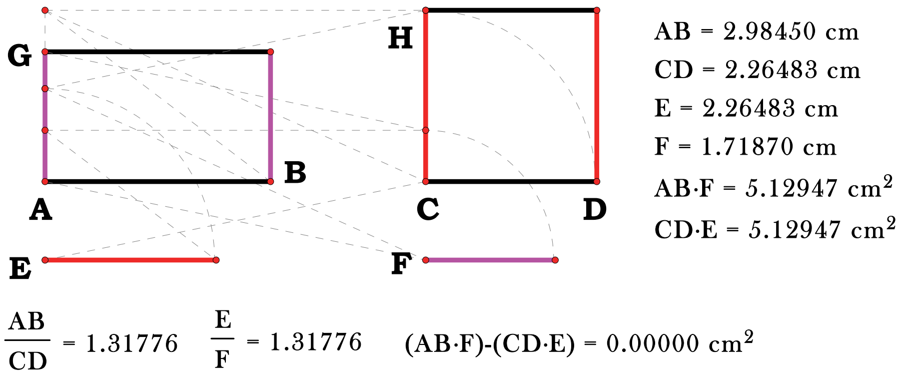
$\Delta ABC = \Delta EAD$.

THEREFORE ETC.

Q. E. D.

Proposition 16.

If four straight lines be proportional, the rectangle contained by the extremes is equal, to the rectangle contained by the means; and, if the rectangle contained by the extremes be equal, to the rectangle contained by the means, the four straight lines will be proportional.



Let,

  the four straight lines *AB*, *CD*, *E*, *F* be proportional,

so that,

  as *AB* is to *CD*,
  so is *E* to *F*;

I say that;

  AB ⊠ *F* = *CD* ⊠ *E*.

Let,

  AG, *CH* be drawn from *A*, *C*,
  at right angles to *AB*, *CD*,

and let,

  AG = *F*, and *CH* = *E*.

Let,

  ⊟*BG*, ⊟*DH*, be completed.

Then since,

  as *AB* is to *CD*,
  so is *E* to *F*, while
  E = *CH*, and
  F = *AG*,

therefore,

  as *AB* is to *CD*,
  so is *CH* to *AG*.

Therefore,

  in ⊟*BG*, ⊟*DH*, the sides about

THE EQUAL ANGLES ARE RECIPROCALLY PROPORTIONAL.

[VI. 14] BUT,

THOSE EQUIANGULAR PARALLELOGRAMS IN WHICH
THE SIDES ABOUT
THE EQUAL ANGLES ARE RECIPROCALLY PROPORTIONAL
ARE EQUAL;

THEREFORE,

$\boxminus BG = \boxminus DH$. AND,

$\boxtimes BG = AB \boxtimes F$,

FOR,

$AG = F$; AND

$\boxtimes DH = CD \boxtimes E$,

FOR,

$E = CH$;

THEREFORE,

$AB \boxtimes F = CD \boxtimes E$.

NEXT, LET,

$AB \boxtimes F = CD \boxtimes E$;

I SAY THAT;

THE FOUR STRAIGHT LINES WILL BE PROPORTIONAL,

SO THAT,

AS AB IS TO CD,
SO IS E TO F.

FOR,

WITH THE SAME CONSTRUCTION,

SINCE,

$AB \boxtimes F = CD \boxtimes E$, AND

$AB \boxtimes F = \boxtimes BG$,

FOR,

$AG = F$, AND

$CD \boxtimes E = \boxtimes DH$,

FOR,

$CH = E$,

THEREFORE,

$\boxtimes BG = \boxtimes DH$.

AND,

THEY ARE EQUIANGULAR.

[VI. 14]

BUT,

IN EQUAL AND EQUIANGULAR PARALLELOGRAMS
THE SIDES ABOUT
THE EQUAL ANGLES ARE RECIPROCALLY PROPORTIONAL.

THEREFORE,

AS AB IS TO CD,
SO IS CH TO AG.

BUT,

$CH = E$, AND AG TO F;

THEREFORE,

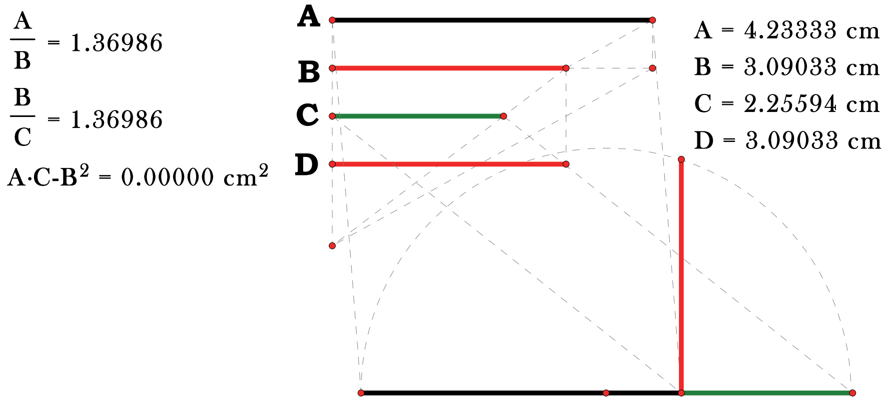
AS AB IS TO CD,
SO IS E TO F.

THEREFORE ETC.

Q. E. D.

Proposition 17.

If three straight lines be proportional, the rectangle contained by the extremes is equal, to the square, on the mean; and, if the rectangle contained by the extremes be equal, to the square, on the mean, the three straight lines will be proportional.



Let,

the three straight lines, A, B, C, be proportional,

so that,

as A is to B,
so is B to C;

I say that;

$A \boxtimes C = \boxdot B$.

Let,

$D = B$.

Then, since,

as A is to B,
so is B to C, and
$B = D$,

therefore,

as A is to B,
so is D to C.

[vi. 16] But,

if four straight lines be proportional,
the rectangle contained by the extremes equals
the rectangle contained by the means.

Therefore,

$A \boxtimes C = B \boxtimes D$.

But,

$B \boxtimes D = \boxdot B$, for,

$B = D$;

THEREFORE,

$A \boxtimes C = \boxdot B$.

NEXT, LET,

$A \boxtimes C = \boxdot B$.

I SAY THAT;
AS A IS TO B,
SO IS B TO C.

FOR,

WITH THE SAME CONSTRUCTION, SINCE,

$A \boxtimes C = \boxdot B$, WHILE, $\boxdot B = B \boxtimes D$,

FOR,
$B = A$,

THEREFORE,

$A \boxtimes C = B \boxtimes D$.

[VI. 16] BUT,
IF THE RECTANGLE CONTAINED BY
THE EXTREMES BE EQUAL TO THAT CONTAINED BY THE MEANS,
THE FOUR STRAIGHT LINES ARE PROPORTIONAL.

THEREFORE,
AS A IS TO B
SO IS D TO C.

BUT,
$B = D$;

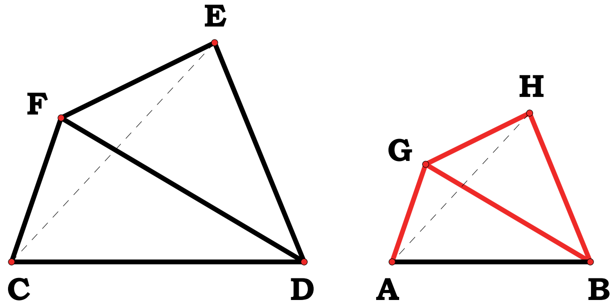
THEREFORE,
AS A IS TO B,
SO IS B TO C.

THEREFORE ETC.

Q. E. D.

PROPOSITION 18.

ON A GIVEN STRAIGHT LINE TO DESCRIBE A RECTILINEAL FIGURE SIMILAR AND SIMILARLY SITUATED TO A GIVEN RECTILINEAL FIGURE.



LET,

AB BE GIVEN, AND

CE THE GIVEN RECTILINEAL FIGURE;

THUS IT IS REQUIRED,

TO DESCRIBE ON AB,

A RECTILINEAL FIGURE SIMILAR, AND

SIMILARLY SITUATED TO THE RECTILINEAL FIGURE, CE.

LET,

DF BE JOINED, AND

ON AB, AND

AT A, B, ON IT,

LET,

$\angle GAB = \angle FCD$,

[I. 23] AND,

$\angle ABG = \angle CDF$.

[I. 32] THEREFORE,

$\angle CFD = \angle AGB$;

THEREFORE,

ΔFCD, IS EQUIANGULAR WITH ΔGAB.

THEREFORE, PROPORTIONALLY,

AS FD IS TO GB,

SO IS FC TO GA, AND,

CD TO AB.

AGAIN,

ON BG, AND AT B, G, ON IT,

[I. 23] LET,

$\angle BGH = \angle DFE$, AND

$\angle GBH = \angle FDE$.

[I. 32] THEREFORE, REMAINING

$\angle$AT $E = \angle$AT H;

THEREFORE,

ΔFDE, IS EQUIANGULAR WITH ΔGBH;

[VI. 4] THEREFORE, PROPORTIONALLY,
AS FD IS TO GB,
SO IS FE TO GH, AND
ED TO HB.

BUT,
IT WAS, ALSO, PROVED THAT,
AS FD IS TO GB,
SO IS FC TO GA, AND
CD TO AB;

THEREFORE ALSO,
AS FC IS TO AG,
SO IS CD TO AB, AND
FE TO GH, AND FURTHER
ED TO HB.

AND, SINCE,

$\angle CFD = \angle AGB$, AND $\angle DFE = \angle BGH$,

THEREFORE,

$\angle CFE = \angle AGH$.

FOR THE SAME REASON,

$\angle CDE = \angle ABH$. AND,

$\angle$AT $C = \angle$AT A, AND $\angle$AT $E = \angle$AT H.

THEREFORE,
AH IS EQUIANGULAR WITH CE; AND THEY HAVE
THE SIDES ABOUT THEIR EQUAL ANGLES PROPORTIONAL;

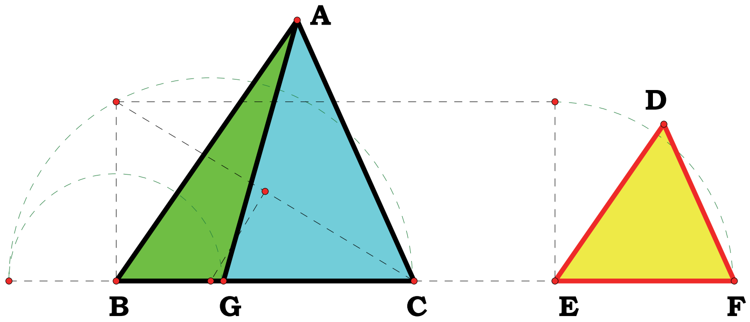
[VI. DEF. 1] THEREFORE,
THE RECTILINEAL FIGURE, AH, IS SIMILAR TO
THE RECTILINEAL FIGURE, CE.

THEREFORE,
ON AB, THE RECTILINEAL FIGURE, AH,
HAS BEEN DESCRIBED SIMILAR, AND SIMILARLY SITUATED TO
THE GIVEN RECTILINEAL FIGURE, CE.

Q. E. F.

PROPOSITION 19.

SIMILAR TRIANGLES ARE TO ONE ANOTHER IN THE DUPLICATE RATIO OF THE CORRESPONDING SIDES.



AB = 4.19816 cm
AC = 3.78037 cm
DE = 2.52792 cm
DF = 2.27635 cm
BC = 3.93700 cm
EF = 2.37067 cm
BG = 1.42750 cm

$\frac{AB}{BC} = 1.06633$ $\quad \frac{BC}{EF} = 1.66071$ $\quad \frac{AB}{DE} = 1.66071$

$\frac{DE}{EF} = 1.06633$ $\quad \frac{EF}{BG} = 1.66071$ $\quad \frac{EF}{BG} = 1.66071$

Area $\triangle EDF = 2.46255 \text{ cm}^2$

Area $\triangle BAG = 2.46255 \text{ cm}^2$

Area $\triangle BCA = 6.79165 \text{ cm}^2$

$(\text{Area } \triangle EDF)-(\text{Area } \triangle BAG) = 0.00000 \text{ cm}^2$

$$\sqrt{\frac{(\text{Area } \triangle BCA)}{(\text{Area } \triangle EDF)}} = 1.66071$$

$$\frac{(\text{Area } \triangle BCA)}{(\text{Area } \triangle BAG)} = 2.75797$$

$$\sqrt{\frac{BC}{BG}} - \frac{BC}{EF} = 0.00000$$

$$\frac{BC}{BG} = 2.75797 \qquad \frac{BC}{EF} - \sqrt{\frac{(\text{Area } \triangle BCA)}{(\text{Area } \triangle EDF)}} = 0.00000$$

LET,

ABC, *DEF*, BE SIMILAR TRIANGLES HAVING

∠AT *B*, EQUAL TO ∠AT *E*,

AND SUCH THAT,

AS *AB* IS TO *BC*,
SO IS *DE* TO *EF*,

[V. DEF. 11] SO THAT,

BC CORRESPONDS TO *EF*;

I SAY THAT;

Δ*ABC*, HAS TO Δ*DEF*, A RATIO DUPLICATE OF

THAT WHICH *BC* HAS TO *EF*

FOR LET,

A THIRD PROPORTIONAL, *BG*, BE TAKEN TO *BC*, *EF*,

[VI. 11] SO THAT,

AS *BC* IS TO *EF*,
SO IS *EF* TO *BG*;

AND LET,

AG BE JOINED.

SINCE THEN,

AS *AB* IS TO *BC*,
SO IS *DE* TO *EF*,

[V. 16] THEREFORE, ALTERNATELY,
AS AB IS TO DE,
SO IS BC TO EF.

BUT,
AS BC IS TO EF,
SO IS EF TO BG;

[V. 11] THEREFORE ALSO,
AS AB IS TO DE,
SO IS EF TO BG.

THEREFORE,

IN ΔABG, ΔDEF, THE SIDES ABOUT

THE EQUAL ANGLES ARE RECIPROCALLY PROPORTIONAL.

[VI. 15] BUT,
THOSE TRIANGLES WHICH HAVE
ONE ANGLE EQUAL TO ONE ANGLE, AND
IN WHICH THE SIDES ABOUT
THE EQUAL ANGLES ARE RECIPROCALLY PROPORTIONAL,
ARE EQUAL;

THEREFORE,

$\Delta ABG = \Delta DEF$.

[V. DEF. 9]

NOW SINCE,
AS BC IS TO EF,
SO IS EF TO BG, AND
IF THREE STRAIGHT LINES BE PROPORTIONAL,
THE FIRST HAS TO THE THIRD, A RATIO DUPLICATE OF
THAT WHICH IT HAS TO THE SECOND,

THEREFORE,
BC HAS TO BG,
A RATIO DUPLICATE OF THAT WHICH CB HAS TO EF.

[VI. 1] BUT,
AS CB IS TO BG,
SO IS ΔABC, TO ΔABG;

THEREFORE,

ΔABC, ALSO, HAS TO ΔABG,

A RATIO DUPLICATE OF THAT WHICH BC HAS TO EF.

BUT,

$\Delta ABG = \Delta DEF$;

THEREFORE,

ΔABC, ALSO, HAS TO ΔDEF,

A RATIO DUPLICATE OF THAT WHICH BC HAS TO EF.

THEREFORE ETC.

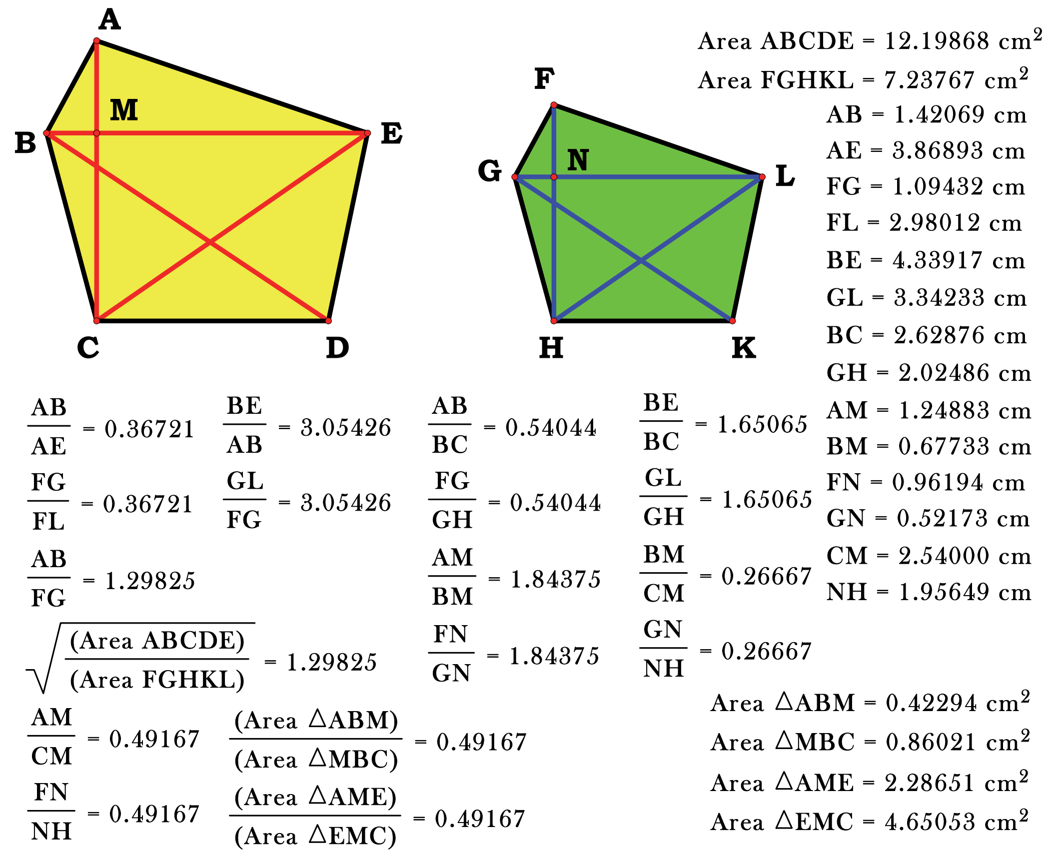
PORISM.

FROM THIS IT IS MANIFEST THAT, IF THREE STRAIGHT LINES BE PROPORTIONAL, THEN, AS THE FIRST IS TO THE THIRD, SO IS THE FIGURE DESCRIBED ON THE FIRST TO THAT WHICH IS SIMILAR AND SIMILARLY DESCRIBED ON THE SECOND.

Q. E. D.

Proposition 20.

Similar polygons are divided into similar triangles, and into triangles equal in multitude and in the same ratio as the wholes, and the polygon has to the polygon a ratio duplicate of that which the corresponding side has to the corresponding side.



LET,

ABCDE, *FGHKL* BE SIMILAR POLYGONS,

AND LET,

AB CORRESPOND TO *FG*;

I SAY THAT;

THE POLYGONS, *ABCDE*, *FGHKL*, ARE DIVIDED INTO SIMILAR TRIANGLES, AND

INTO TRIANGLES EQUAL IN MULTITUDE, AND

IN THE SAME RATIO AS THE WHOLES, AND

THE POLYGON, *ABCDE*, HAS TO THE POLYGON, *FGHKL*,

A RATIO DUPLICATE OF THAT WHICH *AB* HAS TO *FG*.

LET,

BE, *EC*, *GL*, *LH*, BE JOINED.

[VI. DEF. 1] NOW, SINCE,

THE POLYGON, *ABCDE*, IS SIMILAR TO

THE POLYGON, *FGHKL*,

$\angle BAE = \angle GFL$; AND
AS BA IS TO AE,
SO IS GF TO FL.

SINCE THEN,

ΔABE, ΔFGL ARE TWO TRIANGLES HAVING
ONE ANGLE EQUAL TO ONE ANGLE, AND
THE SIDES ABOUT THE EQUAL ANGLES PROPORTIONAL,

[VI. 6] THEREFORE,

ΔABE, IS EQUIANGULAR WITH ΔFGL;

[VI. 4 AND DEF. 1] SO THAT,
IT IS, ALSO, SIMILAR; THEREFORE,

$\angle ABE = \angle FGL$. BUT,

$\angle ABC = \angle FGH$, BECAUSE,
OF THE SIMILARITY OF THE POLYGONS; THEREFORE,

$\angle EBC = \angle LGH$.

AND, SINCE, BECAUSE OF,

THE SIMILARITY OF ΔABE, ΔFGL,
AS EB IS TO BA,
SO IS LG TO GF,

AND MOREOVER ALSO, BECAUSE OF,
THE SIMILARITY OF THE POLYGONS,
AS AB IS TO BC,
SO IS FG TO GH,

[V. 22] THEREFORE, *EX AEQUALI*,
AS EB IS TO BC,
SO IS LG TO GH, THAT IS,
THE SIDES ABOUT THE EQUAL ANGLES,

$\angle EBC = \angle LGH$, ARE PROPORTIONAL;

[VI. 6] THEREFORE,

ΔEBC, IS EQUIANGULAR WITH ΔLGH,

[VI. 4 AND DEF. 1] SO THAT,

ΔEBC, IS, ALSO, SIMILAR TO ΔLGH.

FOR THE SAME REASON,

ΔECD, IS, ALSO, SIMILAR TO ΔLHK.

THEREFORE,
THE SIMILAR POLYGONS,

ABCDE, *FGHKL*, HAVE BEEN DIVIDED
INTO SIMILAR TRIANGLES, AND
INTO TRIANGLES EQUAL IN MULTITUDE.

I SAY THAT;
THEY ARE, ALSO, IN THE SAME RATIO AS THE WHOLES,

THAT IS, IN SUCH MANNER THAT;
THE TRIANGLES ARE PROPORTIONAL, AND

ΔABE, ΔEBC, ΔECD, ARE ANTECEDENTS, WHILE

ΔFGL, ΔLGH, ΔLHK, ARE THEIR CONSEQUENTS,

AND THAT,
THE POLYGON, *ABCDE*, HAS TO THE POLYGON, *FGHKL*
A RATIO DUPLICATE OF THAT WHICH
THE CORRESPONDING SIDE HAS TO THE CORRESPONDING SIDE,

THAT IS,
AB TO *FG*.

FOR LET,
AC, *FH*, BE JOINED.

[VI. 6] THEN SINCE,
BECAUSE OF THE SIMILARITY OF THE POLYGONS,

$\angle ABC = \angle FGH$, AND

AS *AB* IS TO *BC*,
SO IS *FG* TO *GH*,

ΔABC, IS EQUIANGULAR WITH ΔFGH,

THEREFORE,

$\angle BAC = \angle GFH$, AND $\angle BCA = \angle GHF$.

AND, SINCE,

$\angle BAM = \angle GFN$, AND $\angle ABM = \angle FGN$,

[I. 32] THEREFORE,

$\angle AMB = \angle FNG$;

THEREFORE,

ΔABM, IS EQUIANGULAR WITH ΔFGN.

SIMILARLY, WE CAN PROVE THAT;

ΔBMC, IS, ALSO, EQUIANGULAR WITH ΔGNH.

THEREFORE, PROPORTIONALLY,
AS *AM* IS TO *MB*,
SO IS *FN* TO *NG*, AND

as BM is to MC,
so is GN to NH;

so that, in addition, *ex aequali*,
as AM is to MC,
so is FN to NH.

But,
as AM is to MC,
so is ΔABM to ΔMBC, and
ΔAME to ΔEMC;

[vi. 1] for,
they are to one another as their bases.

[v. 12] Therefore also,
as one of the antecedents is to
one of the consequents,
so are all the antecedents to all the consequents;

therefore,
as ΔAMB, is to BMC,
so is ABE to CBE.

But,
as AMB is to BMC,
so is AM to MC;

therefore also,
as AM is to MC,
so is ΔABE, to ΔEBC.

For the same reason also,
as FN is to NH,
so is ΔFGL, to ΔGLH.

And,
as AM is to MC,
so is FN to NH;

therefore also,
as ΔABE, is to ΔBEC,
so is ΔFGL, to ΔGLH;

and, alternately,
as ΔABE, is to ΔFGL,
so is ΔBEC, to ΔGLH.

SIMILARLY WE CAN PROVE THAT;

IF *BD*, *GK* BE JOINED,

AS ΔBEC, IS TO ΔLGH,

SO, ALSO, IS ΔBCD, TO ΔLHK.

AND SINCE,

AS ΔABE, IS TO ΔFGL,

SO IS *EBC* TO *LGH*, AND FURTHER,
BCD TO *LHK*,

[V. 12] THEREFORE ALSO,
AS ONE OF THE ANTECEDENTS IS TO ONE OF
THE CONSEQUENTS,
SO ARE ALL THE ANTECEDENTS TO ALL THE CONSEQUENTS;

THEREFORE,

AS ΔABE, IS TO ΔFGL,

SO IS THE POLYGON, *ABCDE*, TO THE POLYGON, *FGHKL*.

BUT,

ΔABE HAS TO ΔFGL,

A RATIO DUPLICATE OF THAT WHICH
THE CORRESPONDING SIDE, *AB*, HAS TO
THE CORRESPONDING SIDE, *FG*;

[VI. 19] FOR,
SIMILAR TRIANGLES ARE IN THE DUPLICATE RATIO OF
THE CORRESPONDING SIDES.

THEREFORE,
THE POLYGON, *ABCDE*, ALSO, HAS TO THE POLYGON, *FGHKL*,
A RATIO DUPLICATE OF THAT WHICH
THE CORRESPONDING SIDE, *AB*, HAS TO
THE CORRESPONDING SIDE, *FG*.

THEREFORE ETC.

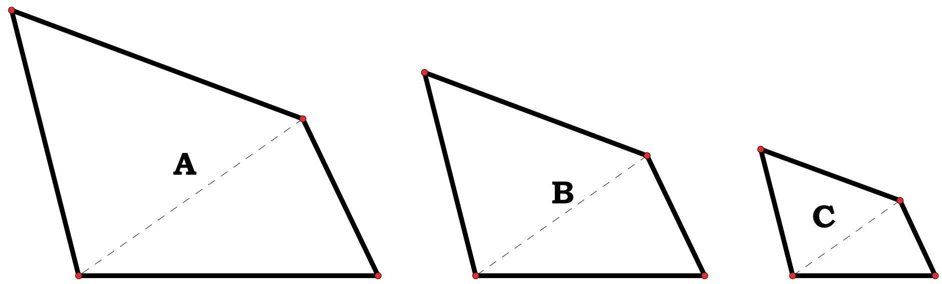
PORISM.

SIMILARLY, ALSO, IT CAN BE PROVED IN THE CASE OF QUADRILATERALS THAT THEY ARE IN THE DUPLICATE RATIO OF THE CORRESPONDING SIDES. AND IT WAS, ALSO, PROVED IN THE CASE OF TRIANGLES; THEREFORE ALSO, GENERALLY, SIMILAR RECTILINEAL FIGURES ARE TO ONE ANOTHER IN THE DUPLICATE RATIO OF THE CORRESPONDING SIDES.

Q. E. D.

PROPOSITION 21.

FIGURES WHICH ARE SIMILAR TO THE SAME RECTILINEAL FIGURE ARE, ALSO, *SIMILAR TO ONE ANOTHER*.



FOR LET,

EACH, OF THE RECTILINEAL FIGURES, A, B, BE SIMILAR TO C;

I SAY THAT;

A IS, ALSO, SIMILAR TO B.

[VI. DEF. I]

FOR, SINCE,

A IS SIMILAR TO C,

IT IS EQUIANGULAR WITH IT, AND

HAS THE SIDES ABOUT THE EQUAL ANGLES PROPORTIONAL.

AGAIN, SINCE,

B IS SIMILAR TO C, IT IS EQUIANGULAR WITH IT, AND HAS THE SIDES ABOUT THE EQUAL ANGLES PROPORTIONAL.

THEREFORE,

EACH, OF THE FIGURES, A, B, IS EQUIANGULAR WITH C,

AND WITH,

C, HAS THE SIDES ABOUT THE EQUAL ANGLES PROPORTIONAL;

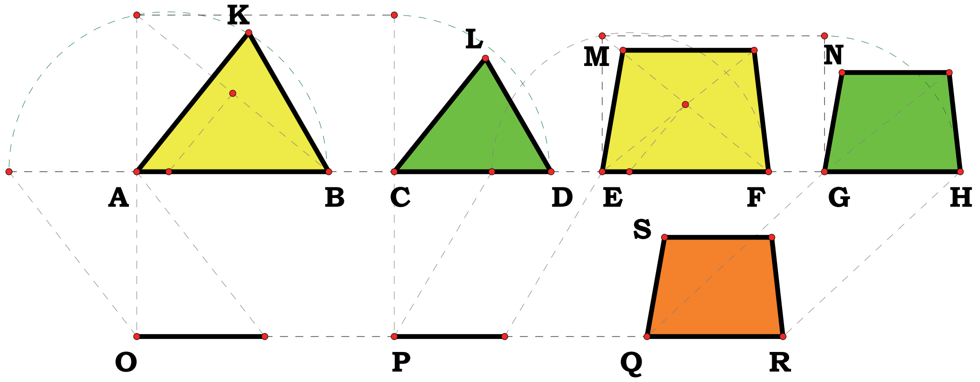
THEREFORE,

A IS SIMILAR TO B.

Q. E. D.

PROPOSITION 22.

IF FOUR STRAIGHT LINES BE PROPORTIONAL, THE RECTILINEAL FIGURES SIMILAR AND SIMILARLY DESCRIBED UPON THEM WILL, ALSO, *BE PROPORTIONAL; AND, IF THE RECTILINEAL FIGURES SIMILAR AND SIMILARLY DESCRIBED UPON THEM BE PROPORTIONAL, THE STRAIGHT LINES WILL THEMSELVES*, ALSO, *BE PROPORTIONAL.*



AB = 2.54000 cm
CD = 2.07433 cm
EF = 2.20133 cm
GH = 1.79776 cm
Area $\triangle$KAB = 2.33871 cm^2
Area $\triangle$LCD = 1.55979 cm^2
Area MEFX = 3.16666 cm^2
Area NGHY = 2.11199 cm^2
O = 1.69404 cm
P = 1.46817 cm
QR = 1.79776 cm
Area SQRZ = 2.11199 cm^2

$$\frac{AB}{CD} = 1.22449 \qquad \frac{(\text{Area } \triangle KAB)}{(\text{Area } \triangle LCD)} = 1.49938$$

$$\frac{EF}{GH} = 1.22449 \qquad \frac{(\text{Area } MEFX)}{(\text{Area } NGHY)} = 1.49938$$

$$\frac{CD}{O} = 1.22449 \qquad \frac{AB}{O} = 1.49938$$

$$\frac{GH}{P} = 1.22449 \qquad \frac{EF}{P} = 1.49938$$

$$\frac{EF}{QR} = 1.22449 \qquad \frac{(\text{Area } MEFX)}{(\text{Area } SQRZ)} = 1.49938$$

LET,

THE FOUR STRAIGHT LINES,
AB, *CD*, *EF*, *GH* BE PROPORTIONAL,

SO THAT,

AS *AB* IS TO *CD*,
SO IS *EF* TO *GH*,

AND LET,

THERE BE DESCRIBED, ON *AB*, *CD*,
THE SIMILAR AND SIMILARLY SITUATED
RECTILINEAL FIGURES, *KAB*, *LCD*, AND ON
EF, *GH*, THE SIMILAR AND SIMILARLY SITUATED
RECTILINEAL FIGURES, *MF*, *NH*;

I SAY THAT;

AS *KAB* IS TO *LCD*,
SO IS *MF* TO *NH*.

[VI. 11] FOR LET,

THERE BE TAKEN A THIRD PROPORTIONAL, *O* TO *AB*, *CD*,
AND A THIRD PROPORTIONAL, *P* TO *EF*, *GH*.

THEN SINCE,
AS *AB* IS TO *CD*,
SO IS *EF* TO *GH*, AND
AS *CD* IS TO *O*,
SO IS *GH* TO *P*,

[V. 22] THEREFORE, *EX AEQUALI*,
AS *AB* IS TO *O*,
SO IS *EF* TO *P*.

[VI. 19, POR.] BUT,
AS *AB* IS TO *O*,
SO IS *KAB* TO *LCD*, AND
AS *EF* IS TO *P*,
SO *MF* IS TO *NH*;

[V. 11] THEREFORE ALSO,
AS *KAB* IS TO *LCD*,
SO IS *MF* TO *NH*.

NEXT, LET,
MF BE TO *NH*,
AS *KAB* IS TO *LCD*;

I SAY, ALSO, THAT;
AS *AB* IS TO *CD*,
SO IS *EF* TO *GH*.

FOR,
IF *EF* IS NOT TO *GH*,
AS *AB* TO *CD*,

[VI. 12] LET,
EF BE TO *QR*,
AS *AB* TO *CD*,

[VI. 18] AND LET,
ON *QR*,
THE RECTILINEAL FIGURE, *SR*, BE DESCRIBED SIMILAR, AND
SIMILARLY SITUATED TO EITHER OF THE TWO, *MF*, *NH*.

SINCE THEN,
AS *AB* IS TO *CD*,
SO IS *EF* TO *QR*, AND
THERE HAVE BEEN DESCRIBED, ON *AB*, *CD*,
THE SIMILAR AND SIMILARLY SITUATED FIGURES,
KAB, *LCD*, AND ON *EF*, *QR*,
THE SIMILAR AND SIMILARLY SITUATED FIGURES, *MF*, *SR*,

THEREFORE,

AS KAB IS TO LCD,
SO IS MF TO SR.

BUT ALSO, BY HYPOTHESIS,

AS KAB IS TO LCD,
SO IS MF TO NH;

[V. 11] THEREFORE ALSO,

AS MF IS TO SR,
SO IS MF TO NH.

THEREFORE,

MF HAS THE SAME RATIO TO EACH, OF
THE FIGURES, NH, SR;

[V. 9] THEREFORE,

$NH = SR$.

BUT,

IT IS, ALSO, SIMILAR AND SIMILARLY SITUATED TO IT;

THEREFORE,

$GH = QR$.

AND, SINCE,

AS AB IS TO CD,
SO IS EF TO QR, WHILE,
$QR = GH$,

THEREFORE,

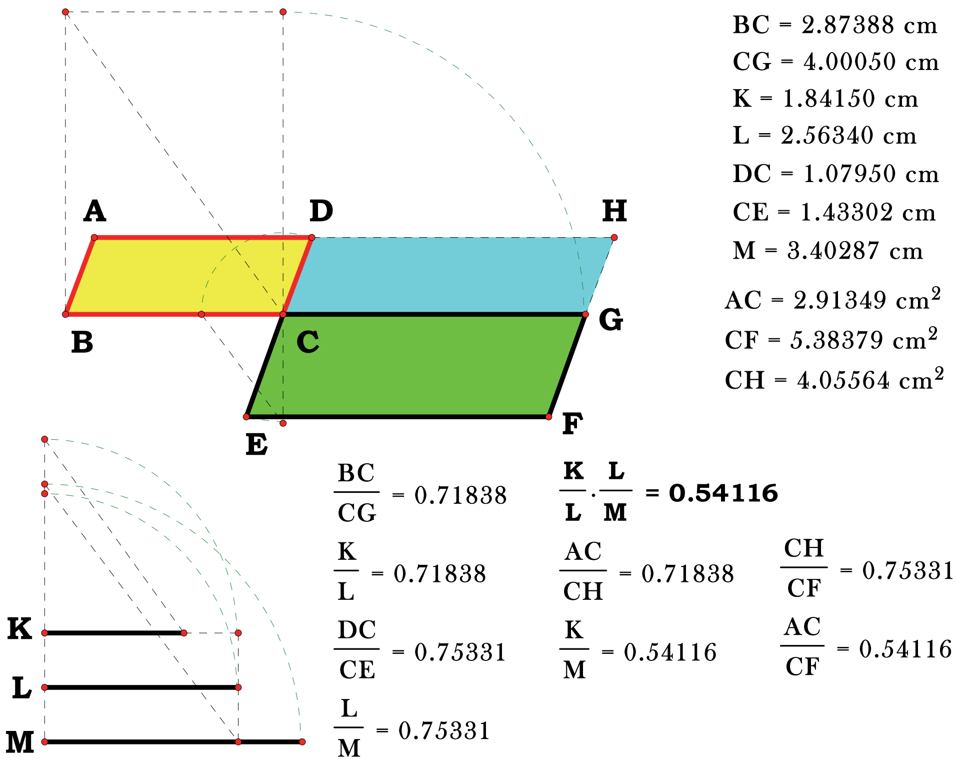
AS AB IS TO CD,
SO IS EF TO GH.

THEREFORE ETC.

Q. E. D.

Proposition 23.

Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides.



Let,

⊟*AC*, ⊟*CF*, be equiangular having ∠*BCD* = ∠*ECG*;

I say that;

⊟ *AC* has to ⊟ *CF*,

the ratio compounded of the ratios of the sides.

For let,

them be placed so that; *BC* is collinear with *CG*;

therefore,

DC is, also, collinear with *CE*.

Let,

⊟*DG*, be completed;

let,

K, be set out,

[vi. 12] and let it be contrived that;

as *BC* is to *CG*,

so is *K* to *L*, and

as *DC* is to *CE*, so is *L* to *M*.

Then,

the ratios of *K* to *L*, and

OF L TO M ARE THE SAME AS THE RATIOS OF THE SIDES,

NAMELY,
OF BC TO CG AND
OF DC TO CE.

BUT,
THE RATIO, OF K TO M, IS COMPOUNDED OF
THE RATIO, OF K TO L, AND OF THAT OF L TO M;

SO THAT, ALSO,
K HAS TO M,
THE RATIO COMPOUNDED OF THE RATIOS OF THE SIDES.

[VI. 1]

NOW SINCE,
AS BC IS TO CG,

SO IS ⊟AC, TO ⊟CH, WHILE

AS BC IS TO CG,
SO IS K TO L,

[V. 11] THEREFORE ALSO,
AS K IS TO L,

SO IS ⊟AC TO ⊟CH.

[VI. 1] AGAIN, SINCE,
AS DC IS TO CE,

SO ⊟ CH TO ⊟CF, WHILE

AS DC IS TO CE,
SO IS L TO M,

[V. 11] THEREFORE ALSO,
AS L IS TO M,

SO IS ⊟CH TO ⊟CF.

SINCE THEN IT WAS PROVED THAT;
AS K IS TO L,

SO IS ⊟AC TO ⊟CH, AND

AS L IS TO M,

SO IS ⊟ CH TO ⊟CF,

THEREFORE, *EX AEQUALI*,
AS K IS TO M,

SO IS ⊟AC TO ⊟ CF.

BUT,
K HAS TO M,

THE RATIO COMPOUNDED OF THE RATIOS OF THE SIDES;

THEREFORE,

⊟AC, ALSO, HAS TO ⊟CF,

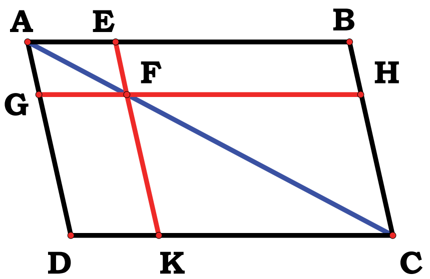
THE RATIO COMPOUNDED OF THE RATIOS OF THE SIDES.

THEREFORE ETC.

Q. E. D.

PROPOSITION 24.

IN ANY PARALLELOGRAM THE PARALLELOGRAMS ABOUT THE DIAMETER ARE SIMILAR BOTH TO THE WHOLE AND TO ONE ANOTHER.



BE = 3.09218 cm
AE = 1.16232 cm
AF = 1.49261 cm
CF = 3.97089 cm
AB = 4.25450 cm
DG = 1.90724 cm
AG = 0.71691 cm
AD = 2.62415 cm

CD = 4.25450 cm
FG = 1.16232 cm
AC = 5.46350 cm
BC = 2.62415 cm
EF = 0.71691 cm

$\frac{BE}{AE} = 2.66036$ $\quad \frac{AB}{AE} = 3.66036$ $\quad \frac{AB}{AD} = 1.62128$ $\quad \frac{AD}{CD} = 0.61680$

$\frac{CF}{AF} = 2.66036$ $\quad \frac{AD}{AG} = 3.66036$ $\quad \frac{AE}{AG} = 1.62128$ $\quad \frac{AG}{FG} = 0.61680$

$\frac{DG}{AG} = 2.66036$ $\quad \frac{AC}{BC} = 2.08200$ $\quad \frac{CD}{BC} = 1.62128$ $\quad \frac{BC}{AB} = 0.61680$

$\frac{CD}{AC} = 0.77871$ $\quad \frac{AF}{EF} = 2.08200$ $\quad \frac{FG}{EF} = 1.62128$ $\quad \frac{EF}{AE} = 0.61680$

$\frac{FG}{AF} = 0.77871$

LET,

⊟*ABCD*,

AND,

AC ITS DIAMETER,

AND LET,

⊟*EG*, ⊟*HK*, BE ABOUT *AC*;

I SAY THAT;

EACH, OF ⊟*EG*, ⊟*HK* IS SIMILAR BOTH

TO THE WHOLE, ⊟*ABCD*, AND TO THE OTHER.

FOR, SINCE,

EF ∥ *BC*, ONE OF THE SIDES OF Δ*ABC*,

[VI. 2] PROPORTIONALLY,

AS *BE* IS TO *EA*,

SO IS *CF* TO *FA*.

AGAIN, SINCE,

FG ∥ *CD*, ONE OF THE SIDES OF Δ*ACD*,

[VI. 2] PROPORTIONALLY,

as CF is to FA,
so is DG to GA.

but, it was proved that;
as CF is to FA, so also,
is BE to EA;

therefore also,
as BE is to EA,
so is DG to GA,

[v. 18] and therefore, *componendo*,
as BA is to AE,
so is DA to AG,

[v. 16] and, alternately,
as BA is to AD,
so is EA to AG.

Therefore,

in ⊟ $ABCD$, ⊟EG, the sides about

the $\angle BAD$, are proportional.

and, since,

$GF \parallel DC$,

$\angle AFG = \angle DCA$; and

$\angle DAC$, is common to ΔADC, ΔAGF;

therefore,

ΔADC, is equiangular with ΔAGF.

For the same reason,

ΔACB, is, also, equiangular with ΔAFE, and

⊟$ABCD$, is equiangular with ⊟ FG.

Therefore, proportionally,
as AD is to DC,
so is AG to GF,
as DC is to CA,
so is GF to FA,
as AC is to CB,
so is AF to FE, and further
as CB is to BA,
so is FE to EA.

And, since it was proved that;
as DC is to CA,

so is GF to FA, and,
as AC is to CB,
so is AF to FE,

[v. 22] therefore, *ex aequali*,
as DC is to CB,
so is GF to FE.

Therefore,

in ⊟$ABCD$, ⊟EG, the sides about

the equal angles are proportional;

[vi. Def. 1] therefore,

⊟$ABCD$, is similar to ⊟EG.

For the same reason,

⊟ $ABCD$, is, also, similar to ⊟KH;

therefore,

each, of ⊟EG, ⊟HK, is similar to $ABCD$.

[vi. 21] But,
figures similar to
the same rectilineal figure are, also, similar to
one another;

therefore,

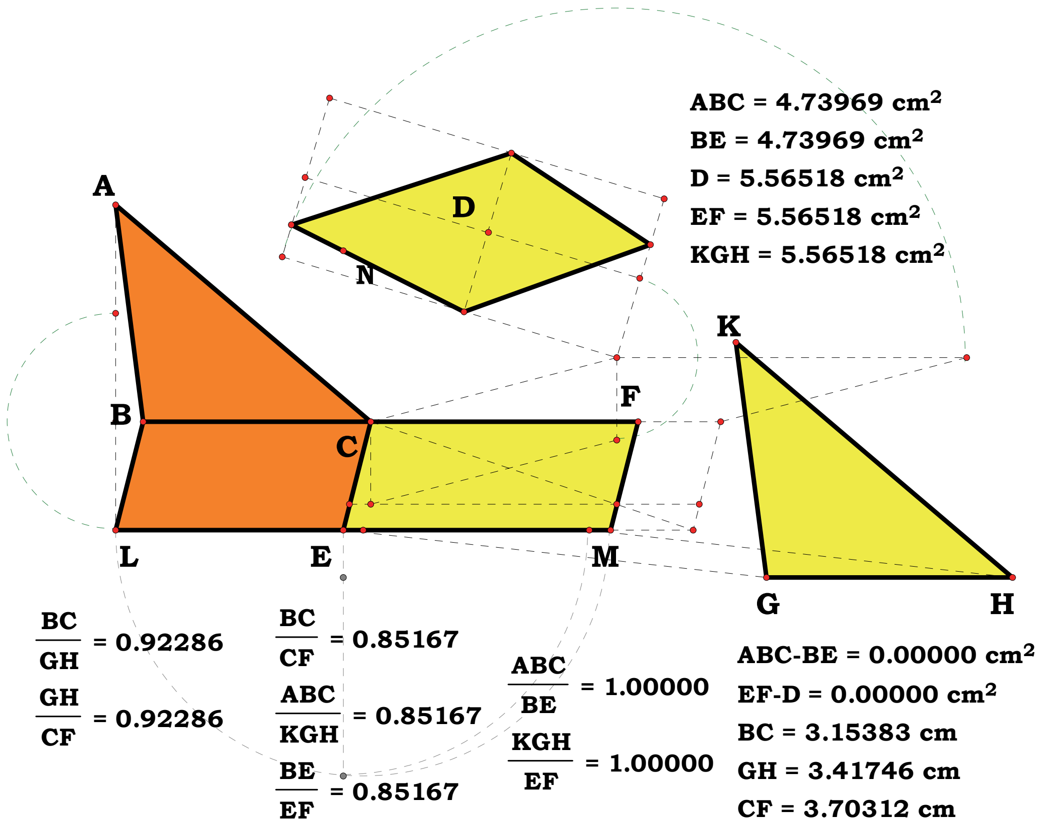
⊟EG, is, also, similar to ⊟ HK.

Therefore etc.

Q. E. D.

Proposition 25.

To construct one and the same figure similar to a given rectilineal figure and equal to another given rectilineal figure.



Let,

ABC, be the given rectilineal figure to which the figure to be constructed must be similar, and

D, that to which it must be equal;

Thus it is required,

to construct one and the same figure similar to ABC and equal to D,

[I. 44] Let,

there be applied to BC, ⊟ $BE = \Delta ABC$,

[I. 45] and,

to CE, ⊟$CM = D$, in $\angle FCE = \angle CBL$,

Therefore,

BC is collinear with CF, and,

LE collinear with EM.

[VI. 13] Now let, $(GH = \sqrt{BC \times CF})$

GH be taken a mean proportional to BC, CF.

[VI. 18] and let,

on GH,
KGH be described similar and,
similarly situated to ABC.

Then, since,
as BC is to GH,
so is GH to CF,

[vi. 19, Por.] and,
if three straight lines be proportional,
as the first is to the third,
so is the figure on the first to the similar and,
similarly situated figure described on the second,

therefore,
as BC is to CF,
so is ΔABC to ΔKGH.

[vi. 1] But,
as BC is to CF, so also,
is the ⊟ BE to ⊟ EF.

Therefore also,
as ΔABC is to ΔKGH,
so is ⊟ BE to ⊟EF;

[v. 16]

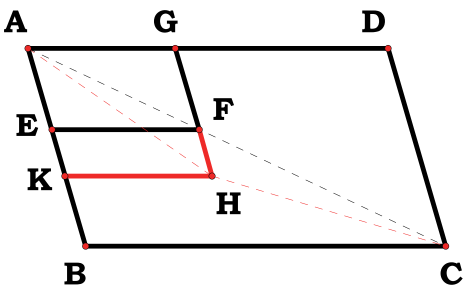
therefore, alternately,
as ΔABC is to ⊟ BE,
so is ΔKGH to ⊟ EF. But,
ΔABC = ⊟BE; therefore,
ΔKGH = ⊟ EF. But,
⊟EF = D;

therefore,
$KGH = D$. And,
KGH is, also, similar to ABC.

Therefore,
one and the same figure,
KGH, has been constructed similar to
the given rectilineal figure, ABC, and
equal to the other given figure, D.

Q. E. D.

PROPOSITION 26.



IF FROM A PARALLELOGRAM THERE BE TAKEN AWAY A PARALLELOGRAM SIMILAR AND SIMILARLY SITUATED TO THE WHOLE AND HAVING A COMMON ANGLE WITH IT, IT IS ABOUT THE SAME DIAMETER WITH THE WHOLE.

FOR LET,

FROM THE PARALLELOGRAM, *ABCD*, THERE BE TAKEN AWAY
THE PARALLELOGRAM, *AF*, SIMILAR AND
SIMILARLY SITUATED TO *ABCD*, AND
HAVING ∠*DAB*, COMMON WITH IT;

I SAY THAT;

ABCD IS ABOUT THE SAME DIAMETER, WITH *AF*.

FOR SUPPOSE IT IS NOT, BUT, IF POSSIBLE, LET,

AHC BE THE DIAMETER < OF *ABCD*, >

LET,

GF BE PRODUCED, AND CARRIED THROUGH TO *H*,

[I. 31] AND LET,

HK BE DRAWN, THROUGH *H*, PARALLEL TO
EITHER OF THE STRAIGHT LINES, *AD*, *BC*.

SINCE, THEN,

ABCD IS ABOUT THE SAME DIAMETER WITH *KG*,

[VI. 24] THEREFORE,

AS *DA* IS TO *AB*,
SO IS GA TO *AK*.

BUT ALSO,

BECAUSE OF THE SIMILARITY OF *ABCD*, *EG*,
AS *DA* IS TO *AB*,
SO IS *GA* TO *AE*;

[V. 11] THEREFORE ALSO,

AS *GA* IS TO *AK*,
SO IS *GA* TO *AE*.

THEREFORE,

GA HAS THE SAME RATIO TO EACH, OF
THE STRAIGHT LINES, *AK*, *AE*.

[V. 9] THEREFORE,

AE = *AK*,

THE LESS TO THE GREATER: WHICH,
IS IMPOSSIBLE.

THEREFORE,
ABCD CANNOT BUT BE ABOUT
THE SAME DIAMETER, WITH *AF*;

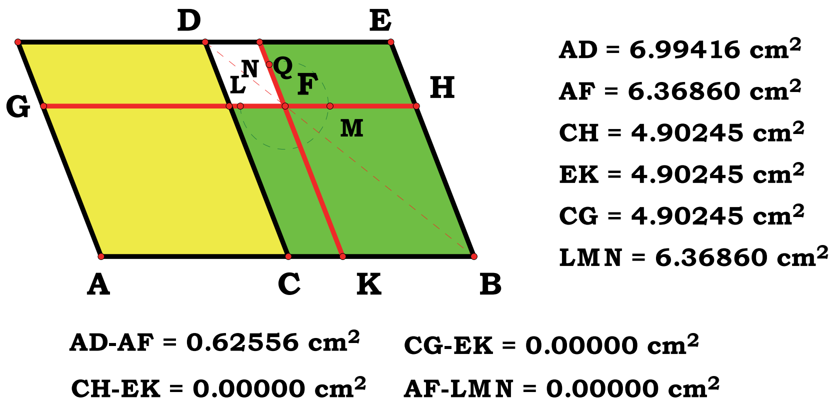
THEREFORE,
THE PARALLELOGRAM, *ABCD*, IS ABOUT
THE SAME DIAMETER WITH THE PARALLELOGRAM, *AF*.

THEREFORE ETC.

Q. E. D.

Proposition 27.

Of all the parallelograms applied to the same straight line and deficient by parallelogrammic figures similar and similarly situated to that described on the half of the straight line, that parallelogram is greatest which is applied to the half of the straight line and is similar to the defect.



Let,

AB be bisected at *C*;

let,

there be applied to *AB*,

the ⊟*AD*, deficient by ⊡*DB*, described on

the half of *AB*, that is, *CB*;

I say that;

of all the parallelograms, applied to *AB*, and deficient by parallelogrammic figures similar, and similarly situated to *DB*,

AD is greatest.

For let,

there be applied to the straight line, *AB*,

the parallelogram, *AF*, deficient by

the parallelogrammic figure, *FB*, similar and,

similarly situated to *DB*;

I say that;

AD is greater than *AF*.

[vi. 26]

For, since,

the parallelogram, *DB*, is similar to

the parallelogram, *FB*,

they are about the same diameter.

Let,

THEIR DIAMETER, *DB*, BE DRAWN,

AND LET,

THE FIGURE BE DESCRIBED.

[I. 43] THEN, SINCE,
$CF = FE$, AND,
FB IS COMMON,

THEREFORE,

⊟CH = ⊟ KE.

[I. 36] BUT,
$CH = CG$, SINCE,
$AC = CB$.

THEREFORE,
$GC = EK$.

LET,
CF BE ADDED TO EACH;

THEREFORE,

⊟AF = THE GNOMON, *LMN*;

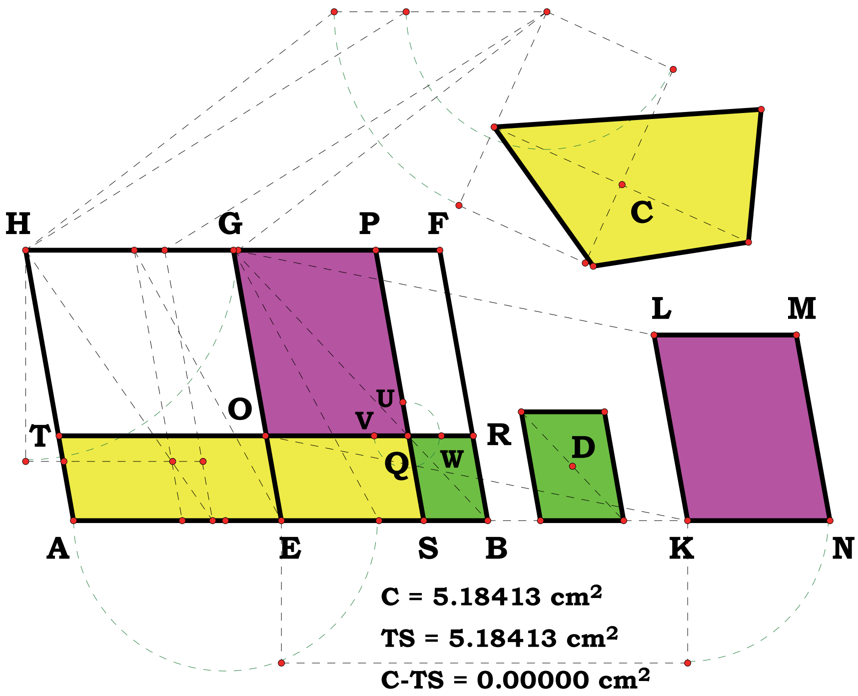
SO THAT,

⊟DB, THAT IS, *AD*, IS GREATER THAN ⊟AF.

THEREFORE ETC.

PROPOSITION 28.

TO A GIVEN STRAIGHT LINE TO APPLY A PARALLELOGRAM EQUAL TO A GIVEN RECTILINEAL FIGURE AND DEFICIENT BY A PARALLELOGRAMMIC FIGURE SIMILAR TO A GIVEN ONE: THUS THE GIVEN RECTILINEAL FIGURE MUST NOT BE GREATER THAN THE PARALLELOGRAM DESCRIBED ON THE HALF OF THE STRAIGHT LINE AND SIMILAR TO THE DEFECT.



LET,

AB BE THE GIVEN STRAIGHT LINE,
C, THE GIVEN RECTILINEAL FIGURE TO WHICH
THE FIGURE TO BE APPLIED, TO *AB*, IS REQUIRED,
TO BE EQUAL, NOT BEING GREATER THAN
THE PARALLELOGRAM, DESCRIBED, ON THE HALF, OF *AB*, AND
SIMILAR TO THE DEFECT, AND
D, THE PARALLELOGRAM TO WHICH THE DEFECT IS REQUIRED,
TO BE SIMILAR;

THUS IT IS REQUIRED,

TO APPLY TO THE GIVEN STRAIGHT LINE, *AB*,
A PARALLELOGRAM EQUAL
TO THE GIVEN RECTILINEAL FIGURE, *C*, AND, DEFICIENT BY
A FIGURE WHICH IS SIMILAR TO ⊟*D*.

[VI. 18] LET,

AB BE BISECTED AT THE POINT, *E*,

AND LET,

ON *EB*,

⊟*EBFG* BE DESCRIBED SIMILAR AND

SIMILARLY SITUATED TO ⊟*D*;

LET,

⊟*AG*, BE COMPLETED.

IF THEN,

AG = *C*,

THAT WHICH WAS ENJOINED WILL HAVE BEEN DONE;

FOR,

THERE HAS BEEN APPLIED TO *AB*,

⊟*AG*, EQUAL TO

THE GIVEN RECTILINEAL FIGURE, *C*, AND,

DEFICIENT BY ⊟*GB*,

WHICH IS SIMILAR TO *D*.

BUT, IF NOT, LET,

HE, BE GREATER THAN *C*.

NOW,

HE = *GB*;

THEREFORE,

GB IS, ALSO, GREATER THAN *C*.

[VI. 25] LET,

KLMN BE CONSTRUCTED AT ONCE EQUAL TO
THE EXCESS, BY WHICH *GB*, IS GREATER THAN *C*, AND
SIMILAR AND SIMILARLY SITUATED, TO *D*.

BUT,

D IS SIMILAR TO *GB*;

[VI. 21] THEREFORE,

KM IS, ALSO, SIMILAR TO *GB*.

LET, THEN,

KL CORRESPOND TO *GE*, AND,
LM TO GF.

NOW, SINCE,

GB = *C*, *KM*,

THEREFORE,

GB > *KM*;

THEREFORE ALSO,

GE > *KL*, AND
GF > *LM*.

LET,

$GO = KL$, AND
$GP = LM$.

AND LET,

$\boxminus OGPQ$, BE COMPLETED;

THEREFORE,
IT IS EQUAL AND SIMILAR TO KM.

[VI. 21] THEREFORE,
GQ IS, ALSO, SIMILAR TO GB;

[VI. 26] THEREFORE,
GQ IS ABOUT THE SAME DIAMETER, WITH GB.

LET,
GQB BE THEIR DIAMETER,

AND LET,
THE FIGURE BE DESCRIBED.

THEN, SINCE,
$BG = C$, KM, AND IN THEM
$GQ = KM$,

THEREFORE,
THE REMAINDER,
THE GNOMON, UWV = THE REMAINDER, C.

AND, SINCE,
$PR = OS$,

LET,
QB BE ADDED TO EACH;

THEREFORE,
THE WHOLE, PB = THE WHOLE, OB.

BUT,
$OB = TE$,

[I. 36] SINCE,
THE SIDES, $AE = EB$;

THEREFORE,
$TE = PB$.

LET,
OS BE ADDED TO EACH;

THEREFORE,
THE WHOLE, TS =
THE WHOLE, THE GNOMON, VWU.

BUT,

THE GNOMON, $VWU = C$;

THEREFORE,

$TS = C$.

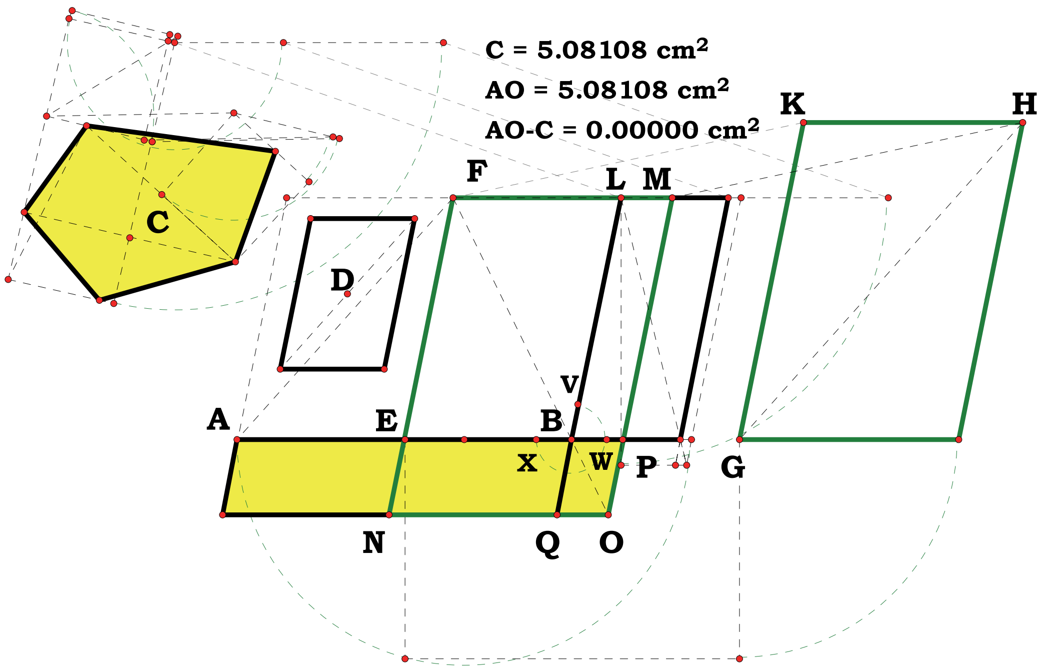
THEREFORE,

TO THE GIVEN STRAIGHT LINE, AB, THERE HAS BEEN APPLIED
THE PARALLELOGRAM, ST, EQUAL TO
THE GIVEN RECTILINEAL FIGURE, C, AND
DEFICIENT BY A PARALLELOGRAMMIC FIGURE, QB,
WHICH IS SIMILAR TO D.

Q. E. F.

PROPOSITION 29.

TO A GIVEN STRAIGHT LINE TO APPLY A PARALLELOGRAM EQUAL TO A GIVEN RECTILINEAL FIGURE AND EXCEEDING BY A PARALLELOGRAMMIC FIGURE SIMILAR TO A GIVEN ONE.



LET,

AB BE GIVEN,

C, THE GIVEN RECTILINEAL FIGURE TO WHICH

THE FIGURE TO BE APPLIED TO *AB*, IS REQUIRED TO BE EQUAL,

AND,

D, THAT TO WHICH THE EXCESS IS REQUIRED TO BE SIMILAR;

THUS IT IS REQUIRED,

TO APPLY TO *AB*,

A PARALLELOGRAM EQUAL TO THE RECTILINEAL FIGURE, *C*,

AND,

EXCEEDING BY A PARALLELOGRAMMIC FIGURE SIMILAR TO *D*.

LET,

AB BE BISECTED AT *E*;

LET,

THERE BE DESCRIBED, ON *EB*,

⊟*BF*, SIMILAR AND, SIMILARLY SITUATED, TO *D*;

[VI. 25] AND LET,

⊟*GH* = ⊟*BF* + *C*, AND

SIMILAR AND SIMILARLY SITUATED TO *D*.

LET,

KH CORRESPOND TO FL, AND
KG TO FE.

NOW, SINCE,

$\boxminus GH > \boxminus FB$,

THEREFORE,
$KH > FL$, AND $KG > FE$.

LET,
FL, FE BE PRODUCED,

LET,
$FLM = KH$, AND $FEN = KG$,

AND LET,

$\boxminus MN$ BE COMPLETED;

THEREFORE,

$\boxminus MN$ IS BOTH EQUAL AND SIMILAR TO $\boxminus GH$.

BUT,

$\boxminus GH$ IS SIMILAR TO $\boxminus EL$;

[VI. 21] THEREFORE,

$\boxminus MN$ IS, ALSO, SIMILAR TO $\boxminus EL$;

[VI. 26] THEREFORE,

$\boxminus EL$ IS ABOUT THE SAME DIAMETER WITH $\boxminus MN$.

LET,
THEIR DIAMETER, FO, BE DRAWN,

AND LET,
THE FIGURE BE DESCRIBED.

SINCE,

$\boxminus GH = \boxminus EL + C$, WHILE

$\boxminus GH = \boxminus MN$,

THEREFORE,

$\boxminus MN = \boxminus EL + C$.

LET,
EL BE SUBTRACTED FROM EACH;

THEREFORE,
THE REMAINDER, THE GNOMON, $XWV = C$.

[I. 36] NOW, SINCE,

$AE = EB$,

$\boxminus AN = \boxminus NB$,

[I. 43] THAT IS, TO $\boxminus LP$.

LET,

EO BE ADDED TO EACH;

THEREFORE,

THE WHOLE, $\boxminus AO$ = THE GNOMON, VWX.

BUT,

THE GNOMON, $VWX = C$;

THEREFORE,

$\boxminus AO = C$.

[VI. 24] THEREFORE,

TO AB,

THERE HAS BEEN APPLIED $\boxminus AO$,

EQUAL TO THE GIVEN RECTILINEAL FIGURE, C, AND

EXCEEDING BY $\boxminus QP$,

WHICH IS SIMILAR TO D,

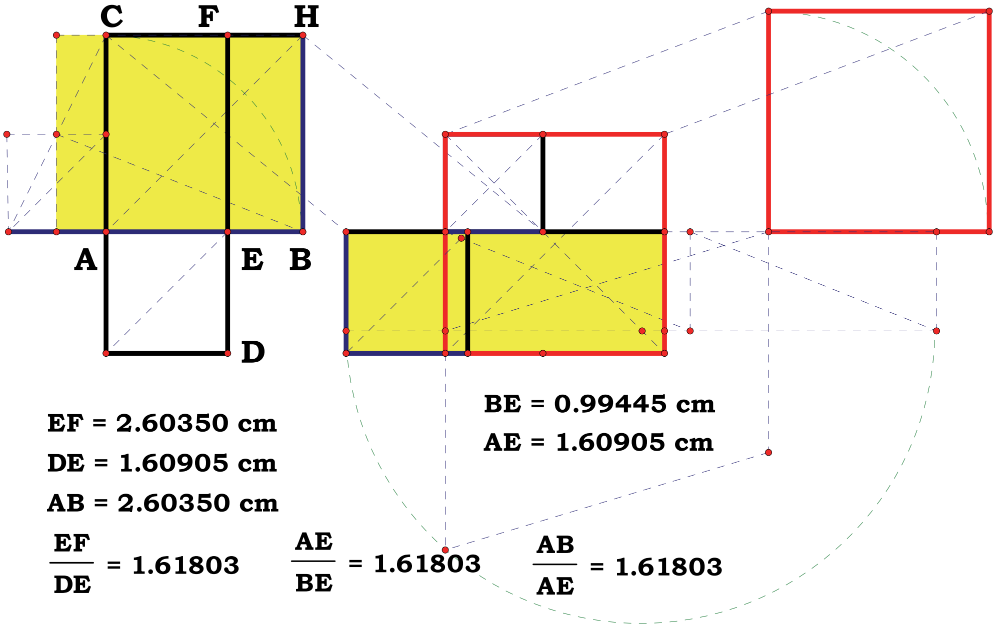
SINCE,

$\boxminus PQ$ IS, ALSO, SIMILAR TO $\boxminus EL$.

Q. E. F.

PROPOSITION 30.

TO CUT A GIVEN FINITE STRAIGHT LINE IN EXTREME AND MEAN RATIO.



LET,

AB BE GIVEN;

THUS IT IS REQUIRED,

TO CUT *AB*, IN EXTREME AND MEAN RATIO.

LET,

ON *AB*,

⊡*BC*, BE DESCRIBED;

[VI. 29] AND LET,

THERE BE APPLIED TO *AC*,

⊟ *CD*, EQUAL TO *BC*, AND

EXCEEDING BY THE FIGURE, *AD*, SIMILAR TO *BC*.

NOW,

BC IS A SQUARE;

THEREFORE,

AD IS, ALSO, A SQUARE.

AND, SINCE,

BC = *CD*,

LET,

CE BE SUBTRACTED FROM EACH;

THEREFORE,

THE REMAINDERS, *BF* = *AD*.

BUT,

IT IS, ALSO, EQUIANGULAR WITH IT;

[VI. 14] THEREFORE,

IN *BF*, *AD*, THE SIDES ABOUT
THE EQUAL ANGLES ARE RECIPROCALLY PROPORTIONAL;

THEREFORE,

AS *FE* IS TO *ED*,
SO IS *AE* TO *EB*.

BUT,

$FE = AB$, AND
$ED = AE$.

THEREFORE,

AS *BA* IS TO *AE*,
SO IS *AE* TO *EB*.

AND,

$AB > AE$;

THEREFORE,

$AE > EB$.

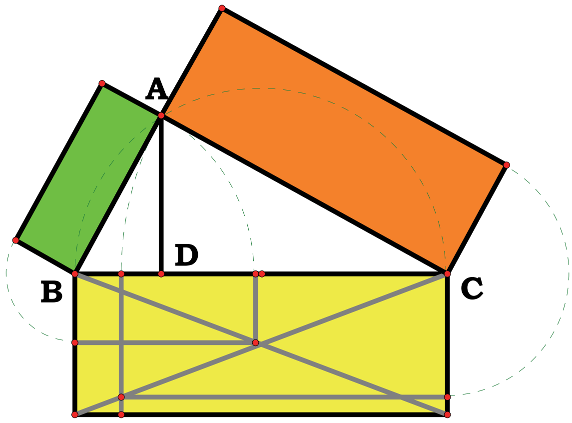
THEREFORE,

AB, HAS BEEN CUT IN EXTREME, AND
MEAN RATIO AT *E*, AND
THE GREATER SEGMENT OF IT IS *AE*.

Q. E. F.

Proposition 31.

In right-angled triangles the figure on the side subtending the right angle is equal, to the similar and similarly described figures on the sides containing the right angle.



AB = 2.15416 cm²
AC = 7.06344 cm²
BC = 9.21761 cm²
BC-(AC+AB) = 0.00000 cm²

Let,

$\triangle ABC$,

be a right-angled triangle having

$\llcorner BAC$, right;

I say that;

the figure, on BC equals the similar and similarly described figures, on $BA + AC$.

Let,

AD be drawn perpendicular.

[vi. 8] Then since,

in $\triangle ABC$, $AD \perp BC$,

$\triangle ABD$, $\triangle ADC$, adjoining the perpendicular are similar both to $\triangle ABC$, and to one another.

And, since,

ABC is similar to ABD,

[vi. Def. 1] therefore,

as CB is to BA,
so is AB to BD.

[vi. 19, Por.] And, since,

three straight lines are proportional,
as the first is to the third,
so is the figure on the first to

THE SIMILAR AND SIMILARLY DESCRIBED FIGURE ON
THE SECOND,

THEREFORE,

AS CB IS TO BD,
SO IS THE FIGURE ON CB TO
THE SIMILAR AND SIMILARLY DESCRIBED FIGURE, ON BA.

FOR THE SAME REASON ALSO,

AS BC IS TO CD,
SO IS THE FIGURE, ON BC, TO THAT, ON CA;

SO THAT, IN ADDITION,

AS BC IS TO BD, DC,
SO IS THE FIGURE, ON BC TO THE SIMILAR, AND
SIMILARLY DESCRIBED FIGURES, ON BA, AC.

BUT,

$\boxdot BC = \boxdot BD + \boxdot DC$;

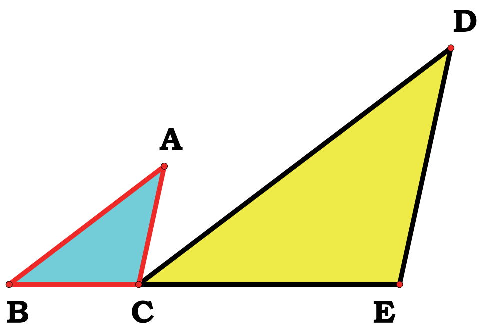
THEREFORE,

THE FIGURE, ON BC = THE SIMILAR, AND
SIMILARLY DESCRIBED FIGURES, ON BA, AC.

THEREFORE ETC.

Q. E. D.

Proposition 32.



If two triangles having two sides proportional to two sides be placed together at one angle so that their corresponding sides are, also, parallel, the remaining sides of the triangles will be in a straight line.

Let,

ABC, *DCE*, be two triangles having
the two sides, *BA*, *AC*, proportional to
the two sides, *DC*, *DE*,

so that,

as *AB* is to *AC*,
so is *DC* to *DE*, and
AB parallel to *DC*, and
AC to *DE*;

I say that;

BC is in a straight line with *CE*.

For, since,

AB is parallel to *DC*,

[I. 29]

and,

the straight line *AC* has fallen upon them,
the alternate angles,
BAC, *ACD*, are equal to one another.

For the same reason,

$\angle CDE = \angle ACD$;

so that,

$\angle BAC = \angle CDE$.

And, since,

ABC, *DCE* are two triangles having one angle,

$\angle$at *A*, equal to one angle,

$\angle$at *D*, and

the sides about the equal angles proportional,

so that,

as *BA* is to *AC*,

SO IS CD TO DE,

[VI. 6]

THEREFORE,

ΔABC, IS EQUIANGULAR WITH

ΔDCE;

THEREFORE,

$\angle ABC = \angle DCE$.

BUT,

$\angle ACD =$

$\angle BAC$;

THEREFORE,

THE WHOLE ANGLE, $ACE =$
THE TWO ANGLES, ABC, BAC.

LET,

$\angle ACB$, BE ADDED TO EACH;

THEREFORE,

THE ANGLES, ACE, ACB, ARE EQUAL TO
THE ANGLES, BAC, ACB, CBA.

[I. 32]

BUT,

THE ANGLES, BAC, ABC, ACB ARE EQUAL TO
TWO RIGHT ANGLES;

THEREFORE,

THE ANGLES, ACE, ACB, ARE, ALSO, EQUAL TO TWO
RIGHT ANGLES.

THEREFORE,

WITH A STRAIGHT LINE, AC, AND
AT THE POINT, C, ON IT,
THE TWO STRAIGHT LINES, BC, CE,
NOT LYING ON THE SAME SIDE MAKE THE ADJACENT ANGLES,
ACE, ACB, EQUAL TO TWO RIGHT ANGLES;

[I. 14]

THEREFORE,

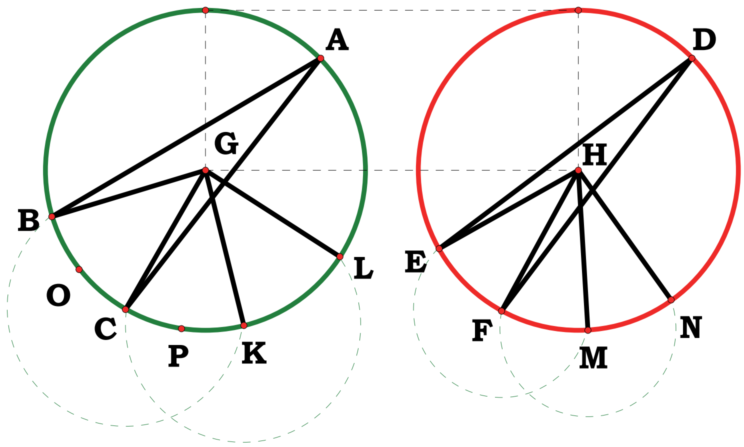
BC IS IN A STRAIGHT LINE WITH CE.

THEREFORE ETC.

Q. E. D.

Proposition 33.

In equal circles angles have the same ratio as the circumferences on which they stand, whether they stand at the centres or at the circumferences.



Let,

ABC, *DEF* be equal circles,

and let,

∠*BGC*, ∠*EHF*, be angles at their centres, *G*, *H*, and

∠*BAC*, ∠*EDF*, angles at the circumferences;

I say that;

as the circumference, *BC*, is to the circumference, *EF*,

so is ∠*BGC*, to ∠*EHF*, and

∠*BAC*, to ∠*EDF*.

For let,

any number of consecutive circumferences, *CK*, *KL*, be made equal to the circumference, *BC*,

and any number of consecutive circumferences, *FM*, *MN*, equal to the circumference, *EF*;

and let,

GK, *GL*, *HM*, *HN*, be joined.

[iii. 27] Then, since,

the circumferences, *BC*, *CK*, *KL*, are equal to one another,

∠*BGC*, ∠*CGK*, ∠*KGL*, are, also, equal to one another;

therefore,

whatever multiple the circumference, *BL* is of *BC*,

THAT MULTIPLE, ALSO, IS $\angle BGL$ OF $\angle BGC$.

FOR THE SAME REASON ALSO,
WHATEVER MULTIPLE THE CIRCUMFERENCE,
NE IS OF EF, THAT MULTIPLE, ALSO, IS
$\angle NHE$ OF $\angle EHF$.

[III. 27] IF THEN,
THE CIRCUMFERENCES, $BL = EN$,
$\angle BGL = \angle EHN$;

IF,
THE CIRCUMFERENCES, $BL > EN$,
$\angle BGL > \angle EHN$;
AND, IF LESS, LESS.

THERE BEING THEN FOUR MAGNITUDES,
TWO CIRCUMFERENCES, BC, EF, AND
TWO ANGLES, $\angle BGC$, $\angle EHF$, THERE HAVE BEEN TAKEN, OF
THE CIRCUMFERENCE, BC, AND $\angle BGC$, EQUIMULTIPLES,

NAMELY,
THE CIRCUMFERENCE, BL, AND $\angle BGL$, AND
OF THE CIRCUMFERENCE, EF, AND $\angle EHF$, EQUIMULTIPLES,

NAMELY,
THE CIRCUMFERENCE, EN, AND $\angle EHN$.

AND IT HAS BEEN PROVED THAT; IF,
THE CIRCUMFERENCE, BL, IS IN EXCESS OF
THE CIRCUMFERENCE, EN,
$\angle BGL$, IS, ALSO, IN EXCESS OF $\angle EHN$;
IF EQUAL, EQUAL; AND,
IF LESS, LESS.

[V. DEF. 5] THEREFORE,
AS THE CIRCUMFERENCE, BC IS TO EF,
SO IS $\angle BGC$ TO $\angle EHF$.

BUT,
AS $\angle BGC$ IS TO $\angle EHF$,
SO IS $\angle BAC$ TO $\angle EDF$;

FOR,
THEY ARE DOUBLES RESPECTIVELY.

THEREFORE ALSO,

AS THE CIRCUMFERENCE, *BC*, IS TO THE CIRCUMFERENCE, *EF*,

SO IS $\angle BGC$, TO $\angle EHF$, AND

$\angle BAC$, TO $\angle EDF$.

THEREFORE ETC.

Q. E. D.

BOOK VII.

OF

EUCLID'S ELEMENTS

TRANSLATED FROM THE TEXT OF HEIBERG

BY

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K. C. B., K. C. V. O., F. R. S.,

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2013 *EDITION*

REVISED WITH SUBTRACTIONS

REFORMATTED AND ABRIDGED
FOR STUDY OF
***THE ELEMENTS*.**

BY JOHN CLARK.

BOOK VII.

Definitions.

1. AN **UNIT** IS THAT BY VIRTUE OF WHICH EACH, OF THE THINGS THAT EXIST IS CALLED ONE.

2. A **NUMBER** IS A MULTITUDE COMPOSED OF UNITS.

3. A NUMBER IS **A PART** OF A NUMBER, THE LESS OF THE GREATER, WHEN IT MEASURES THE GREATER;

4. BUT **PARTS** WHEN IT DOES NOT MEASURE IT.

5. THE GREATER NUMBER IS A **MULTIPLE** OF THE LESS WHEN IT IS MEASURED BY THE LESS.

6. AN **EVEN NUMBER** IS THAT WHICH IS DIVISIBLE INTO TWO EQUAL PARTS.

7. AN **ODD NUMBER** IS THAT WHICH IS NOT DIVISIBLE INTO TWO EQUAL PARTS, OR THAT WHICH DIFFERS BY AN UNIT FROM AN EVEN NUMBER.

8. AN **EVEN-TIMES EVEN NUMBER** IS THAT WHICH IS MEASURED BY AN EVEN NUMBER ACCORDING TO AN EVEN NUMBER.

9. AN **EVEN-TIMES ODD NUMBER** IS THAT WHICH IS MEASURED BY AN EVEN NUMBER ACCORDING TO AN ODD NUMBER.

10. AN **ODD-TIMES ODD NUMBER** IS THAT WHICH IS MEASURED BY AN ODD NUMBER ACCORDING TO AN ODD NUMBER.

11. A **PRIME NUMBER** IS THAT WHICH IS MEASURED BY AN UNIT ALONE.

12. NUMBERS **PRIME TO ONE ANOTHER** ARE THOSE WHICH ARE MEASURED BY AN UNIT ALONE AS A COMMON MEASURE.

13. A **COMPOSITE NUMBER** IS THAT WHICH IS MEASURED BY SOME NUMBER.

14. NUMBERS **COMPOSITE TO ONE ANOTHER** ARE THOSE WHICH ARE MEASURED BY SOME NUMBER AS A COMMON MEASURE.

15. A NUMBER IS SAID TO **MULTIPLY** A NUMBER WHEN THAT WHICH IS MULTIPLIED IS ADDED TO ITSELF AS MANY TIMES AS THERE ARE UNITS IN THE OTHER, AND THUS SOME NUMBER IS PRODUCED.

16. AND, WHEN TWO NUMBERS HAVING MULTIPLIED ONE ANOTHER MAKE SOME NUMBER, THE NUMBER SO PRODUCED IS CALLED **PLANE**, AND ITS **SIDES** ARE THE NUMBERS WHICH HAVE MULTIPLIED ONE ANOTHER.

17. AND, WHEN THREE NUMBERS HAVING MULTIPLIED ONE ANOTHER MAKE SOME NUMBER, THE NUMBER SO PRODUCED IS **SOLID**, AND ITS **SIDES** ARE THE NUMBERS WHICH HAVE MULTIPLIED ONE ANOTHER.

18. A **SQUARE NUMBER** IS EQUAL MULTIPLIED BY EQUAL, OR A NUMBER WHICH IS CONTAINED BY TWO EQUAL NUMBERS.

19. AND A **CUBE** IS EQUAL MULTIPLIED BY EQUAL AND AGAIN BY EQUAL, OR A NUMBER WHICH IS CONTAINED BY THREE EQUAL NUMBERS.

20. NUMBERS ARE **PROPORTIONAL** WHEN THE FIRST IS THE SAME MULTIPLE, OR THE SAME PART, OR THE SAME PARTS, OF THE SECOND THAT THE THIRD IS OF THE FOURTH.

21. **SIMILAR PLANE** AND **SOLID** NUMBERS ARE THOSE WHICH HAVE THEIR SIDES PROPORTIONAL.

22. A **PERFECT NUMBER** IS THAT WHICH EQUALS ITS OWN PARTS.

Notes.

Definition 1. *An* UNIT *is that by virtue of which each, of the things that exist is called one.*

NOTES.

DEFINITION 2. *A* NUMBER *IS A MULTITUDE COMPOSED OF UNITS*.

NOTES.

DEFINITION 3. *A NUMBER IS* A PART *OF A NUMBER*, *THE LESS OF THE GREATER*, *WHEN IT MEASURES THE GREATER*;

NOTES.

DEFINITION 4. *BUT* PARTS *WHEN IT DOES NOT MEASURE IT*.

NOTES.

DEFINITION 5. *THE GREATER NUMBER IS A* MULTIPLE *OF THE LESS WHEN IT IS MEASURED BY THE LESS*.

Notes.

Definitions 6. *An* even number *is that which is divisible into two equal parts.*

Notes.

Definitions 7. *An* odd number *is that which is not divisible into two equal parts, or that which differs by an unit from an even number.*

NOTES.

DEFINITION 8. *AN* EVEN-TIMES EVEN NUMBER *IS THAT WHICH IS MEASURED BY AN EVEN NUMBER ACCORDING TO AN EVEN NUMBER*.

Notes.

Definition 9. *An* even-times odd number *is that which is measured by an even number according to an odd number*.

Notes.

Definition 10. *An* odd-times odd number *is that which is measured by an odd number according to an odd number.*

NOTES.

DEFINITION 11. *A* PRIME NUMBER *IS THAT WHICH IS MEASURED BY AN UNIT ALONE.*

NOTES.

DEFINITION 12. *NUMBERS* PRIME TO ONE ANOTHER *ARE THOSE WHICH ARE MEASURED BY AN UNIT ALONE AS A COMMON MEASURE.*

Notes.

Definition 13. *A* composite number *is that which is measured by some number*.

Notes.

Definition 14. *Numbers* composite to one another *are those which are measured by some number as a common measure.*

Notes.

Definition 15. *A number is said to* multiply *a number when that which is multiplied is added to itself as many times as there are units in the other, and thus some number is produced.*

NOTES.

DEFINITION 16. *AND, WHEN TWO NUMBERS HAVING MULTIPLIED ONE ANOTHER MAKE SOME NUMBER, THE NUMBER SO PRODUCED IS CALLED* PLANE, *AND ITS* SIDES *ARE THE NUMBERS WHICH HAVE MULTIPLIED ONE ANOTHER.*

Notes.

Definition 17. *And, when three numbers having multiplied one another make some number, the number so produced is* solid, *and its* sides *are the numbers which have multiplied one another.*

NOTES.

DEFINITION 18. *A* SQUARE NUMBER *IS EQUAL MULTIPLIED BY EQUAL, OR A NUMBER WHICH IS CONTAINED BY TWO EQUAL NUMBERS.*

NOTES.

DEFINITION 19. *AND A* CUBE *IS EQUAL MULTIPLIED BY EQUAL AND AGAIN BY EQUAL, OR A NUMBER WHICH IS CONTAINED BY THREE EQUAL NUMBERS.*

Notes.

Definition 20. *Numbers are* proportional *when the first is the same multiple, or the same part, or the same parts, of the second that the third is of the fourth.*

Notes.

Definition 21. Similar plane *and* solid *numbers are those which have their sides proportional*.

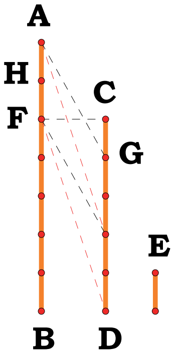
Notes.

Definition 22. *A* perfect number *is that which equals its own parts.*

BOOK VII.

PROPOSITIONS.

PROPOSITION 1.



TWO UNEQUAL NUMBERS BEING SET OUT, AND THE LESS BEING CONTINUALLY SUBTRACTED IN TURN FROM THE GREATER, IF THE NUMBER WHICH IS LEFT NEVER MEASURES THE ONE BEFORE IT UNTIL AN UNIT IS LEFT, THE ORIGINAL NUMBERS WILL BE PRIME TO ONE ANOTHER.

FOR,

THE LESS OF TWO UNEQUAL NUMBERS,
AB, *CD*, BEING CONTINUALLY SUBTRACTED FROM THE GREATER,

LET,

THE NUMBER WHICH IS LEFT NEVER MEASURE
THE ONE BEFORE IT UNTIL AN UNIT IS LEFT;

I SAY THAT;

AB, *CD* ARE PRIME TO ONE ANOTHER,

THAT IS,

THAT AN UNIT ALONE MEASURES *AB*, *CD*.

FOR,

IF *AB*, *CD* ARE NOT PRIME TO ONE ANOTHER,
SOME NUMBER WILL MEASURE THEM.

LET,

A NUMBER MEASURE THEM,

AND LET,

IT BE *E*;

LET,

CD, MEASURING *BF*,
LEAVE *FA* LESS THAN ITSELF,

LET,

AF, MEASURING *DG*,
LEAVE *GC* LESS THAN ITSELF,

AND LET,

GC, MEASURING *FH*,
LEAVE AN UNIT *HA*.

SINCE, THEN,

E MEASURES *CD*, AND

CD MEASURES *BF*,

THEREFORE,

E, ALSO, MEASURES *BF*.

BUT,

IT, ALSO, MEASURES THE WHOLE *BA*;

THEREFORE,

IT WILL, ALSO, MEASURE THE REMAINDER, *AF*.

BUT,

AF MEASURES *DG*;

THEREFORE,

E, ALSO, MEASURES *DG*.

BUT,

IT, ALSO, MEASURES THE WHOLE, *DC*,

THEREFORE,

IT WILL, ALSO, MEASURE THE REMAINDER, *CG*.

BUT,

CG MEASURES *FH*;

THEREFORE,

E, ALSO, MEASURES *FH*.

BUT,

IT, ALSO, MEASURES THE WHOLE, *FA*;

THEREFORE,

IT WILL, ALSO, MEASURE THE REMAINDER, THE UNIT, *AH*,
THOUGH IT IS A NUMBER:

WHICH,

IS IMPOSSIBLE.

THEREFORE,

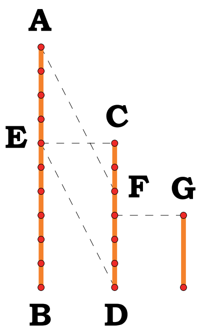
NO NUMBER WILL MEASURE THE NUMBERS *AB*, *CD*;

[VII. DEF. 12] THEREFORE,

AB, *CD* ARE PRIME TO ONE ANOTHER.

Q. E. D.

PROPOSITION 2.



GIVEN TWO NUMBERS NOT PRIME TO ONE ANOTHER, TO FIND THEIR GREATEST COMMON MEASURE.

LET,
AB, *CD* BE THE TWO GIVEN NUMBERS
NOT PRIME TO ONE ANOTHER.

THUS IT IS REQUIRED,
TO FIND THE GREATEST COMMON MEASURE OF *AB*, *CD*.

IF NOW,
CD MEASURES *AB*, AND
IT, ALSO, MEASURES ITSELF,
CD IS A COMMON MEASURE OF *CD*, *AB*, AND
IT IS MANIFEST THAT IT IS, ALSO, THE GREATEST;

FOR,
NO GREATER NUMBER THAN *CD* WILL MEASURE *CD*.

BUT,
IF *CD* DOES NOT MEASURE *AB*,

THEN,
THE LESS OF THE NUMBERS, *AB*, *CD*,
BEING CONTINUALLY SUBTRACTED FROM THE GREATER,
SOME NUMBER WILL BE LEFT WHICH WILL MEASURE
THE ONE BEFORE IT.

[VII. 1] FOR,
AN UNIT WILL NOT BE LEFT;

OTHERWISE,
AB, *CD* WILL BE PRIME TO ONE ANOTHER,

WHICH,
IS CONTRARY TO THE HYPOTHESIS.

THEREFORE,
SOME NUMBER WILL BE LEFT, WHICH
WILL MEASURE THE ONE BEFORE IT.

NOW LET,
CD, MEASURING *BE*,
LEAVE *EA* LESS THAN ITSELF,

LET,
EA, MEASURING *DF*,
LEAVE *FC* LESS THAN ITSELF,

AND LET,

CF MEASURE AE.

SINCE THEN,

CF MEASURES AE, AND
AE MEASURES DF,

THEREFORE,

CF WILL, ALSO, MEASURE DF.

BUT,

IT, ALSO, MEASURES ITSELF;

THEREFORE,

IT WILL, ALSO, MEASURE THE WHOLE, CD.

BUT,

CD MEASURES BE;

THEREFORE,

CF, ALSO, MEASURES BE.

BUT,

IT, ALSO, MEASURES EA;

THEREFORE,

IT WILL, ALSO, MEASURE THE WHOLE, BA.

BUT,

IT, ALSO, MEASURES CD;

THEREFORE,

CF MEASURES AB, CD.

THEREFORE,

CF IS A COMMON MEASURE OF AB, CD.

I SAY NEXT;

THAT IT IS, ALSO, THE GREATEST.

FOR,

IF CE IS NOT THE GREATEST COMMON MEASURE OF AB, CD,
SOME NUMBER WHICH IS GREATER THAN CF,
WILL MEASURE THE NUMBERS AB, CD.

LET,

SUCH A NUMBER MEASURE THEM,

AND LET,

IT BE G.

NOW, SINCE,

G MEASURES CD, WHILE
CD MEASURES BE,
G, ALSO, MEASURES BE.

But,

it, also, measures the whole, BA;

therefore,

it will, also, measure the remainder, AE.

But,

AE measures DF;

therefore,

G will, also, measure DF.

But,

it, also, measures the whole, DC;

therefore,

it will, also, measure the remainder, CF,

that is,

the greater will measure the less:

which,

is impossible.

Therefore,

no number, which is greater than CF, will measure the numbers AB, CD;

therefore,

CF is the greatest common measure of AB, CD.

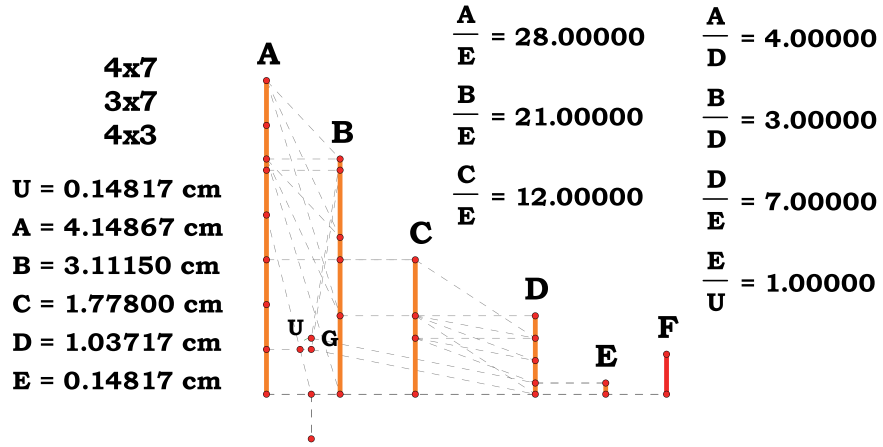
Porism.

From this it is manifest that, if a number measure two numbers, it will, also, measure their greatest common measure.

Q. E. D.

PROPOSITION 3.

GIVEN THREE NUMBERS NOT PRIME TO ONE ANOTHER, TO FIND THEIR GREATEST COMMON MEASURE.



LET,

A, *B*, *C* BE THE THREE GIVEN NUMBERS
NOT PRIME TO ONE ANOTHER;

THUS IT IS REQUIRED,

TO FIND THE GREATEST COMMON MEASURE OF *A*, *B*, *C*.

[VII. 2] FOR LET,

THE GREATEST COMMON MEASURE, *D*,
OF THE TWO NUMBERS, *A*, *B*, BE TAKEN;

THEN,

D EITHER MEASURES, OR
DOES NOT MEASURE, *C*.

FIRST, LET IT,

MEASURE IT.

BUT,

IT MEASURES *A*, *B* ALSO;

THEREFORE,

D MEASURES *A*, *B*, *C*;

THEREFORE,

D IS A COMMON MEASURE OF *A*, *B*, *C*.

I SAY THAT;

IT IS, ALSO, THE GREATEST.

FOR,

IF *D* IS NOT THE GREATEST COMMON MEASURE OF *A*, *B*, *C*,
SOME NUMBER WHICH IS GREATER THAN *D*,
WILL MEASURE THE NUMBERS, *A*, *B*, *C*.

LET,

SUCH A NUMBER MEASURE THEM,

AND LET,
IT BE E.

SINCE THEN,
E MEASURES A, B, C,
IT WILL, ALSO, MEASURE A, B;

[VII. 2, POR.] THEREFORE,
IT WILL, ALSO, MEASURE
THE GREATEST COMMON MEASURE OF A, B.

BUT,
THE GREATEST COMMON MEASURE OF A, B IS D;

THEREFORE,
E MEASURES D,
THE GREATER THE LESS:

WHICH,
IS IMPOSSIBLE.

THEREFORE,
NO NUMBER, WHICH IS GREATER THAN D,
WILL MEASURE THE NUMBERS, A, B, C.

THEREFORE,
D IS THE GREATEST COMMON MEASURE OF A, B, C.

NEXT, LET,
D NOT MEASURE C;

I SAY FIRST THAT;
C, D ARE NOT PRIME TO ONE ANOTHER.

FOR, SINCE,
A, B, C ARE NOT PRIME TO ONE ANOTHER,
SOME NUMBER WILL MEASURE THEM.

[VII. 2, POR.] NOW,
THAT WHICH MEASURES A, B, C,
WILL, ALSO, MEASURE A, B, AND
WILL MEASURE D, THE GREATEST COMMON MEASURE OF A, B.

BUT,
IT MEASURES C ALSO;

THEREFORE,
SOME NUMBER WILL MEASURE THE NUMBERS, D, C;

THEREFORE,
D, C ARE NOT PRIME TO ONE ANOTHER.

[VII. 2] LET,

THEN THEIR GREATEST COMMON MEASURE, *E*, BE TAKEN.

THEN, SINCE,
E MEASURES *D*, AND
D MEASURES *A*, *B*,

THEREFORE,
E, ALSO, MEASURES *A*, *B*.

BUT,
IT MEASURES *C*, ALSO;
THEREFORE MEASURES *A*, *B*, *C*,

THEREFORE,
E IS A COMMON MEASURE OF *A*, *B*, *C*.

I SAY NEXT THAT;
IT IS, ALSO, THE GREATEST.

FOR,
IF *E* IS NOT THE GREATEST COMMON MEASURE OF *A*, *B*, *C*,
SOME NUMBER WHICH IS GREATER THAN *E* WILL MEASURE
THE NUMBERS, *A*, *B*, *C*.

LET,
SUCH A NUMBER MEASURE THEM,

AND LET,
IT BE *F*.

NOW, SINCE,
F MEASURES *A*, *B*, *C*,
IT, ALSO, MEASURES *A*, *B*;

[VII. 2, POR.] THEREFORE,
IT WILL, ALSO, MEASURE
THE GREATEST COMMON MEASURE OF *A*, *B*.

BUT,
THE GREATEST COMMON MEASURE OF *A*, *B* IS *D*;

THEREFORE,
F MEASURES *D*.

AND,
IT MEASURES *C* ALSO;

THEREFORE,
F MEASURES *D*, *C*;

[VII. 2, POR.] THEREFORE,
IT WILL, ALSO, MEASURE
THE GREATEST COMMON MEASURE OF *D*, *C*.

BUT,

THE GREATEST COMMON MEASURE OF D, C IS E;

THEREFORE,

F MEASURES E,

THE GREATER THE LESS:

WHICH,

IS IMPOSSIBLE.

THEREFORE,

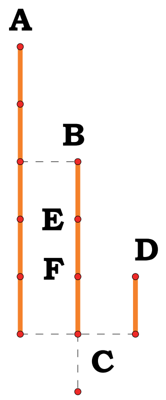
NO NUMBER, WHICH IS GREATER THAN E, WILL MEASURE THE NUMBERS, A, B, C;

THEREFORE,

E IS THE GREATEST COMMON MEASURE OF A, B, C.

Q. E. D.

Proposition 4.



Any number is either a part or parts of any number, the less of the greater.

Let,
A, *BC* be two numbers,

and let,
BC be the less;

I say that;
BC is either a part, or parts, of *A*.

For,
A, *BC* are either prime to one another or not.

First, let,
A, *BC* be prime to one another.

Then,
if *BC* be divided into the units in it,
each unit of those in *BC* will be some part of *A*;

so that,
BC is parts of *A*.

Next let,
A, *BC* not be prime to one another;

then,
BC either measures, or does not measure, *A*.

now,
If *BC* measures *A*,
BC is a part of *A*.

[vii. 2] But, if not, let,
the greatest common measure, *D* of *A*, *BC*, be taken;

and let,
BC be divided into the numbers equal to *D*,

namely,
BE, *EF*, *FC*;

Now, since,
D measures *A*,
D is a part of *A*.

But,
D equals each, of the numbers, *BE*, *EF*, *FC*;

therefore,

EACH, OF THE NUMBERS, *BE*, *EF*, *FC*, IS, ALSO, A PART OF *A*;

SO THAT,

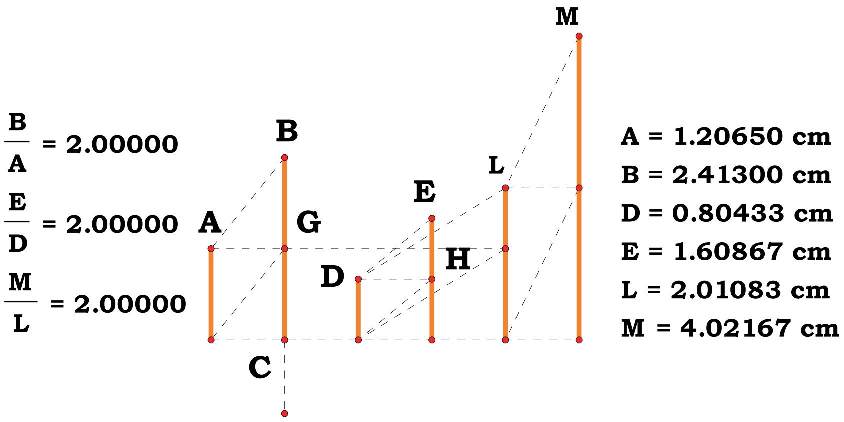
BC IS PARTS OF *A*.

THEREFORE ETC.

Q. E. D.

PROPOSITION 5.

IF A NUMBER BE A PART OF A NUMBER, AND ANOTHER BE THE SAME PART OF ANOTHER, THE SUM WILL, ALSO, *BE THE SAME PART OF THE SUM THAT THE ONE IS OF THE ONE.*



FOR LET,

THE NUMBER, *A*, BE A PART, OF *BC*, AND ANOTHER
D, THE SAME PART, OF ANOTHER, *EF*, THAT *A* IS OF *BC*;

I SAY THAT;

THE SUM, OF *A*, *D*, IS, ALSO, THE SAME PART OF
THE SUM, OF *BC*, *EF*, THAT *A* IS OF *BC*.

FOR SINCE,

WHATEVER PART, *A*, IS OF *BC*,
D IS, ALSO, THE SAME PART OF *EF*,

THEREFORE,

AS MANY NUMBERS AS THERE ARE IN *BC* EQUAL TO *A*,
SO MANY NUMBERS ARE THERE, ALSO, IN *EF* EQUAL TO *D*.

LET,

BC BE DIVIDED INTO THE NUMBERS EQUAL TO *A*,

NAMELY,

BG, *GC*, AND
EF INTO THE NUMBERS EQUAL TO *D*,

NAMELY,

EH, *HF*;

THEN,

THE MULTITUDE, OF *BG*, *GC*, WILL BE EQUAL TO
THE MULTITUDE, OF *EH*, *HF*.

AND, SINCE,

BG = *A*, AND
EH TO *D*,

THEREFORE,

BG, EH are, also, equal to A, D.

For the same reason,

GC, HF are, also, equal to A, D.

Therefore,

as many numbers as there are in BC equal to A,
so many are there, also, in BC, EF equal to A, D.

Therefore,

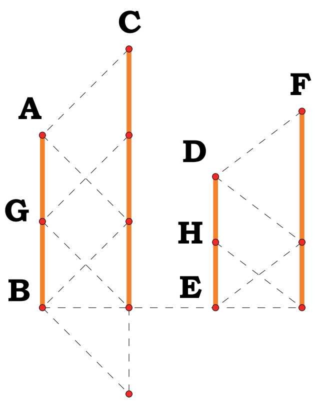
whatever multiple, BC, is of A,
the same multiple, also, is
the sum, of BC, EF, of the sum, of A, D.

Therefore,

whatever part, A, is of BC,
the same part, also, is
the sum, of A, D, of the sum, of BC, EF.

Q. E. D.

Proposition 6.



If a number be parts of a number, and another be the same parts of another, the sum will, also, be the same parts of the sum that the one is of the one.

For let,
the number, *AB*,
be parts of the number *C*,

and another,
DB, the same parts of another, *F*,
that *AB* is of *C*;

I say that;
the sum, of *AB*, *DE*, is, also,
the same parts of the sum, of *C*, *F*, that *AB* is of *C*.

For since,
whatever parts, *AB*, is of *C*,
DE is, also, the same parts, of *F*,

therefore,
as many parts, of *C*, as there are in *AB*,
so many parts, of *F*, are there, also, in *DE*.

Let,
AB be divided into the parts of *C*,

namely,
AG, *GB*, and
DE into the parts of *F*,

namely,
DH, *HE*;

thus,
the multitude, of *AG*, *GB*, will be equal to
the multitude, of *DH*, *HE*.

And since,
whatever part, *AG*, is of *C*,
the same part is *DH* of *F* also,

[vii. 5] therefore,
whatever part, *AG*, is of *C*,
the same part, also, is
the sum of *AG*, *DH*, of the sum, of *C*, *F*.

For the same reason,

WHATEVER PART, GB, IS OF C, THE SAME PART, ALSO, IS THE SUM, OF GB, HE, OF THE SUM, OF C, F.

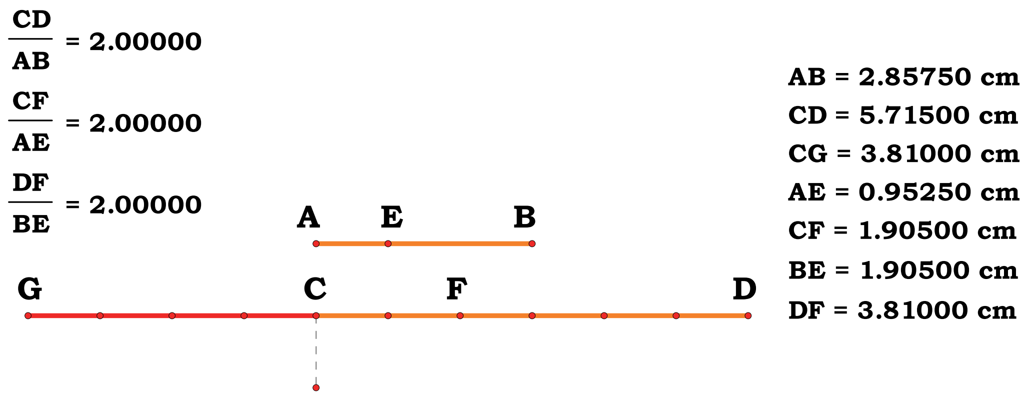
THEREFORE,

WHATEVER PARTS, AB IS OF C, THE SAME PARTS, ALSO, IS THE SUM, OF AB, DE, OF THE SUM OF C, F.

Q. E. D.

Proposition 7.

If a number be that part of a number, which a number subtracted is of a number subtracted, the remainder will, also, *be the same part of the remainder that the whole is of the whole.*



For let,

the number, *AB*, be that part of the number, *CD*,

which,

AE, subtracted is of *CF*, subtracted;

I say that;

the remainder, *EB*, is, also, the same part of the remainder, *FD*, that the whole, *AB*, is of the whole, *CD*.

For let,

whatever part, *AE*, is of *CF*,
the same part, also, *EB* be of *CG*.

Now since,

whatever part, *AE*, is of *CF*,
the same part, also, is *EB* of *CG*,

[vii. 5] therefore,

whatever part, *AE*, is of *CF*,
the same part, also, is *AB* of *GF*.

But, by hypothesis,

whatever part, *AE*, is of *CF*,
the same part, also, is *AB* of *CD*;

therefore,

whatever part, *AB*, is of *GF*,
the same part is it of *CD* also;

therefore,

GF = *CD*.

Let,

CF BE SUBTRACTED FROM EACH;

THEREFORE,
THE REMAINDER, *GC*, = THE REMAINDER, *FD*.

NOW SINCE,
WHATEVER PART, *AE*, IS OF *CF*,
THE SAME PART, ALSO, IS *EB* OF *GC*,

WHILE,
GC = *FD*,

THEREFORE,
WHATEVER PART, *AE*, IS OF *CF*,
THE SAME PART, ALSO, IS *EB* OF *FD*.

BUT,
WHATEVER PART, *AE*, IS OF *CF*,
THE SAME PART, ALSO, IS *AB* OF *CD*;

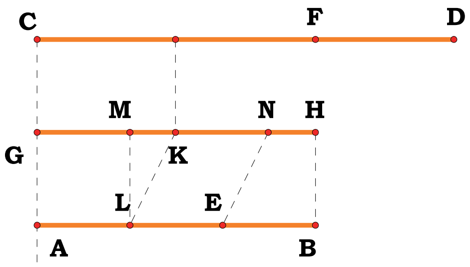
THEREFORE ALSO,
THE REMAINDER, *EB*, IS THE SAME PART OF
THE REMAINDER, *FD*, THAT THE WHOLE, *AB*, IS OF
THE WHOLE, *CD*.

Q. E. D.

PROPOSITION 8.

IF A NUMBER BE THE SAME PARTS OF A NUMBER THAT A NUMBER SUBTRACTED IS OF A NUMBER SUBTRACTED, THE REMAINDER WILL, ALSO, *BE THE SAME PARTS OF THE REMAINDER THAT THE WHOLE IS OF THE WHOLE.*

AB = 3.74650 cm	GH = 3.74650 cm	GM = 1.24883 cm	EL = 1.24883 cm
CD = 5.61975 cm	CD = 5.61975 cm	KM = 0.62442 cm	KN = 1.24883 cm
AE = 2.49767 cm	GK = 1.87325 cm	DF = 1.87325 cm	NH = 0.62442 cm
CF = 3.74650 cm	AL = 1.24883 cm	KH = 1.87325 cm	BE = 1.24883 cm



$\frac{CD}{AB} = 1.50000$ $\frac{CD}{GH} = 1.50000$

$\frac{CF}{AE} = 1.50000$ $\frac{CD}{GK} = 3.00000$

$\frac{CD\text{-}CF}{AB\text{-}AE} = 1.50000$ $\frac{CF}{AL} = 3.00000$

$\frac{CF}{GM} = 3.00000$ $\frac{CF}{EL} = 3.00000$ $\frac{DF}{KM+NH} = 1.50000$ $\frac{CD}{AB} = 1.50000$

$\frac{DF}{KM} = 3.00000$ $\frac{CF}{KN} = 3.00000$ $\frac{CD}{GH} = 1.50000$

$\frac{CD}{KH} = 3.00000$ $\frac{DF}{NH} = 3.00000$ $\frac{DF}{BE} = 1.50000$

FOR LET,

THE NUMBER, *AB*, BE THE SAME PARTS OF THE NUMBER, *CD*,
THAT *AE*, SUBTRACTED, IS OF *CF*, SUBTRACTED;

I SAY THAT;

THE REMAINDER, *EB*, IS, ALSO, THE SAME PARTS OF
THE REMAINDER, *FD*,
THAT THE WHOLE, *AB*, IS OF THE WHOLE, *CD*.

FOR LET,

GH BE MADE EQUAL TO *AB*.

THEREFORE,

WHATEVER PARTS, *GH*, IS OF *CD*,
THE SAME PARTS, ALSO, IS *AE* OF *CF*.

LET,

GH BE DIVIDED INTO THE PARTS OF *CD*,

NAMELY,

GK, *KH*, AND
AE INTO THE PARTS OF *CF*,

NAMELY,

AL, *LE*;

THUS,

THE MULTITUDE, OF GK, KH, WILL BE EQUAL TO
THE MULTITUDE, OF AL, LE.

NOW SINCE,
WHATEVER PART, GK, IS OF CD,
THE SAME PART, ALSO, IS AL OF CF, WHILE
CD IS GREATER THAN CF,

THEREFORE,
GK IS, ALSO, GREATER THAN AL.

LET,
GM BE MADE EQUAL TO AL.

THEREFORE,
WHATEVER PART, GK, IS OF CD,
THE SAME PART, ALSO, IS GM OF CF;

[VII. 7] THEREFORE ALSO,
THE REMAINDER, MK, IS THE SAME PART OF
THE REMAINDER, FD, THAT
THE WHOLE, GK, IS OF THE WHOLE, CD.

AGAIN, SINCE,
WHATEVER PART, KH, IS OF CD,
THE SAME PART, ALSO, IS EL OF CF,

WHILE,
CD IS GREATER THAN CF,

THEREFORE,
HK IS, ALSO, GREATER THAN EL.

LET,
KN BE MADE EQUAL TO EL.

THEREFORE,
WHATEVER PART, KH, IS OF CD,
THE SAME PART, ALSO, IS KN OF CF;

[VII. 7] THEREFORE,
ALSO THE REMAINDER, NH, IS
THE SAME PART, OF THE REMAINDER, FD, THAT
THE WHOLE, KH, IS OF THE WHOLE, CD.

BUT,
THE REMAINDER, MK, WAS, ALSO, PROVED TO BE
THE SAME PART, OF THE REMAINDER, FD, THAT
THE WHOLE, GK, IS OF THE WHOLE, CD;

THEREFORE,
ALSO THE SUM OF MK, NH IS
THE SAME PARTS, OF DF, THAT

THE WHOLE, *HG*, IS OF THE WHOLE, *CD*.

BUT,

THE SUM, OF *MK*, *NH*, = *EB*, AND
HG = *BA*;

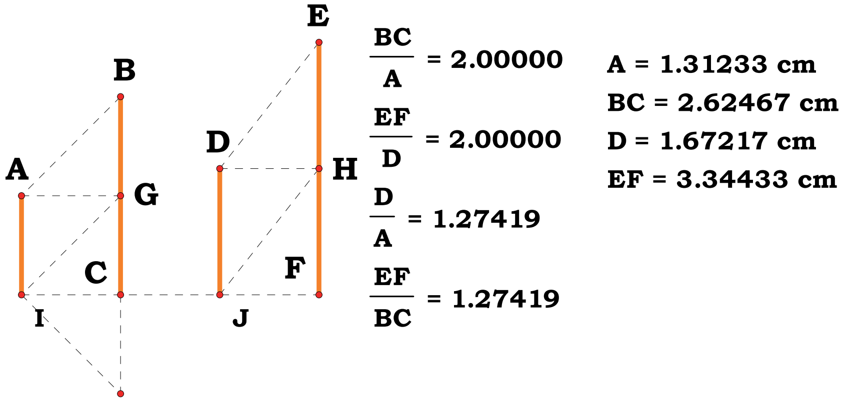
THEREFORE,

THE REMAINDER, *EB*, IS THE SAME PARTS OF
THE REMAINDER, *FD*, THAT
THE WHOLE, *AB*, IS OF THE WHOLE, *CD*.

Q. E. D.

PROPOSITION 9.

IF A NUMBER BE A PART OF A NUMBER, AND ANOTHER BE THE SAME PART OF ANOTHER, ALTERNATELY ALSO, WHATEVER PART OR PARTS THE FIRST IS OF THE THIRD, THE SAME PART, OR THE SAME PARTS, WILL THE SECOND, ALSO, *BE OF THE FOURTH.*



FOR LET,

THE NUMBER, *A*, BE A PART OF THE NUMBER, *BC*,

AND ANOTHER,

D, THE SAME PART OF ANOTHER, *EF*, THAT *A* IS OF *BC*;

I SAY THAT;

ALTERNATELY ALSO,
WHATEVER PART OR PARTS, *A*, IS OF *D*,
THE SAME PART OR PARTS IS *BC* OF *EF* ALSO.

FOR SINCE,

WHATEVER PART, *A*, IS OF *BC*,
THE SAME PART, ALSO, IS *D* OF *EF*,

THEREFORE,

AS MANY NUMBERS AS THERE ARE IN *BC* EQUAL TO *A*,
SO MANY, ALSO, ARE THERE IN *EF* EQUAL TO *D*.

LET,

BC BE DIVIDED INTO THE NUMBERS EQUAL TO *A*,

NAMELY,

BG, *GC*, AND
EF INTO THOSE EQUAL TO *D*,

NAMELY,

EH, *HE*;

THUS,

THE MULTITUDE, OF *BG*, *GC*, WILL BE EQUAL TO
THE MULTITUDE, OF *EH*, *HF*.

NOW, SINCE,

THE NUMBERS, *BG*, *GC*, ARE EQUAL TO ONE ANOTHER, AND

THE NUMBERS, *EH*, *HF*, ARE, ALSO, EQUAL TO ONE ANOTHER,

WHILE,
THE MULTITUDE, OF *BG*, *GC*, =
THE MULTITUDE, OF *EH*, *HF*,

THEREFORE,
WHATEVER PART OR PARTS, *BG*, IS OF *EH*,
THE SAME PART OR THE SAME PARTS IS *GC* OF *HF* ALSO;

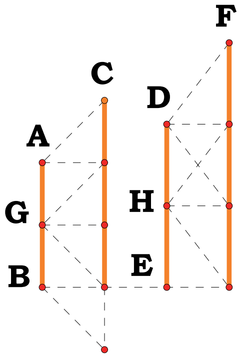
[VII. 5, 6] SO THAT, IN ADDITION,
WHATEVER PART OR PARTS, *BG*, IS OF *EH*,
THE SAME PART ALSO, OR THE SAME PARTS, IS
THE SUM, *BC*, OF THE SUM, *EF*.

BUT,
BG = *A*, AND
EH TO *D*;

THEREFORE,
WHATEVER PART OR PARTS, *A*, IS OF *D*,
THE SAME PART OR THE SAME PARTS IS *BC* OF *EF* ALSO.

Q. E. D.

Proposition 10.



If a number be parts of a number, and another be the same parts of another, alternately also, whatever parts or part the first is of the third, the same parts or the same part will the second, also, *be of the fourth.*

For let,

the number, *AB*, be parts of

the number, *C*, and another,

DE, the same parts of another, *F*;

I say that;

alternately also,

whatever parts or part, *AB*, is of *DE*,

the same parts or the same part is *C* of *F* also.

For since,

whatever parts, *AB*, is of *C*,

the same parts, also, is *DE* of *F*,

therefore,

as many parts, of *C*, as there are in *AB*,

so many parts, also, of *F*, are there in *DE*.

Let,

AB be divided into the parts, of *C*,

namely,

AG, *GB*, and

DE into the parts, of *F*,

namely,

DH, *HE*;

thus,

the multitude, of *AG*, *GB*, will be equal to

the multitude, of *DH*, *HE*.

Now since,

whatever part, *AG*, is of *C*,

the same part, also, is *DH* of *F*,

[vii. 9] alternately also,

whatever part or parts, *AG*, is of *DH*,

the same part or the same parts is *C* of *F* also.

For the same reason, also,

whatever part or parts, *GB*, is of *HE*,

THE SAME PART OR THE SAME PARTS IS C OF F ALSO;

[VII. 5, 6] SO THAT, IN ADDITION,

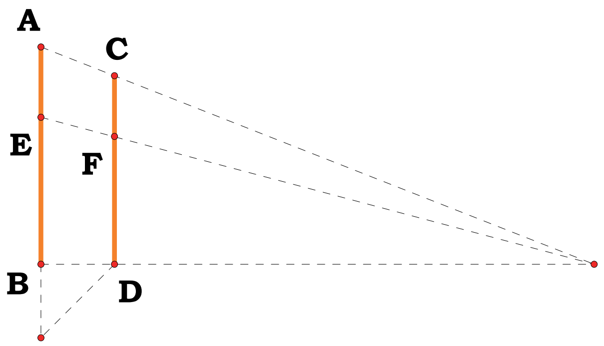
WHATEVER PARTS OR PART, AB, IS OF DE,

THE SAME PARTS, ALSO, OR THE SAME PART, IS C OF F.

Q. E. D.

PROPOSITION 11.

IF, AS WHOLE IS TO WHOLE, SO IS A NUMBER SUBTRACTED TO A NUMBER SUBTRACTED, THE REMAINDER WILL, ALSO, *BE TO THE REMAINDER AS WHOLE TO WHOLE.*



SO LET,

AS THE WHOLE, *AB*, IS TO THE WHOLE, *CD*,
AE, SUBTRACTED BE TO *CF*, SUBTRACTED;

I SAY THAT;

THE REMAINDER, *EB*, IS, ALSO, TO THE REMAINDER, *FD*,
AS THE WHOLE, *AB*, TO THE WHOLE, *CD*.

[VII. DEF. 20] SINCE,

AS *AB* IS TO *CD*,
SO IS *AE* TO *CF*,
WHATEVER PART OR PARTS, *AB*, IS OF *CD*,
THE SAME PART OR THE SAME PARTS IS *AE* OF *CF* ALSO;

[VII. 7, 8] THEREFORE ALSO,

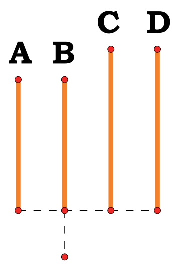
THE REMAINDER, *EB*, IS THE SAME PART OR PARTS, OF *FD*,
THAT *AB* IS OF *CD*.

[VII. DEF. 20] THEREFORE,

AS *EB* IS TO *FD*,
SO IS *AB* TO *CD*.

Q. E. D.

Proposition 12.



If there be as many numbers as we please in proportion, then, as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents.

Let,

A, *B*, *C*, *D*,

be as many numbers as we please in proportion,

so that,

as *A* is to *B*,

so is *C* to *D*;

I say that;

as *A* is to *B*,

so are *A*, *C* to *B*, *D*.

[vii. Def. 20] For since,

as *A* is to *B*,

so is *C* to *D*,

whatever part or parts, *A*, is of *B*,

the same part or parts is *C* of *D* also.

[vii. 5, 6] Therefore also,

the sum, of *A*, *C*, is the same part or,

the same parts, of the sum, of *B*, *D*, that *A* is of *B*.

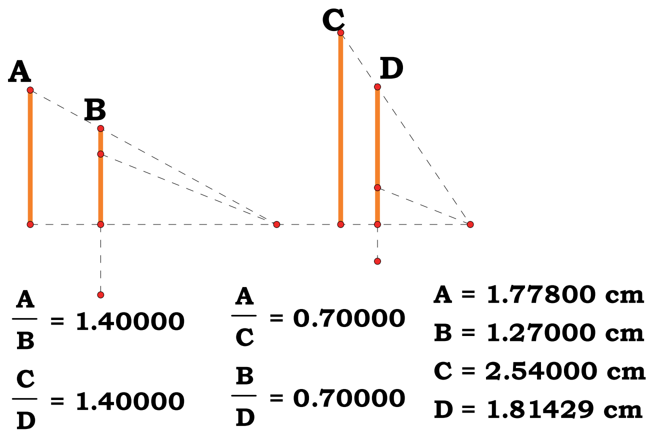
[vii. Def. 20] Therefore,

as *A* is to *B*,

so are *A*, *C* to *B*, *D*.

PROPOSITION 13.

IF FOUR NUMBERS BE PROPORTIONAL, THEY WILL, ALSO, *BE PROPORTIONAL ALTERNATELY.*



LET,

THE FOUR NUMBERS, *A*, *B*, *C*, *D*, BE PROPORTIONAL,

SO THAT,

AS *A* IS TO *B*,
SO IS *C* TO *D*;

I SAY THAT;

THEY WILL, ALSO, BE PROPORTIONAL ALTERNATELY,

SO THAT,

AS *A* IS TO *C*,
SO WILL *B* BE TO *D*.

FOR SINCE,

AS *A* IS TO *B*,
SO IS *C* TO *D*.

[VII. DEF. 20] THEREFORE,

WHATEVER PART OR PARTS, *A*, IS OF *B*,
THE SAME PART OR THE SAME PARTS IS *C* OF *D* ALSO.

[VII. 10] THEREFORE, ALTERNATELY,

WHATEVER PART OR PARTS, *A*, IS OF *C*,
THE SAME PART OR THE SAME PARTS IS *B* OF *D* ALSO.

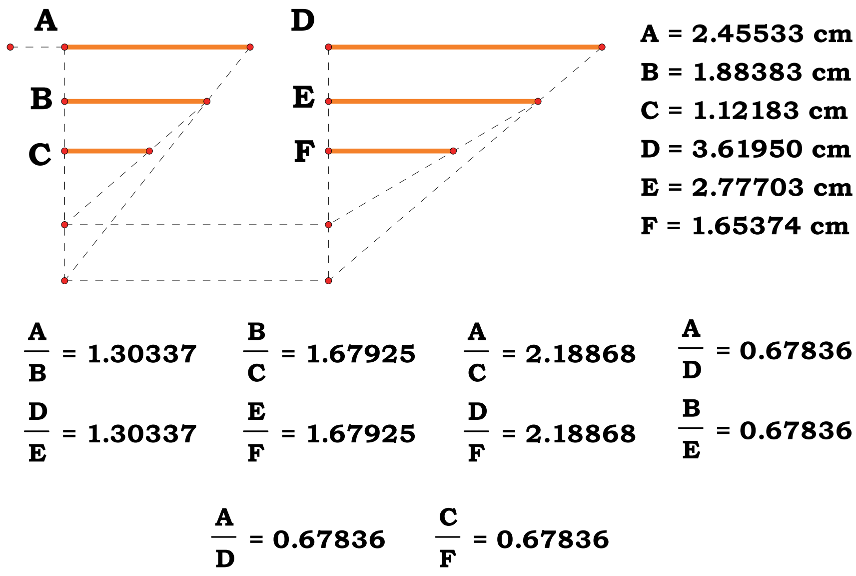
[VII. DEF. 20] THEREFORE,

AS *A* IS TO *C*,
SO IS *B* TO *D*.

Q. E. D.

Proposition 14.

If there be as many numbers as we please, and others equal to them in multitude, which taken two and two are in the same ratio, they will, also, *be in the same ratio* ex aequali.



Let,

there be as many numbers as we please, A, B, C, and,
others equal to them in multitude, D, E, F,
which taken two and two are in the same ratio,

so that,

as A is to B,
so is D to E, and
as B is to C,
so is E to F;

I say that; *ex aequali*,

as A is to C,
so, also, is D to F.

For, since,

as A is to B,
so is D to E,

[vii. 13] therefore, alternately,

as A is to D,
so is B to E.

Again, since,

as B is to C,
so is E to F,

[vii. 13] therefore, alternately,

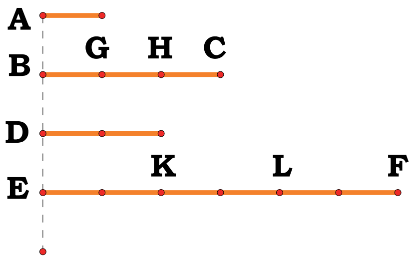
as B is to E,
so is C to F.

But,
as B is to E,
so is A to D;

therefore also,
as A is to D,
so is C to F.

[*Id.*] Therefore, alternately,
as A is to C,
so is D to F.

PROPOSITION 15.



IF AN UNIT MEASURE ANY NUMBER, AND ANOTHER NUMBER MEASURE ANY OTHER NUMBER THE SAME NUMBER OF TIMES, ALTERNATELY ALSO, THE UNIT WILL MEASURE THE THIRD NUMBER THE SAME NUMBER OF TIMES THAT THE SECOND MEASURES THE FOURTH.

FOR LET,

THE UNIT, *A*, MEASURE ANY NUMBER, *BC*,

AND LET,

ANOTHER NUMBER, *D*, MEASURE ANY OTHER NUMBER, *EF*,
THE SAME NUMBER OF TIMES;

I SAY THAT; ALTERNATELY ALSO,

THE UNIT, *A*, MEASURES THE NUMBER, *D*
THE SAME NUMBER OF TIMES THAT *BC* MEASURES *EF*.

FOR, SINCE,

THE UNIT, *A*, MEASURES THE NUMBER, *BC*,
THE SAME NUMBER OF TIMES THAT *D* MEASURES *EF*,

THEREFORE,

AS MANY UNITS AS THERE ARE IN *BC*,
SO MANY NUMBERS EQUAL TO *D* ARE THERE IN *EF*, ALSO.

LET,

BC BE DIVIDED INTO THE UNITS IN IT, *BG*, *GH*, *HC*, AND
EF INTO THE NUMBERS, *EK*, *KL*, *LF*, EQUAL TO *D*.

THUS,

THE MULTITUDE, OF *BG*, *GH*, *HC*, WILL BE EQUAL TO
THE MULTITUDE, OF *EK*, *KL*, *LF*.

AND, SINCE,

THE UNITS, *BG*, *GH*, *HC*, ARE EQUAL TO ONE ANOTHER, AND
THE NUMBERS,
EK, *KL*, *LF*, ARE, ALSO, EQUAL TO ONE ANOTHER,

WHILE,

THE MULTITUDE, OF THE UNITS, *BG*, *GH*, *HC*, =
THE MULTITUDE, OF THE NUMBERS, *EK*, *KL*, *LF*,

THEREFORE,

AS THE UNIT, *BG*, IS TO THE NUMBER, *EK*,
SO WILL THE UNIT, *GH*, BE TO THE NUMBER, *KL*, AND,
THE UNIT, *HC*, TO THE NUMBER, *LF*.

[VII. 12] THEREFORE ALSO,

AS ONE OF THE ANTECEDENTS IS TO
ONE OF THE CONSEQUENTS,
SO WILL ALL THE ANTECEDENTS BE TO ALL THE CONSEQUENTS;

THEREFORE,

AS THE UNIT, *BG*, IS TO THE NUMBER, *EK*,
SO IS *BC* TO *EF*.

BUT,

THE UNIT, *BG*, = THE UNIT, *A*, AND
THE NUMBER, *EK*, TO THE NUMBER, *D*.

THEREFORE,

AS THE UNIT, *A*, IS TO THE NUMBER, *D*,
SO IS *BC* TO *EF*.

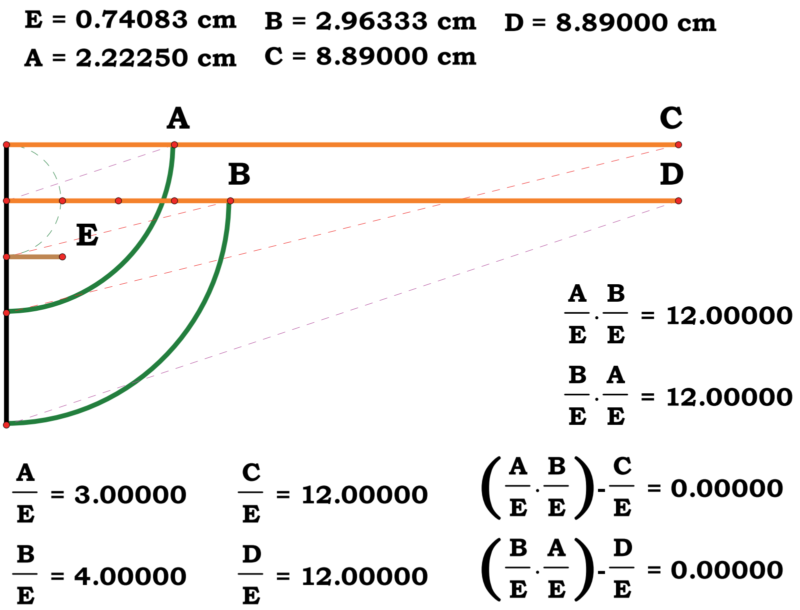
THEREFORE,

THE UNIT, *A*, MEASURES THE NUMBER, *D*,
THE SAME NUMBER OF TIMES THAT *BC* MEASURES *EF*.

Q. E. D.

Proposition 16.

If two numbers, by multiplying one another, make certain numbers, the numbers so produced will be equal to one another.



Let,

A, *B*, be two numbers,

and let,

A, by multiplying *B*, make *C*, and
B, by multiplying *A*, make *D*;

I say that;

C = *D*.

For, since,

A, by multiplying *B*, has made *C*,

therefore,

B measures *C*, according to the units in *A*.

But,

the unit, *E*, also, measures the number, *A*, according to the units in it;

therefore,

the unit, *E*, measures *A*,
the same number of times that *B* measures *C*.

[vii. 15] Therefore, alternately,

the unit, *E*, measures the number, *B*,
the same number of times that *A* measures *C*.

Again, since,

B, by multiplying *A*, has made *D*,

THEREFORE,

A MEASURES D, ACCORDING TO THE UNITS IN B.

BUT,

THE UNIT, E, ALSO, MEASURES B, ACCORDING TO THE UNITS IN IT;

THEREFORE,

THE UNIT, E, MEASURES THE NUMBER, B, THE SAME NUMBER OF TIMES THAT A MEASURES D.

BUT,

THE UNIT, E, MEASURED THE NUMBER, B, THE SAME NUMBER OF TIMES THAT A MEASURES C;

THEREFORE,

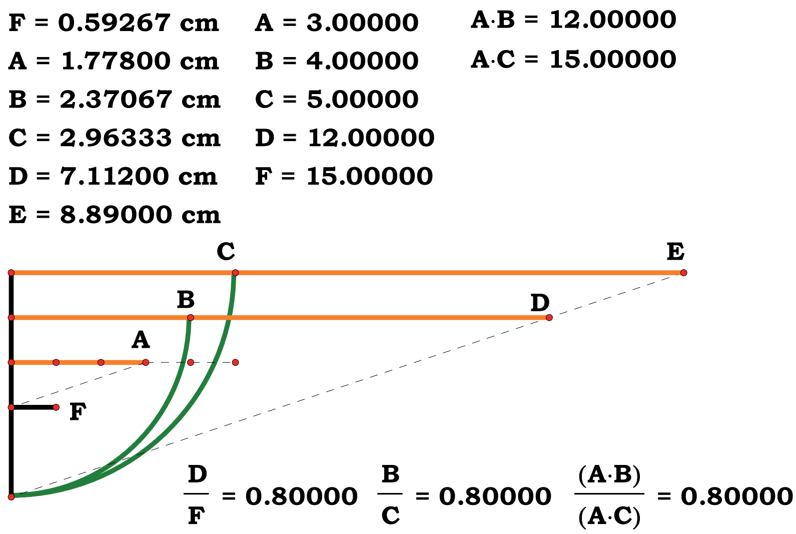
A MEASURES EACH, OF THE NUMBERS, C, D, THE SAME NUMBER OF TIMES.

THEREFORE $C = D$.

Q. E. D.

Proposition 17.

If a number, by multiplying two numbers, make certain numbers, the numbers so produced will have the same ratio as the numbers multiplied.



For let,

    the number, *A*, by multiplying
    the two numbers, *B*, *C*, make *D*, *E*;

I say that;

    as *B* is to *C*,
    so is *D* to *E*.

For, since,

    A, by multiplying *B*, has made *D*,

therefore,

    B measures *D*, according to the units in *A*.

But,

    the unit, *F*, also, measures the number, *A*,
    according to the units in it;

therefore,

    the unit, *F*, measures the number, *A*,
    the same number of times that *B* measures *D*.

[vii. Def. 20] Therefore,

    as the unit, *F*, is to the number, *A*,
    so is *B* to *D*.

For the same reason,

    as the unit, *F*, is to the number, *A*,
    so, also, is *C* to *E*;

therefore also,

    as *B* is to *D*,

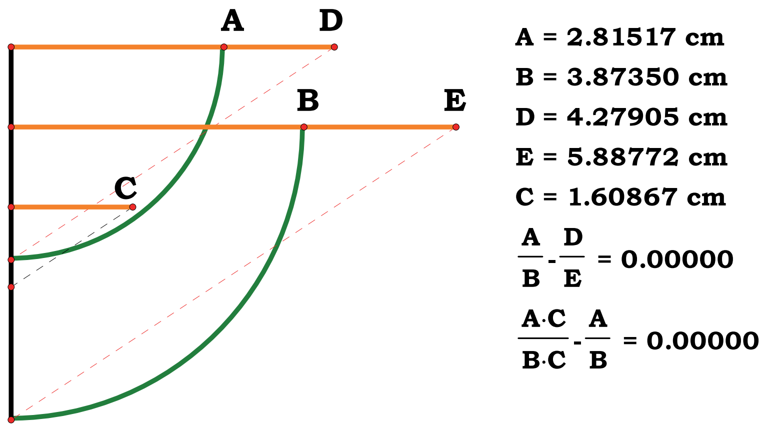
SO IS *C* TO *E*.

[VII. 13] THEREFORE, ALTERNATELY,
AS *B* IS TO *C*,
SO IS *D* TO *E*.

Q. E. D.

PROPOSITION 18.

IF TWO NUMBERS, BY MULTIPLYING ANY NUMBER, MAKE CERTAIN NUMBERS, THE NUMBERS SO PRODUCED WILL HAVE THE SAME RATIO AS THE MULTIPLIERS.



FOR LET,
TWO NUMBERS, *A*, *B*, BY MULTIPLYING ANY NUMBER *C*,
MAKE *D*, *E*;

I SAY THAT;
AS *A* IS TO *B*,
SO IS *D* TO *E*.

FOR, SINCE,
A, BY MULTIPLYING *C*, HAS MADE *D*,

[VII. 16] THEREFORE ALSO,
C, BY MULTIPLYING *A*, HAS MADE *D*.

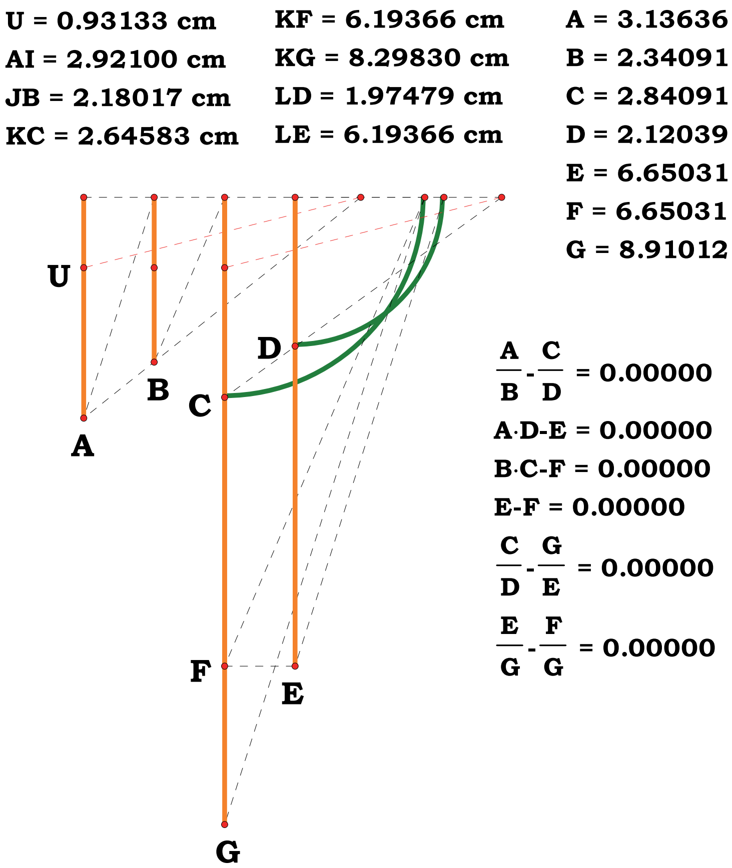
FOR THE SAME REASON ALSO,
C, BY MULTIPLYING *B*, HAS MADE *E*.

THEREFORE,
THE NUMBER, *C*, BY MULTIPLYING
THE TWO NUMBERS, *A*, *B*, HAS MADE *D*, *E*.

[VII. 17] THEREFORE,
AS *A* IS TO *B*,
SO IS *D* TO *E*.

Proposition 19.

If four numbers be proportional the number produced from the first and fourth will be equal to the number produced from the second and third; and, if the number produced from the first and fourth be equal to that produced from the second and third, the four numbers will be proportional.



LET,

A, *B*, *C*, *D*, BE FOUR NUMBERS IN PROPORTION,

SO THAT,

AS *A* IS TO *B*,
SO IS *C* TO *D*;

AND LET,

A, BY MULTIPLYING *D*, MAKE *E*,

AND LET,

B, BY MULTIPLYING *C*, MAKE *F*.

I SAY THAT;

E = *F*.

FOR LET,

A, BY MULTIPLYING *C*, MAKE *G*.

SINCE, THEN,

A, BY MULTIPLYING *C*, HAS MADE *G*, AND
BY MULTIPLYING, *D*, HAS MADE *E*,
THE NUMBER, *A*, BY MULTIPLYING,
THE TWO NUMBERS, *C*, *D*, HAS MADE *G*, *E*.

[VII. 17] THEREFORE,
AS *C* IS TO D,
SO IS *G* TO *E*.

BUT,
AS *C* IS TO *D*,
SO IS *A* TO *B*;

THEREFORE ALSO,
AS *A* IS TO *B*,
SO IS *G* TO *E*.

AGAIN, SINCE,
A, BY MULTIPLYING *C*, HAS MADE *G*,

BUT, FURTHER,
B, HAS ALSO, BY MULTIPLYING *C*, MADE *F*,
THE TWO NUMBERS, *A*, *B*,
BY MULTIPLYING A CERTAIN NUMBER, *C*, HAVE MADE *G*, *F*.

[VII. 18] THEREFORE,
AS *A* IS TO *B*,
SO IS *G* TO *F*.

BUT FURTHER,
AS *A* IS TO *B*,
SO IS *G* TO *E* ALSO;

THEREFORE ALSO,
AS *G* IS TO *E*,
SO IS *G* TO *F*.

THEREFORE,
G HAS TO EACH, OF THE NUMBERS, *E*, *F*, THE SAME RATIO;

[CF. V. 9] THEREFORE,
$E = F$.

AGAIN, LET,
E BE EQUAL TO *F*;

I SAY THAT;
AS *A* IS TO *B*,
SO IS *C* TO *D*.

FOR,
WITH THE SAME CONSTRUCTION,

SINCE,

$E = F$,

[CF. V. 7] THEREFORE,

AS G IS TO E,
SO IS G TO F.

[VII. 17] BUT,

AS G IS TO E,
SO IS C TO D,

[VII. 18] AND,

AS G IS TO F,
SO IS A TO B.

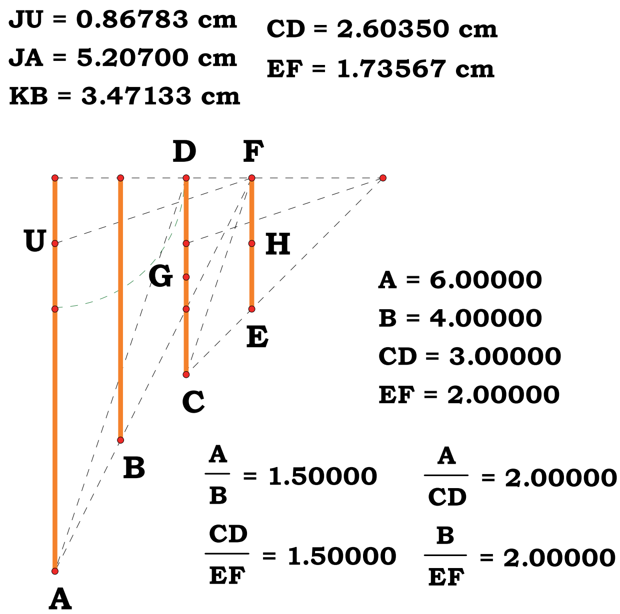
THEREFORE ALSO,

AS A IS TO B,
SO IS C TO D.

Q. E. D.

PROPOSITION 20.

THE LEAST NUMBERS OF THOSE WHICH HAVE THE SAME RATIO WITH THEM MEASURE THOSE WHICH HAVE THE SAME RATIO THE SAME NUMBER OF TIMES, THE GREATER THE GREATER AND THE LESS THE LESS.



FOR LET,

CD, *EF* BE THE LEAST NUMBERS OF THOSE WHICH HAVE THE SAME RATIO WITH *A*, *B*;

I SAY THAT;

CD MEASURES *A* THE SAME NUMBER OF TIMES THAT *EF* MEASURES *B*.

NOW,

CD IS NOT PARTS OF *A*.

FOR, IF POSSIBLE, LET IT BE SO;

[VII. 13 AND DEF. 20] THEREFORE,

EF IS, ALSO, THE SAME PARTS OF *B*, THAT *CD* IS OF *A*.

THEREFORE,

AS MANY PARTS, OF *A*, AS THERE ARE IN *CD*, SO MANY PARTS, OF *B*, ARE THERE, ALSO, IN *EF*.

LET,

CD BE DIVIDED INTO THE PARTS, OF *A*,

NAMELY,

CG, *GD*, AND

EF INTO THE PARTS, OF *B*,

NAMELY,

EH, *HE*;

THUS,

THE MULTITUDE, OF *CG*, *GD*, WILL BE EQUAL TO
THE MULTITUDE, OF *EH*, *HF*.

NOW, SINCE,

THE NUMBERS, *CG*, *GD*, ARE EQUAL TO ONE ANOTHER, AND
THE NUMBERS, *EH*, *HF*, ARE, ALSO, EQUAL TO ONE ANOTHER,

WHILE,

THE MULTITUDE, OF *CG*, *GD*, =
THE MULTITUDE, OF *EH*, *HF*,

THEREFORE,

AS *CG* IS TO *EH*,
SO IS *GD* TO *HF*.

[VII. 12] THEREFORE ALSO,

AS ONE OF THE ANTECEDENTS IS TO
ONE OF THE CONSEQUENTS,
SO WILL ALL THE ANTECEDENTS BE TO ALL THE CONSEQUENTS.

THEREFORE,

AS *CG* IS TO *EH*,
SO IS *CD* TO *EF*.

THEREFORE,

CG, *EH* ARE IN THE SAME RATIO WITH *CD*, *EF*,
BEING LESS THAN THEY:

WHICH,

IS IMPOSSIBLE,

FOR BY HYPOTHESIS,

CD, *EF* ARE THE LEAST NUMBERS OF THOSE
WHICH HAVE THE SAME RATIO WITH THEM.

THEREFORE,

CD IS NOT PARTS OF *A*;

[VII. 4] THEREFORE,

IT IS A PART OF IT.

[VII. 13 AND DEF. 20] AND,

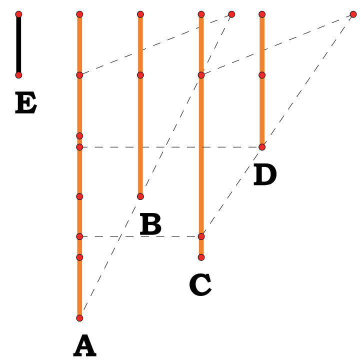
EF IS THE SAME PART, OF *B*, THAT *CD* IS OF *A*;

THEREFORE,

CD MEASURES *A*,
THE SAME NUMBER OF TIMES THAT *EF* MEASURES *B*.

Q. E. D.

PROPOSITION 21.



NUMBERS PRIME TO ONE ANOTHER ARE THE LEAST OF THOSE WHICH HAVE THE SAME RATIO WITH THEM.

LET,

A, *B* BE NUMBERS PRIME TO ONE ANOTHER;

I SAY THAT;

A, *B* ARE THE LEAST OF THOSE WHICH HAVE THE SAME RATIO WITH THEM.

FOR,

IF NOT, THERE WILL BE SOME NUMBERS LESS THAN *A*, *B* WHICH ARE IN THE SAME RATIO WITH *A*, *B*.

LET,

THEM BE *C*, *D*.

[VII. 20] SINCE THEN,

THE LEAST NUMBERS OF THOSE WHICH HAVE THE SAME RATIO MEASURE THOSE WHICH HAVE THE SAME RATIO THE SAME NUMBER OF TIMES,
THE GREATER THE GREATER AND,
THE LESS THE LESS,

THAT IS,

THE ANTECEDENT THE ANTECEDENT, AND,
THE CONSEQUENT THE CONSEQUENT,

THEREFORE,

C MEASURES *A*, THE SAME NUMBER OF TIMES THAT *D* MEASURES *B*.

NOW, LET,

AS MANY TIMES AS *C* MEASURES *A*,
SO MANY UNITS THERE BE IN *E*.

THEREFORE,

D, ALSO, MEASURES *B*, ACCORDING TO THE UNITS IN *E*.

AND

SINCE *C* MEASURES *A*, ACCORDING TO THE UNITS IN *E*,

[VII. 16] THEREFORE,

E, ALSO, MEASURES *A*, ACCORDING TO THE UNITS IN *C*.

[VII. 16] FOR THE SAME REASON,

E, ALSO, MEASURES *B*, ACCORDING TO THE UNITS IN *D*.

Therefore,

E measures *A*, *B*, which
are prime to one another:

[vii. Def. 12] which,
is impossible.

Therefore,

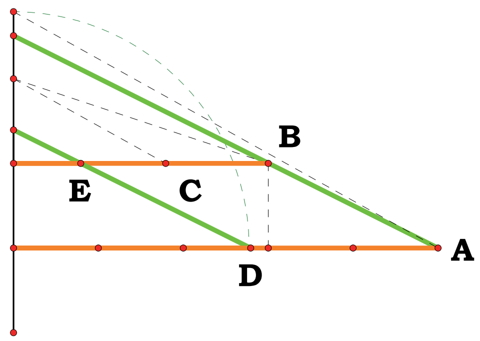
there will be no numbers less than *A*, *B*
which are in the same ratio with *A*, *B*.

Therefore,

A, *B* are the least of those
which have the same ratio with them.

Q. E. D.

PROPOSITION 22.



THE LEAST NUMBERS OF THOSE WHICH HAVE THE SAME RATIO WITH THEM ARE PRIME TO ONE ANOTHER.

LET,

A, B BE
THE LEAST NUMBERS
OF THOSE WHICH HAVE
THE SAME RATIO WITH THEM;

I SAY THAT;

A, B ARE PRIME TO ONE ANOTHER.

FOR,

IF THEY ARE NOT PRIME TO ONE ANOTHER,
SOME NUMBER WILL MEASURE THEM.

LET,

SOME NUMBER MEASURE THEM,

AND LET,

IT BE C.

AND,

AS MANY TIMES AS C MEASURES A,
SO MANY UNITS LET THERE BE IN D, AND
AS MANY TIMES AS C MEASURES B,
SO MANY UNITS LET THERE BE IN E.

[VII. DEF. 15] SINCE,

C MEASURES A, ACCORDING TO THE UNITS IN D,

THEREFORE,

C, BY MULTIPLYING D, HAS MADE A.

FOR THE SAME REASON ALSO,

C, BY MULTIPLYING E, HAS MADE B.

THUS,

THE NUMBER, C, BY MULTIPLYING
THE TWO NUMBERS, D, E, HAS MADE A, B;

[VII. 17] THEREFORE,

AS D IS TO E,
SO IS A TO B;

THEREFORE,

D, E ARE IN THE SAME RATIO WITH A, B,
BEING LESS THAN THEY:

WHICH,

IS IMPOSSIBLE.

THEREFORE,

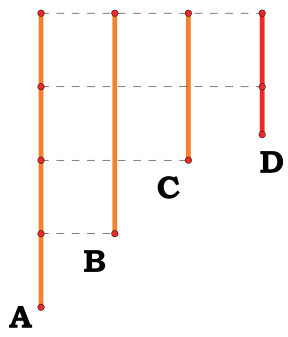
NO NUMBER WILL MEASURE THE NUMBERS, A, B.

THEREFORE,

A, B ARE PRIME TO ONE ANOTHER.

Q. E. D.

PROPOSITION 23.



IF TWO NUMBERS BE PRIME TO ONE ANOTHER, THE NUMBER WHICH MEASURES THE ONE OF THEM WILL BE PRIME TO THE REMAINING NUMBER.

LET,
A, *B* BE TWO NUMBERS
PRIME TO ONE ANOTHER,

AND LET,
ANY NUMBER, *C*, MEASURE *A*;

I SAY THAT;
C, *B* ARE, ALSO, PRIME TO ONE ANOTHER.

FOR,
IF *C*, *B* ARE NOT PRIME TO ONE ANOTHER,
SOME NUMBER WILL MEASURE *C*, *B*.

LET,
A NUMBER MEASURE THEM, AND
LET IT BE *D*.

SINCE,
D MEASURES *C*, AND
C MEASURES *A*,

THEREFORE,
D, ALSO, MEASURES *A*.

BUT,
IT, ALSO, MEASURES *B*;

THEREFORE,
D MEASURES *A*, *B*,
WHICH ARE PRIME TO ONE ANOTHER:

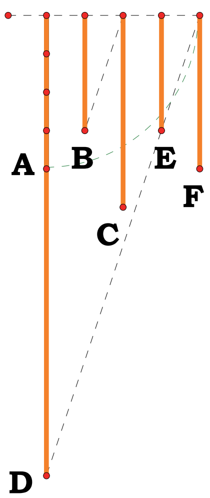
[VII. DEF. 12] WHICH,
IS IMPOSSIBLE.

THEREFORE,
NO NUMBER WILL MEASURE THE NUMBERS, *C*, *B*.

THEREFORE,
C, *B* ARE PRIME TO ONE ANOTHER.

Q. E. D.

Proposition 24.



If two numbers be prime to any number, their product, also, will be prime to the same.

For let,

the two numbers, *A*, *B*
be prime to any number, *C*,

and let,

A, by multiplying *B*, make *D*;

I say that;

C, *D* are prime to one another.

For,

if *C*, *D* are not prime to one another,
some number will measure *C*, *D*.

Let,

a number measure them,

and let,

it be *E*.

Now,

since *C*, *A* are prime to one another,
and a certain number, *E*, measures *C*,

[vii. 23] therefore,

A, *E* are prime to one another.

As many times, then, let,

as *E* measures *D*,
so many units there be in *F*;

[vii. 16] therefore,

F, also, measures *D*, according to the units in *E*.

[vii. Def. 15] Therefore,

E, by multiplying *F*, has made *D*.

But, further,

A, by multiplying *B*, has, also, made *D*;

therefore,

the product, of *E*, *F*, = the product, of *A*, *B*.

[vii. 19] But,

if the product of the extremes
be equal to that of the means,
the four numbers are proportional;

THEREFORE,

AS E IS TO A,
SO IS B TO F.

[VII. 21] BUT,

A, E ARE PRIME TO ONE ANOTHER,
NUMBERS WHICH ARE PRIME TO ONE ANOTHER ARE, ALSO,
THE LEAST OF THOSE WHICH HAVE THE SAME RATIO, AND
THE LEAST NUMBERS OF THOSE WHICH HAVE
THE SAME RATIO WITH THEM MEASURE THOSE
WHICH HAVE THE SAME RATIO THE SAME NUMBER OF TIMES,
THE GREATER THE GREATER, AND,
THE LESS THE LESS,

[VII. 20] THAT IS,

THE ANTECEDENT THE ANTECEDENT, AND,
THE CONSEQUENT THE CONSEQUENT;

THEREFORE,

E MEASURES B.

BUT,

IT, ALSO, MEASURES C;

THEREFORE,

E MEASURES B, C,
WHICH ARE PRIME TO ONE ANOTHER:

[VII. DEF. 12] WHICH,

IS IMPOSSIBLE.

THEREFORE,

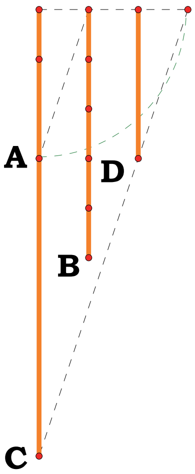
NO NUMBER WILL MEASURE THE NUMBERS, C, D.

THEREFORE,

C, D ARE PRIME TO ONE ANOTHER.

Q. E, D.

PROPOSITION 25.



IF TWO NUMBERS BE PRIME TO ONE ANOTHER, THE PRODUCT OF ONE OF THEM INTO ITSELF WILL BE PRIME TO THE REMAINING ONE.

LET,

A, *B* BE TWO NUMBERS PRIME TO ONE ANOTHER,

AND LET,

A, BY MULTIPLYING ITSELF, MAKE *C*:

I SAY THAT;

B, *C* ARE PRIME TO ONE ANOTHER.

FOR LET,

D BE MADE EQUAL TO *A*.

SINCE,

A, *B* ARE PRIME TO ONE ANOTHER, AND

$A = D$,

THEREFORE,

D, *B* ARE, ALSO, PRIME TO ONE ANOTHER.

THEREFORE,

EACH, OF THE TWO NUMBERS, *D*, *A*, IS PRIME TO *B*;

[VII. 24] THEREFORE,

THE PRODUCT, OF *D*, *A*, WILL, ALSO, BE PRIME, TO *B*.

BUT,

THE NUMBER WHICH IS THE PRODUCT, OF *D*, *A*, IS *C*.

THEREFORE,

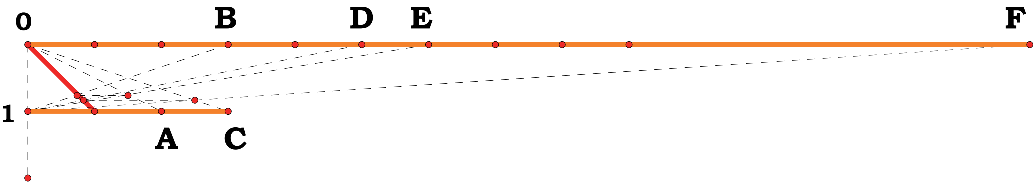
C, *B* ARE PRIME TO ONE ANOTHER.

Q. E. D.

Proposition 26.

If two numbers be prime to two numbers, both to each, their products, also, *will be prime to one another.*

01 = 0.88900 cm	**0D = 4.44500 cm**	**A = 2.00000**	**E = 6.00000**
1A = 1.77800 cm	**0E = 5.33400 cm**	**B = 3.00000**	**F = 15.00000**
0B = 2.66700 cm	**0F = 13.33500 cm**	**C = 3.00000**	**A·B-E = 0.00000**
1C = 2.66700 cm		**D = 5.00000**	**C·D-F = 0.00000**



For let,

the two numbers,

A, B, be prime to the two numbers, C, D;

both to each,

and let,

A, by multiplying B, make E,

and let,

C, by multiplying D, make F;

I say that;

E, F are prime to one another.

[VII. 24] For, since,

each, of the numbers, A, B, is prime to C,

therefore,

the product of A, B will, also, be prime to C.

But,

the product, of A, B, is E;

therefore,

E, C are prime to one another.

For the same reason,

E, D are, also, prime to one another.

Therefore,

each, of the numbers, C, D, is prime to E.

[VII. 24] Therefore,

the product, of C, D, will, also, be prime, to E.

But,

the product, of C, D, is F.

Therefore,

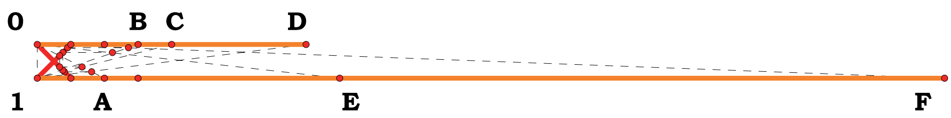
E, F ARE PRIME TO ONE ANOTHER.

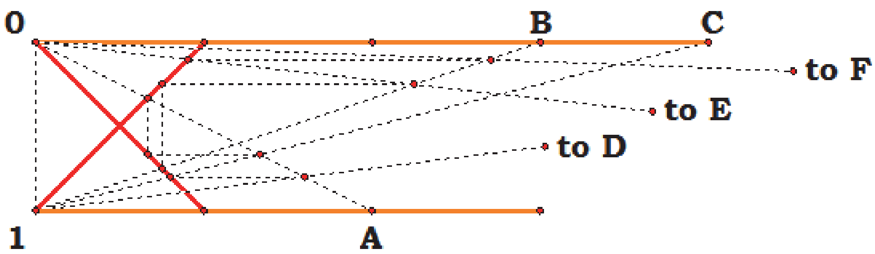
Q. E. D.

Proposition 27.

If two numbers be prime to one another, and each, by multiplying itself, make a certain number, the products will be prime to one another; and, if the original numbers, by multiplying the products, make certain numbers, the latter will, also, be prime to one another [and this is always the case with the extremes].

01 = 0.44450 cm	A = 2.00000	A^2-C = 0.00000
1A = 0.88900 cm	B = 3.00000	A^3-D = 0.00000
0B = 1.33350 cm	C = 4.00000	B^2-E = 0.00000
0C = 1.77800 cm	D = 8.00000	B^3-F = 0.00000
0D = 3.55600 cm	E = 9.00000	
1E = 4.00050 cm	F = 27.00000	
1F = 12.00150 cm		





LET,

A, *B* BE TWO NUMBERS PRIME TO ONE ANOTHER,

LET,

A, BY MULTIPLYING ITSELF, MAKE *C*, AND
BY MULTIPLYING *C*, MAKE *D*,

AND LET,

B, BY MULTIPLYING ITSELF, MAKE *E*, AND
BY MULTIPLYING *E*, MAKE *F*;

I SAY THAT;

BOTH, *C*, *E* AND *D*, *F*, ARE PRIME TO ONE ANOTHER.

[VII. 25] FOR, SINCE,

A, *B* ARE PRIME TO ONE ANOTHER, AND
A, BY MULTIPLYING ITSELF, HAS MADE *C*,

THEREFORE,

C, *B* ARE PRIME TO ONE ANOTHER.

SINCE,

THEN *C*, *B* ARE PRIME TO ONE ANOTHER, AND
B, BY MULTIPLYING ITSELF, HAS MADE *E*,

[*ID.*] THEREFORE,
C, E ARE PRIME TO ONE ANOTHER.

AGAIN, SINCE,
A, B ARE PRIME TO ONE ANOTHER, AND
B, BY MULTIPLYING ITSELF, HAS MADE E,

[*ID.*] THEREFORE,
A, E ARE PRIME TO ONE ANOTHER.

SINCE THEN,
THE TWO NUMBERS, A, C, ARE PRIME TO
THE TWO NUMBERS, B, E, BOTH TO EACH,

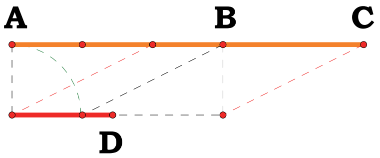
[VII. 26] THEREFORE,
ALSO THE PRODUCT, OF A, C, IS PRIME TO
THE PRODUCT, OF B, E.

AND,
THE PRODUCT, OF A, C, IS D, AND
THE PRODUCT, OF B, E, IS F.

THEREFORE,
D, F ARE PRIME TO ONE ANOTHER.

Q. E. D.

PROPOSITION 28.



IF TWO NUMBERS BE PRIME TO ONE ANOTHER, THE SUM WILL, ALSO, *BE PRIME TO EACH, OF THEM*; *AND, IF THE SUM TO TWO NUMBERS BE PRIME TO ANY ONE OF THEM, THE ORIGINAL NUMBERS WILL*, ALSO, *BE PRIME TO ONE ANOTHER*.

FOR LET,
TWO NUMBERS, AB, BC, PRIME TO ONE ANOTHER BE ADDED;

I SAY THAT;
THE SUM, AC, IS, ALSO, PRIME TO EACH, OF
THE NUMBERS, AB, BC.

FOR,
IF CA, AB ARE NOT PRIME TO ONE ANOTHER,
SOME NUMBER WILL MEASURE CA, AB.

LET,
A NUMBER MEASURE THEM,

AND LET,
IT BE D.

SINCE THEN,
D MEASURES CA, AB,

THEREFORE,
IT WILL, ALSO, MEASURE THE REMAINDER, BC.

BUT,
IT, ALSO, MEASURES BA;

THEREFORE,
D MEASURES AB, BC,
WHICH ARE PRIME TO ONE ANOTHER:

[VII. DEF. 12] WHICH,
IS IMPOSSIBLE.

THEREFORE,
NO NUMBER WILL MEASURE THE NUMBERS, CA, AB;

THEREFORE,
CA, AB ARE PRIME TO ONE ANOTHER.

FOR THE SAME REASON,
AC, CB ARE, ALSO, PRIME TO ONE ANOTHER.

THEREFORE,
CA IS PRIME TO EACH, OF THE NUMBERS, AB, BC.

AGAIN, LET,

CA, AB BE PRIME TO ONE ANOTHER;

I SAY THAT;

AB, BC ARE, ALSO, PRIME TO ONE ANOTHER.

FOR,

IF AB, BC ARE NOT PRIME TO ONE ANOTHER,
SOME NUMBER WILL MEASURE AB, BC.

LET,

A NUMBER MEASURE THEM,

AND LET,

IT BE D.

NOW, SINCE,

D MEASURES EACH, OF THE NUMBERS, AB, BC,
IT WILL, ALSO, MEASURE THE WHOLE, CA.

BUT,

IT, ALSO, MEASURES AB;

THEREFORE,

D MEASURES CA, AB
WHICH ARE PRIME TO ONE ANOTHER:

[VII. DEF. 12] WHICH,

IS IMPOSSIBLE.

THEREFORE,

NO NUMBER WILL MEASURE THE NUMBERS, AB, BC.

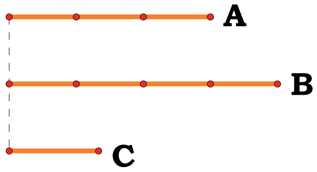
THEREFORE,

AB, BC ARE PRIME TO ONE ANOTHER.

Q. E. D.

Proposition 29.

Any prime number is prime to any number which it does not measure.



Let,

A be a prime number,

and let,

it not measure *B*;

I say that;

B, *A* are prime to one another.

For,

if *B*, *A* are not prime to one another,
some number will measure them.

Let,

C measure them.

Since,

C measures *B*, and
A does not measure *B*,

therefore,

C is not the same with *A*.

Now, since,

C measures *B*, *A*,

therefore,

it, also, measures *A* which is prime,
though it is not the same with it:

which,

is impossible.

Therefore,

no number will measure *B*, *A*.

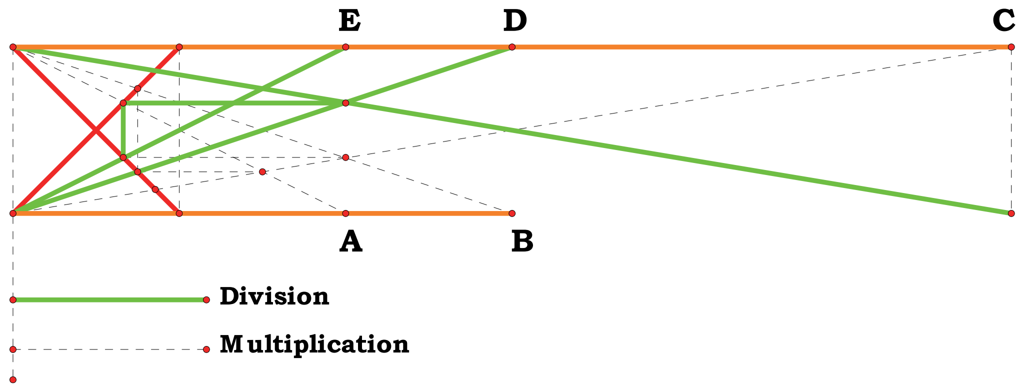
Therefore,

A, *B* are prime to one another.

Q. E. D.

PROPOSITION 30.

IF TWO NUMBERS, BY MULTIPLYING ONE ANOTHER, MAKE SOME NUMBER, AND ANY PRIME NUMBER MEASURE THE PRODUCT, IT WILL, ALSO, *MEASURE ONE OF THE ORIGINAL NUMBERS.*



FOR LET,

THE TWO NUMBERS, *A*, *B*,

BY MULTIPLYING ONE ANOTHER, MAKE *C*,

AND LET,

ANY PRIME NUMBER, *D*, MEASURE *C*;

I SAY THAT;

D MEASURES ONE OF THE NUMBERS, *A*, *B*.

FOR LET,

IT NOT MEASURE A.

NOW *D* IS PRIME;

[VII. 29] THEREFORE,

A, *D* ARE PRIME TO ONE ANOTHER.

AND,

AS MANY TIMES AS *D* MEASURES *C*,

SO MANY UNITS LET THERE BE IN *E*.

SINCE THEN,

D MEASURES *C*, ACCORDING TO THE UNITS IN *E*,

[VII. DEF. 15] THEREFORE,

D, BY MULTIPLYING *E*, HAS MADE *C*.

FURTHER,

A, BY MULTIPLYING *B*, HAS, ALSO, MADE *C*;

THEREFORE,

THE PRODUCT, OF *D*, *E*, = THE PRODUCT, OF *A*, *B*.

[VII. 19] THEREFORE,

AS *D* IS TO A,

SO IS *B* TO *E*.

[VII. 21] BUT,

D, *A* ARE PRIME TO ONE ANOTHER,
PRIMES ARE, ALSO, LEAST, AND
THE LEAST MEASURE THE NUMBERS WHICH HAVE
THE SAME RATIO THE SAME NUMBER OF TIMES,
THE GREATER THE GREATER, AND,
THE LESS THE LESS,

[VII. 20] THAT IS,

THE ANTECEDENT THE ANTECEDENT AND,
THE CONSEQUENT THE CONSEQUENT;

THEREFORE,

D MEASURES *B*.

SIMILARLY WE CAN, ALSO, SHOW THAT,

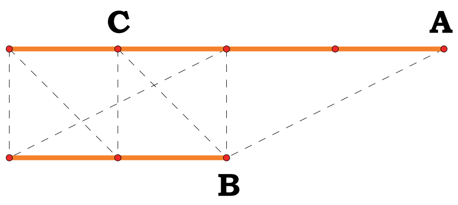
IF *D* DO NOT MEASURE *B*,
IT WILL MEASURE *A*.

THEREFORE,

D MEASURES ONE OF THE NUMBERS, *A*, *B*.

Q. E. D.

PROPOSITION 31.



ANY COMPOSITE NUMBER IS MEASURED BY SOME PRIME NUMBER.

LET,

A BE A COMPOSITE NUMBER;

I SAY THAT;

A IS MEASURED BY SOME PRIME NUMBER.

FOR, SINCE,

A IS COMPOSITE,

SOME NUMBER WILL MEASURE IT.

LET,

A NUMBER MEASURE IT,

AND LET,

IT BE *B*.

NOW,

IF *B* IS PRIME,

WHAT WAS ENJOINED WILL HAVE BEEN DONE.

BUT,

IF IT IS COMPOSITE, SOME NUMBER WILL MEASURE IT.

LET,

A NUMBER MEASURE IT,

AND LET,

IT BE *C*.

THEN, SINCE,

C MEASURES *B*, AND

B MEASURES *A*,

THEREFORE,

C, ALSO, MEASURES *A*.

AND,

IF *C* IS PRIME,

WHAT WAS ENJOINED WILL HAVE BEEN DONE.

BUT,

IF IT IS COMPOSITE, SOME NUMBER WILL MEASURE IT.

THUS,

IF THE INVESTIGATION BE CONTINUED IN THIS WAY,

SOME PRIME NUMBER WILL BE FOUND

WHICH WILL MEASURE THE NUMBER BEFORE IT,

WHICH WILL, ALSO, MEASURE *A*.

FOR,

IF IT IS NOT FOUND,
AN INFINITE SERIES OF NUMBERS WILL MEASURE
THE NUMBER A,

EACH, OF WHICH,
IS LESS THAN THE OTHER:

WHICH,
IS IMPOSSIBLE IN NUMBERS.

THEREFORE,
SOME PRIME NUMBER WILL BE FOUND
WHICH WILL MEASURE THE ONE BEFORE IT,
WHICH WILL, ALSO, MEASURE A.

THEREFORE,
ANY COMPOSITE NUMBER IS MEASURED BY
SOME PRIME NUMBER.

PROPOSITION 32.

ANY NUMBER EITHER IS PRIME OR IS MEASURED BY SOME PRIME NUMBER.

A

LET,

A BE A NUMBER;

I SAY THAT;

A EITHER IS PRIME, OR

IS MEASURED BY SOME PRIME NUMBER.

NOW,

IF *A* IS PRIME,

THAT WHICH WAS A ENJOINED WILL HAVE BEEN DONE.

[VII. 31] BUT,

IF IT IS COMPOSITE,

SOME PRIME NUMBER WILL MEASURE IT.

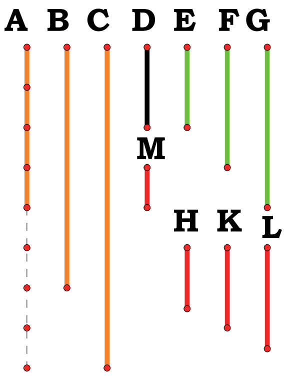
THEREFORE,

ANY NUMBER EITHER IS PRIME, OR

IS MEASURED BY SOME PRIME NUMBER.

Q. E. D.

Proposition 33.



Given as many numbers as we please, to find the least of those which have the same ratio with them.

Let,

A, *B*, *C*, be the given numbers,
as many as we please;

thus it is required,
to find the least of
those which have the same ratio with *A*, *B*, *C*.

Now,
A, *B*, *C* are either prime to one another or not.

[vii. 21] And,
if *A*, *B*, *C* are prime to one another,
they are the least of those which have
the same ratio with them.

[vii. 3] But, if not, let,
D, the greatest common measure, of *A*, *B*, *C*, be taken,

and let,
as many times as *D* measures the numbers, *A*, *B*, *C*,
respectively,
so many units there be in the numbers, *E*, *F*, *G*,
respectively.

[vii. 16] Therefore,
the numbers, *E*, *F*, *G*, measure the numbers, *A*, *B*, *C*,
respectively,
according to the units in *D*.

[vii. Def. 20] Therefore,
E, *F*, *G*, measure *A*, *B*, *C*, the same number of times;

therefore,
E, *F*, *G* are in the same ratio with *A*, *B*, *C*.

I say next that;
they are the least that are in that ratio.

For,
if *E*, *F*, *G* are not the least of those
which have the same ratio with *A*, *B*, *C*,
there will be numbers less than *E*, *F*, *G*,
which are in the same ratio with *A*, *B*, *C*.

Let,

THEM BE H, K, L;

THEREFORE,

H MEASURES A, THE SAME NUMBER OF TIMES THAT
THE NUMBERS, K, L, MEASURE
THE NUMBERS, B, C, RESPECTIVELY.

NOW LET,

AS MANY TIMES AS H MEASURES A,
SO MANY UNITS THERE BE IN M;

THEREFORE,

THE NUMBERS,
K, L, ALSO, MEASURE THE NUMBERS, B, C, RESPECTIVELY,
ACCORDING TO THE UNITS IN M.

AND, SINCE,

H MEASURES A, ACCORDING TO THE UNITS IN M,

[VII. 16] THEREFORE,

M, ALSO, MEASURES A, ACCORDING TO THE UNITS IN H.

FOR THE SAME REASON,

M, ALSO, MEASURES THE NUMBERS, B, C,
ACCORDING TO THE UNITS IN THE NUMBERS, K, L,
RESPECTIVELY;

THEREFORE,

M MEASURES A, B, C.

NOW, SINCE,

H MEASURES A, ACCORDING TO THE UNITS IN M,

[VII. DEF. 15] THEREFORE,

H, BY MULTIPLYING M, HAS MADE A.

FOR THE SAME REASON ALSO,

E, BY MULTIPLYING D, HAS MADE A.

THEREFORE,

THE PRODUCT, OF E, D, = THE PRODUCT, OF H, M.

[VII. 19] THEREFORE,

AS E IS TO H,
SO IS M TO D.

BUT,

E IS GREATER THAN H;

THEREFORE,

M IS, ALSO, GREATER THAN D.

AND,

IT MEASURES A, B, C:

WHICH,

IS IMPOSSIBLE,

FOR,

BY HYPOTHESIS,

D IS THE GREATEST COMMON MEASURE OF A, B, C.

THEREFORE,

THERE CANNOT BE ANY NUMBERS LESS THAN E, F, G,
WHICH ARE IN THE SAME RATIO WITH A, B, C.

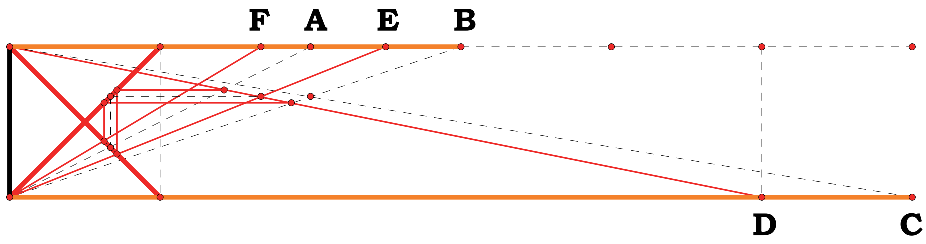
THEREFORE,

E, F, G, ARE THE LEAST OF THOSE
WHICH HAVE THE SAME RATIO WITH A, B, C.

Q. E. D.

PROPOSITION 34.

GIVEN TWO NUMBERS, TO FIND THE LEAST NUMBER WHICH THEY MEASURE.



LET,

A, B BE THE TWO GIVEN NUMBERS;

THUS IT IS REQUIRED,

TO FIND THE LEAST NUMBER WHICH THEY MEASURE.

NOW,

A, B ARE EITHER PRIME TO ONE ANOTHER OR NOT.

FIRST, LET,

A, B BE PRIME TO ONE ANOTHER,

AND LET,

A, BY MULTIPLYING B, MAKE C;

[VII. 16] THEREFORE ALSO,

B, BY MULTIPLYING A, HAS MADE C.

THEREFORE,

A, B MEASURE C.

I SAY NEXT THAT;

IT IS, ALSO, THE LEAST NUMBER THEY MEASURE.

FOR,

IF NOT, A, B WILL MEASURE SOME NUMBER
WHICH IS LESS THAN C.

LET,

THEM MEASURE D.

THEN LET,

AS MANY TIMES AS A MEASURES D,
SO MANY UNITS THERE BE IN E,

AND,

AS MANY TIMES AS B MEASURES D,
SO MANY UNITS LET THERE BE IN F;

[VII. DEF. 15] THEREFORE,

A, BY MULTIPLYING E, HAS MADE D,

AND,

B, BY MULTIPLYING F HAS MADE D;

THEREFORE,

THE PRODUCTS, $A \times E = B \times F$.

[VII. 19] THEREFORE,

AS A IS TO B,

SO IS F TO E.

[VII. 21] BUT,

A, B ARE PRIME, PRIMES ARE, ALSO, LEAST,

[VII. 20] AND,

THE LEAST MEASURE

THE NUMBERS WHICH HAVE THE SAME RATIO

THE SAME NUMBER OF TIMES,

THE GREATER THE GREATER AND,

THE LESS THE LESS;

THEREFORE,

B MEASURES E, AS CONSEQUENT, CONSEQUENT.

AND,

SINCE A, BY MULTIPLYING B, E, HAS MADE C, D,

[VII. 17] THEREFORE,

AS B IS TO E,

SO IS C TO D.

BUT,

B MEASURES E;

THEREFORE,

C, ALSO, MEASURES D,

THE GREATER THE LESS:

WHICH,

IS IMPOSSIBLE.

THEREFORE,

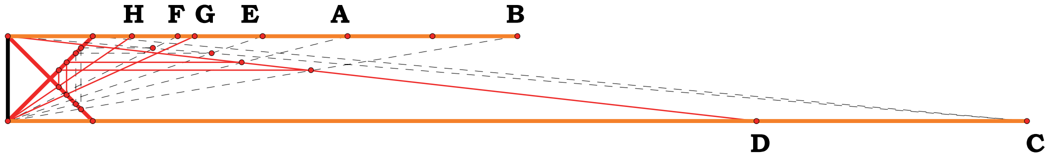
A, B DO NOT MEASURE ANY NUMBER LESS THAN C;

THEREFORE,

C IS THE LEAST THAT IS MEASURED BY A, B.

NEXT, LET,

A, B NOT BE PRIME TO ONE ANOTHER,



[VII. 33] AND LET,

F, E,
THE LEAST NUMBERS OF THOSE WHICH HAVE
THE SAME RATIO WITH A, B BE TAKEN;

[VII. 19] THEREFORE,
THE PRODUCT, OF A, E, = THE PRODUCT, OF B, F.

AND LET,
A, BY MULTIPLYING E, MAKE C;

THEREFORE ALSO,
B, BY MULTIPLYING F, HAS MADE C;

THEREFORE,
A, B MEASURE C.

I SAY NEXT THAT;
IT IS, ALSO, THE LEAST NUMBER THAT THEY MEASURE.

FOR, IF NOT,
A, B WILL MEASURE SOME NUMBER WHICH IS LESS THAN C.

LET,
THEM MEASURE D.

AND LET,
AS MANY TIMES AS A MEASURES D,
SO MANY UNITS THERE BE IN G,

AND LET,
AS MANY TIMES AS B MEASURES D,
SO MANY UNITS THERE BE IN H.

THEREFORE,
A, BY MULTIPLYING G, HAS MADE D, AND
B, BY MULTIPLYING H, HAS MADE D.

THEREFORE,
THE PRODUCT OF A, G = THE PRODUCT OF B, H;

[VII. 19] THEREFORE,
AS A IS TO B,
SO IS H TO G.

BUT,
AS A IS TO B,
SO IS F TO E.

THEREFORE ALSO,
AS F IS TO E,
SO IS H TO G.

BUT,
F, E ARE LEAST,

[VII. 20] AND,

THE LEAST MEASURE THE NUMBERS
WHICH HAVE THE SAME RATIO THE SAME NUMBER OF TIMES,
THE GREATER THE GREATER AND,
THE LESS THE LESS;

THEREFORE,

E MEASURES *G*.

AND, SINCE,

A, BY MULTIPLYING *E*, *G*, HAS MADE *C*, *D*,

[VII. 17] THEREFORE,

AS *E* IS TO *G*,
SO IS *C* TO *D*.

BUT,

E MEASURES *G*;

THEREFORE,

C, ALSO, MEASURES *D*,
THE GREATER THE LESS:

WHICH,

IS IMPOSSIBLE.

THEREFORE,

A, *B* WILL NOT MEASURE ANY NUMBER
WHICH IS LESS THAN *C*.

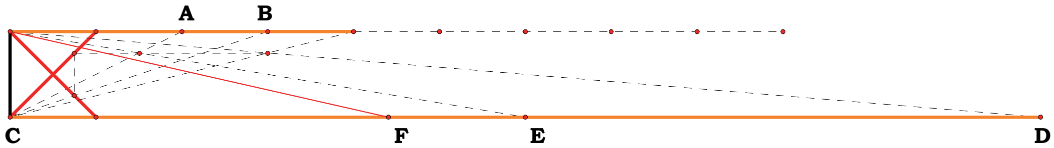
THEREFORE,

C IS THE LEAST THAT IS MEASURED BY *A*, *B*.

Q. E. D.

Proposition 35.

If two numbers measure any number, the least number measured by them will, also, measure the same.



FOR LET,

THE TWO NUMBERS, *A*, *B*, MEASURE ANY NUMBER, *CD*,

AND LET,

E BE THE LEAST THAT THEY MEASURE;

I SAY THAT;

E, ALSO, MEASURES *CD*.

FOR,

IF *E* DOES NOT MEASURE *CD*,

LET,

E MEASURING *DF*, LEAVE *CF*, LESS THAN ITSELF.

NOW, SINCE,

A, *B* MEASURE *E*, AND

E MEASURES *DF*,

THEREFORE,

A, *B* WILL, ALSO, MEASURE *DF*.

BUT,

THEY, ALSO, MEASURE THE WHOLE, *CD*;

THEREFORE,

THEY WILL, ALSO, MEASURE THE REMAINDER, *CF*,

WHICH IS LESS THAN *E*:

WHICH,

IS IMPOSSIBLE.

THEREFORE,

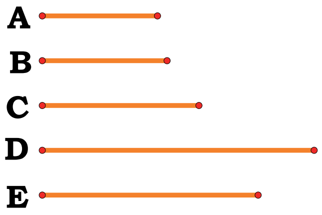
E CANNOT FAIL TO MEASURE *CD*;

THEREFORE,

IT MEASURES IT.

Q. E. D.

PROPOSITION 36.



GIVEN THREE NUMBERS, TO FIND THE LEAST NUMBER WHICH THEY MEASURE.

LET,
A, *B*, *C*, BE
THE THREE GIVEN NUMBERS;

THUS IT IS REQUIRED,
TO FIND THE LEAST NUMBER WHICH THEY MEASURE.

[VII. 34] LET,
D, THE LEAST NUMBER MEASURED BY
THE TWO NUMBERS, *A*, *B*, BE TAKEN.

THEN,
C EITHER MEASURES, OR DOES NOT MEASURE, *D*.

FIRST, LET IT,
MEASURE IT.

BUT,
A, *B*, ALSO, MEASURE *D*;

THEREFORE,
A, *B*, *C* MEASURE *D*.

I SAY NEXT THAT;
IT IS, ALSO, THE LEAST THAT THEY MEASURE.

FOR,
IF NOT, *A*, *B*, *C* WILL MEASURE SOME NUMBER
WHICH IS LESS THAN *D*.

LET,
THEM MEASURE *E*.

SINCE,
A, *B*, *C*, MEASURE *E*,

THEREFORE, ALSO,
A, *B* MEASURE *E*.

[VII. 35] THEREFORE,
THE LEASE NUMBER
MEASURED BY *A*, *B* WILL, ALSO, MEASURE *E*.

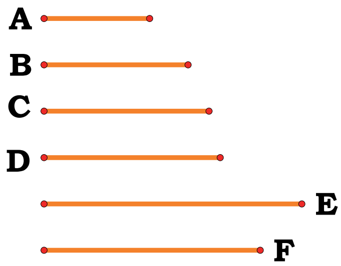
BUT,
D IS THE LEAST NUMBER MEASURED BY *A*, *B*;

THEREFORE,
D WILL MEASURE *E*,

THE GREATER THE LESS:

WHICH,
IS IMPOSSIBLE.

THEREFORE,
A, *B*, *C*, WILL NOT MEASURE ANY NUMBER
WHICH IS LESS THAN *D*;



THEREFORE,
D IS THE LEAST
THAT *A*, *B*, *C* MEASURE.

AGAIN, LET,
C NOT MEASURE *D*,

[VII. 34] AND LET,
E, THE LEAST NUMBER MEASURED BY *C*, *D* BE TAKEN.

SINCE,
A, *B* MEASURE *D*, AND
D MEASURES *E*,

THEREFORE, ALSO,
A, *B* MEASURE *E*.

BUT,
C, ALSO, MEASURES *E*;

THEREFORE ALSO,
A, *B*, *C* MEASURE *E*.

I SAY NEXT THAT;
IT IS, ALSO, THE LEAST THAT THEY MEASURE.

FOR,
IF NOT, *A*, *B*, *C*, WILL MEASURE SOME NUMBER
WHICH IS LESS THAN *E*.

LET,
THEM MEASURE *F*.

SINCE,
A, *B*, *C*, MEASURE *F*,

THEREFORE ALSO,
A, *B* MEASURE *F*;

[VII. 35] THEREFORE,
THE LEAST NUMBER MEASURED BY *A*, *B*,
WILL, ALSO, MEASURE *F*.

BUT,
D IS THE LEAST NUMBER MEASURED BY *A*, *B*;

THEREFORE,

D MEASURES F.

BUT,

C, ALSO, MEASURES F;

THEREFORE,

D, C MEASURE F,

SO THAT,

THE LEAST NUMBER MEASURED BY D, C,
WILL, ALSO, MEASURE F.

BUT,

E IS THE LEAST NUMBER MEASURED BY C, D;

THEREFORE,

E MEASURES F,
THE GREATER THE LESS:

WHICH,

IS IMPOSSIBLE.

THEREFORE,

A, B, C, WILL NOT MEASURE ANY NUMBER
WHICH IS LESS THAN E.

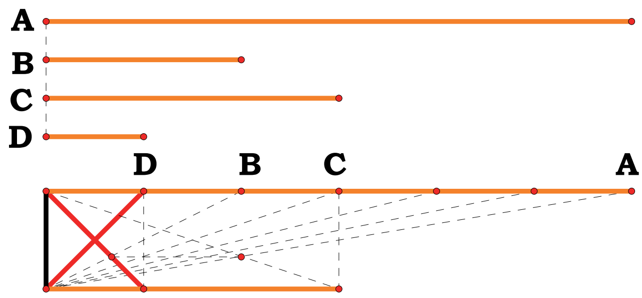
THEREFORE,

E IS THE LEAST THAT IS MEASURED BY A, B, C.

Q. E. D.

PROPOSITION 37.

IF A NUMBER BE MEASURED BY ANY NUMBER, THE NUMBER WHICH IS MEASURED WILL HAVE A PART CALLED BY THE SAME NAME AS THE MEASURING NUMBER.



FOR LET,

THE NUMBER, *A*, BE MEASURED BY ANY NUMBER, *B*;

I SAY THAT;

A HAS A PART CALLED BY THE SAME NAME AS *B*.

FOR LET,

AS MANY TIMES AS *B* MEASURES *A*,
SO MANY UNITS THERE BE IN *C*.

SINCE,

B MEASURES *A*, ACCORDING TO THE UNITS IN *C*,

AND,

THE UNIT, *D*, ALSO, MEASURES THE NUMBER, *C*,
ACCORDING TO THE UNITS IN IT,

THEREFORE,

THE UNIT, *D*, MEASURES THE NUMBER, *C*,
THE SAME NUMBER OF TIMES AS *B* MEASURES *A*.

[VII. 15] THEREFORE, ALTERNATELY,

THE UNIT, *D*, MEASURES THE NUMBER, *B*,
THE SAME NUMBER OF TIMES AS *C* MEASURES A;

THEREFORE,

WHATEVER PART THE UNIT, *D*, IS OF THE NUMBER *B*,
THE SAME PART IS *C* OF *A*, ALSO.

BUT,

THE UNIT, *D*, IS A PART, OF THE NUMBER *B*,
CALLED BY THE SAME NAME AS IT;

THEREFORE,

C IS, ALSO, A PART, OF *A*, CALLED BY THE SAME NAME AS *B*,

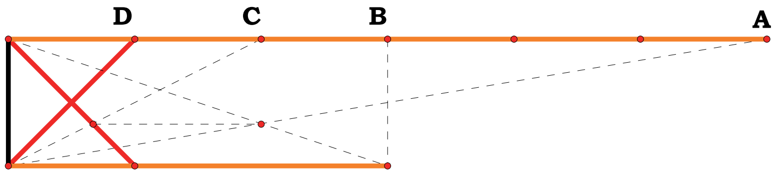
SO THAT,

A HAS A PART *C*,
WHICH IS CALLED BY THE SAME NAME AS *B*.

Q. E. D.

PROPOSITION 38.

IF A NUMBER HAVE ANY PART WHATEVER, IT WILL BE MEASURED BY A NUMBER CALLED BY THE SAME NAME AS THE PART.



FOR LET,

THE NUMBER, *A*, HAVE ANY PART WHATEVER, *B*,

AND LET,

C BE A NUMBER CALLED BY THE SAME NAME AS THE PART *B*;

I SAY THAT;

C MEASURES *A*.

FOR, SINCE,

B IS A PART, OF *A*, CALLED BY THE SAME NAME AS *C*,

AND,

THE UNIT, *D*, IS, ALSO, A PART, OF *C*, CALLED BY THE SAME NAME AS IT,

THEREFORE,

WHATEVER PART THE UNIT, *D*, IS OF THE NUMBER, *C*, THE SAME PART IS *B* OF *A*, ALSO;

THEREFORE,

THE UNIT, *D*, MEASURES THE NUMBER, *C*, THE SAME NUMBER OF TIMES THAT *B* MEASURES *A*.

[VII. 15] THEREFORE, ALTERNATELY,

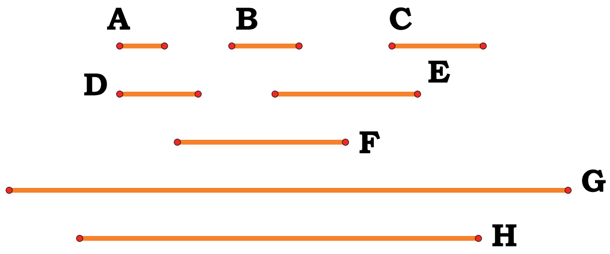
THE UNIT, *D*, MEASURES THE NUMBER, *B*, THE SAME NUMBER OF TIMES THAT *C* MEASURES *A*.

THEREFORE,

C MEASURES *A*.

Q. E. D.

Proposition 39.



To find the number which is the least that will have given parts.

Let,

A, B, C, be the given parts;

thus it is required,

to find the number which is the least that will have the parts, A, B, C.

Let,

D, E, F, be numbers called by the same name as the parts, A, B, C,

[VII. 36] and let,

G, the least number measured by D, E, F, be taken.

[VII. 37] Therefore,

G has parts called by the same name as D, E, F.

But,

A, B, C are parts called by the same name as D E, F;

therefore,

G has the parts A, B, C.

I say next that;

it is, also, the least number that has.

For,

if not, there will be some number less than G, which will have the parts, A, B, C.

Let,

it be H.

Since,

H has the parts, A, B, C,

[VII. 38] therefore,

H will be measured by numbers called by the same name as the parts, A, B, C.

But,

D, E, F are numbers called by the same name as the parts, A, B, C;

therefore,

H IS MEASURED BY *D*, *E*, *F*,

AND,

IT IS LESS THAN *G*:

WHICH,

IS IMPOSSIBLE.

THEREFORE,

THERE WILL BE NO NUMBER LESS THAN *G*,
THAT WILL HAVE THE PARTS, *A*, *B*, *C*.

Q. E. D.

BOOK VIII.

OF

EUCLID'S ELEMENTS

TRANSLATED FROM THE TEXT OF HEIBERG

BY

SIR THOMAS L. HEATH,

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HONORARY FELLOW (SOMETIME FELLOW) OF TRINITY COLLEGE CAMBRIDGE

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REVISED WITH SUBTRACTIONS

REFORMATTED AND ABRIDGED FOR STUDY OF *THE ELEMENTS*.

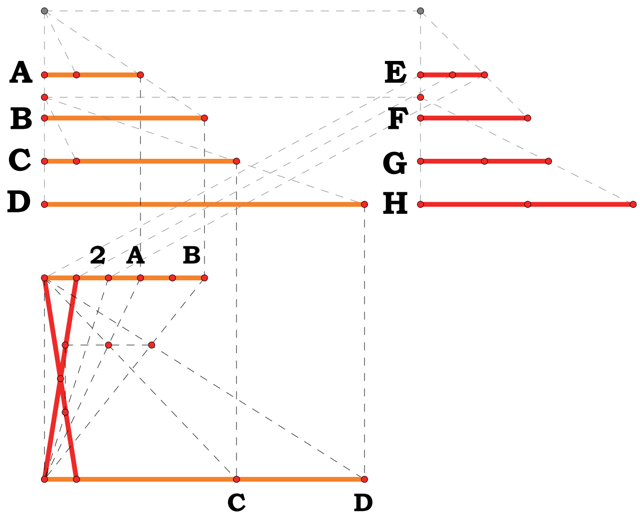
BY JOHN CLARK.

BOOK VIII.

PROPOSITIONS.

PROPOSITION 1.

IF THERE BE AS MANY NUMBERS AS WE PLEASE IN CONTINUED PROPORTION, AND THE EXTREMES OF THEM BE PRIME TO ONE ANOTHER, THE NUMBERS ARE THE LEAST OF THOSE WHICH HAVE THE SAME RATIO WITH THEM.



LET,

THERE BE AS MANY NUMBERS AS WE PLEASE, *A*, *B*, *C*, *D*, IN CONTINUED PROPORTION,

AND LET,

THE EXTREMES OF THEM, *A*, *D*, BE PRIME TO ONE ANOTHER;

I SAY THAT;

A, *B*, *C*, *D* ARE THE LEAST OF THOSE WHICH HAVE THE SAME RATIO WITH THEM.

FOR LET,

IF NOT, *E*, *F*, *G*, *H* BE LESS THAN *A*, *B*, *C*, *D*, AND IN THE SAME RATIO WITH THEM.

NOW, SINCE,

A, *B*, *C*, *D* ARE IN THE SAME RATIO WITH *E*, *F*, *G*,*H*, AND THE MULTITUDE OF THE NUMBERS, *A*, *B*, *C*, *D*, EQUALS THE MULTITUDE OF THE NUMBERS, *E*, *F*, *G*, *H*,

[VII. 14] THEREFORE, *EX AEQUALI*,

AS *A* IS TO *D*,
SO IS *E* TO *H*.

[VII. 21] BUT,

A, *D* ARE PRIME,
PRIMES ARE, ALSO, LEAST, AND
THE LEAST NUMBERS MEASURE THOSE

WHICH HAVE THE SAME RATIO THE SAME NUMBER OF TIMES,
THE GREATER THE GREATER AND,
THE LESS THE LESS,

[VII. 20] THAT IS,
THE ANTECEDENT THE ANTECEDENT AND,
THE CONSEQUENT THE CONSEQUENT.

THEREFORE,
A MEASURES *E*,
THE GREATER THE LESS:

WHICH,
IS IMPOSSIBLE.
THEREFORE, *E*, *F*, *G*, *H*, WHICH ARE LESS THAN *A*, *B*, *C*, *D*,
ARE NOT IN THE SAME RATIO WITH THEM.

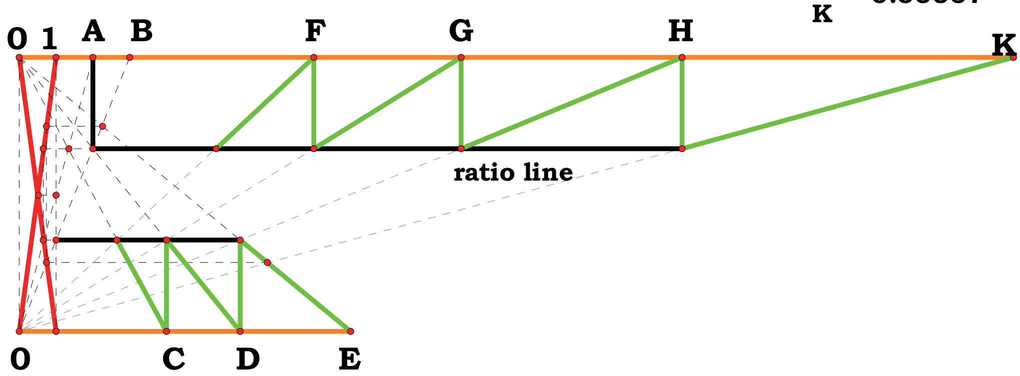
THEREFORE,
A, *B*, *C*, *D*, ARE THE LEAST OF THOSE WHICH HAVE
THE SAME RATIO WITH THEM.

Q. E. D.

PROPOSITION 2.

TO FIND NUMBERS IN CONTINUED PROPORTION, AS MANY AS MAY BE PRESCRIBED, AND THE LEAST THAT ARE IN A GIVEN RATIO.

01 = 0.48683 cm	A = 2.00000	A^2-C = 0.00000	$\frac{C}{D}$ = 0.66667
A0 = 0.97367 cm	B = 3.00000	A·B-D = 0.00000	$\frac{D}{E}$ = 0.66667
B0 = 1.46050 cm	C = 4.00000	B^2-E = 0.00000	$\frac{F}{G}$ = 0.66667
C0 = 1.94733 cm	D = 6.00000	A^3-F = 0.00000	$\frac{G}{H}$ = 0.66667
D0 = 2.92100 cm	E = 9.00000	A·D-G = 0.00000	$\frac{H}{K}$ = 0.66667
E0 = 4.38150 cm	F = 8.00000	A·E-H = 0.00000	
0F = 3.89467 cm	G = 12.00000	B·E-K = 0.00000	
0G = 5.84200 cm	H = 18.00000	$\frac{A}{B}$ = 0.66667	
0H = 8.76300 cm	K = 27.00000		
0K = 13.14450 cm			



LET,

THE RATIO, OF *A* TO *B* BE THE GIVEN RATIO IN LEAST NUMBERS;

THUS IT IS REQUIRED,

TO FIND NUMBERS IN CONTINUED PROPORTION,
AS MANY AS MAY BE PRESCRIBED, AND
THE LEAST THAT ARE IN THE RATIO, OF *A* TO *B*.

LET,

FOUR BE PRESCRIBED;

LET,

A, BY MULTIPLYING ITSELF, MAKE *C*,

AND LET ,

BY MULTIPLYING *B*, IT MAKE *D*;

LET,

B, BY MULTIPLYING ITSELF, MAKE *E*;

FURTHER, LET,

A, BY MULTIPLYING *C*, *D*, *E*, MAKE *F*, *G*, *H*,

AND LET,

B, BY MULTIPLYING *E*, MAKE *K*.

NOW, SINCE,

A, BY MULTIPLYING ITSELF, HAS MADE *C*, AND

by multiplying B, has made D,

[vii. 17] therefore,
as A is to B,
so is C to D.

Again, since,
A, by multiplying B, has made D, and
B, by multiplying itself, has made E,

therefore,
the numbers, A, B, by multiplying B, have made
the numbers, D, E, respectively.

[vii. 18] Therefore,
as A is to B,
so is D to E.

But,
as A is to B,
so is C to D;

therefore also,
as C is to D,
so is D to E.

And, since,
A, by multiplying C, D, has made F, G,

[vii. 17] therefore,
as C is to D,
so is F to G.

But,
as C is to D,
so was A to B;

therefore also,
as A is to B,
so is F to G.

Again, since,
A, by multiplying D, E, has made G, H,

[vii. 17] therefore,
as D is to E,
so is G to H.

But,
as D is to E,
so is A to B.

Therefore also,

as A is to B,
so is G to H.

And, since,
A, B, by multiplying E, have made H, K,

[vii. 18] therefore,
as A is to B,
so is H to K.

But,
as A is to B
so is F to G, and
G to H.

Therefore also,
as F is to G,
so is G to H, and
H to K,

therefore,
C, D, E, and F, G, H, K, are proportional in the ratio, of A to B.

I say next that;
they are the least numbers that are so.

[vii. 22] For, since,
A, B are the least of those
which have the same ratio with them, and
the least of those which have
the same ratio are prime to one another,

therefore,
A, B are prime to one another.

And,
the numbers, A, B, by multiplying themselves,
respectively, have made the numbers, C, E, and
by multiplying the numbers, C, E, respectively,
have made the numbers, F, K;

[vii. 27] therefore,
C, E and F, K are prime to one another, respectively.

[viii. 1] But,
if there be as many numbers as we please
in continued proportion, and
the extremes of them be prime to one another,
they are the least of those which have
the same ratio with them.

THEREFORE,

C, *D*, *E*, AND *F* *G*, *H*, *K*, ARE THE LEAST OF THOSE WHICH HAVE THE SAME RATIO WITH *A*, *B*.

Q. E. D.

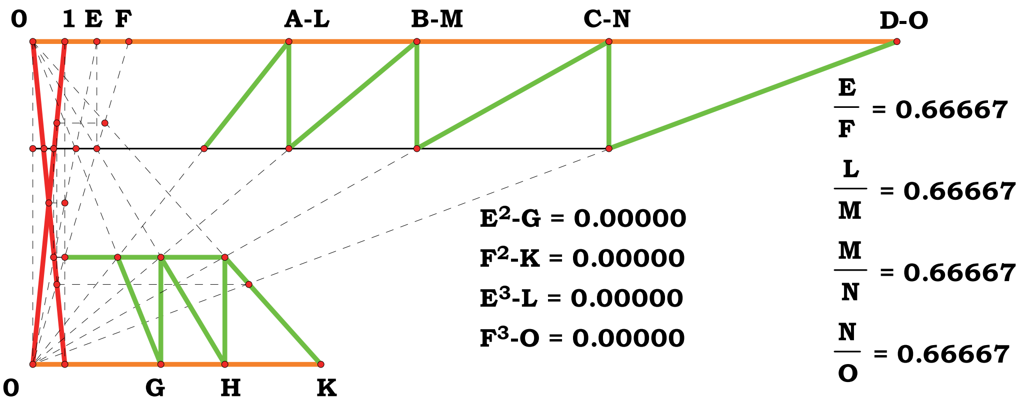
PORISM.

FROM THIS IT IS MANIFEST THAT, IF THREE NUMBERS IN CONTINUED PROPORTION BE THE LEAST OF THOSE WHICH HAVE THE SAME RATIO WITH THEM, THE EXTREMES OF THEM ARE SQUARES, AND, IF FOUR NUMBERS, CUBES.

Proposition 3.

If as many numbers as we please in continued proportion be the least of those which have the same ratio with them, the extremes of them are prime to one another.

01 = 0.42333 cm	0C-N = 7.62000 cm	G = 4.00000	L = 8.00000
0E = 0.84667 cm	0G = 1.69333 cm	H = 6.00000	M = 12.00000
0F = 1.27000 cm	0H = 2.54000 cm	K = 9.00000	N = 18.00000
0A-L = 3.38667 cm	0K = 3.81000 cm	E = 2.00000	O = 27.00000
0B-M = 5.08000 cm	0D-O = 11.43000 cm	F = 3.00000	



Let,

as many numbers as we please, A, B, C, D,
in continued proportion be the least of those
which have the same ratio with them;

I say that;

the extremes of them, A, D, are prime to one another.

[vii. 33] For let,

two numbers, E, F,
the least that are in the ratio, of A, B, C, D, be taken,

[viii. 2] then,

three others, G, H, K, with the same property;

and others,

more by one continually,

until,

the multitude taken becomes equal to
the multitude of the numbers, A, B, C, D.

Let,

them be taken,

and let,

them be L, M, N, O.

[vii. 22] Now, since,

E, F are the least of those

WHICH HAVE THE SAME RATIO WITH THEM,
THEY ARE PRIME TO ONE ANOTHER,

[VIII. 2, POR.] AND, SINCE,
THE NUMBERS, E, F, BY MULTIPLYING THEMSELVES,
RESPECTIVELY, HAVE MADE THE NUMBERS, G, K, AND
BY MULTIPLYING THE NUMBERS, G, K,
RESPECTIVELY, HAVE MADE THE NUMBERS, L, O.

[VII. 27] THEREFORE,
BOTH G, K, AND L, O, ARE PRIME TO ONE ANOTHER,

AND, SINCE,
A, B, C, D ARE THE LEAST OF THOSE WHICH HAVE
THE SAME RATIO WITH THEM, WHILE L, M, N, O ARE
THE LEAST THAT ARE IN THE SAME RATIO WITH A, B, C, D,

AND,
THE MULTITUDE OF THE NUMBERS, A, B, C, D, EQUALS
THE MULTITUDE OF THE NUMBERS, L, M, N, O,

THEREFORE,
THE NUMBERS, A, B, C, D ARE EQUAL TO
THE NUMBERS, L, M, N, O, RESPECTIVELY;

THEREFORE,
$A = L$, AND, $D = O$.

AND,
L, O ARE PRIME TO ONE ANOTHER.

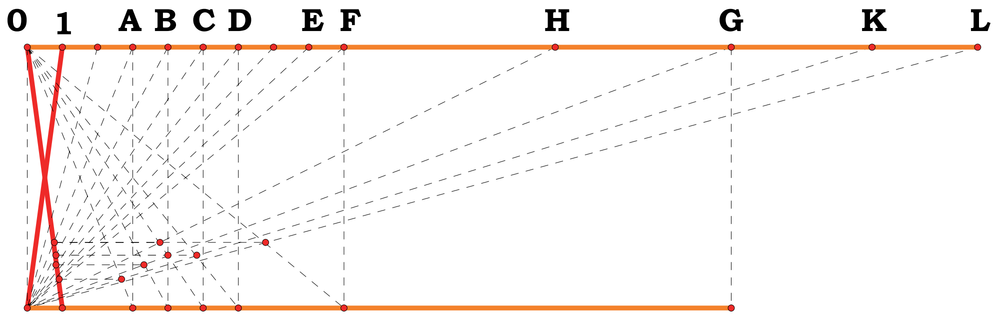
THEREFORE,
A, D ARE, ALSO, PRIME TO ONE ANOTHER.

Q. E. D.

Proposition 4.

Given as many ratios as we please in least numbers, to find numbers in continued proportion which are the least in the given ratios.

A = 3.00000	$\frac{A}{B}$ = 0.75000	$\frac{(A\cdot C)}{(B\cdot C)}$ = 0.75000	
B = 4.00000			
C = 5.00000	$\frac{C}{D}$ = 0.83333	$\frac{(B\cdot C)}{(B\cdot D)}$ = 0.83333	
D = 6.00000			
E = 8.00000	$\frac{E}{F}$ = 0.88889	$\frac{(B\cdot D)}{(A\cdot F)}$ = 0.88889	$\frac{(A\cdot E)}{(A\cdot F)}$ = 0.88889
F = 9.00000			
H = 15.00000	A·C = 15.00000		
G = 20.00000	B·C = 20.00000		
K = 24.00000	B·D = 24.00000	A·E = 24.00000	
L = 27.00000	A·F = 27.00000	A·F = 27.00000	



Let,

THE GIVEN RATIOS IN LEAST NUMBERS BE THAT OF *A* TO *B*,
THAT OF *C* TO *D*, AND,
THAT OF *E* TO *F*;

THUS IT IS REQUIRED,

TO FIND NUMBERS IN CONTINUED PROPORTION WHICH ARE THE LEAST THAT ARE IN THE RATIO, OF *A* TO *B*, IN THE RATIO, OF *C* TO *D*, AND, IN THE RATIO, OF *E* TO *F*.

[VII. 34] Let,

G, THE LEAST NUMBER MEASURED BY *B*, *C*, BE TAKEN.

And let,

AS MANY TIMES AS *B* MEASURES *G*,
SO MANY TIMES, ALSO, *A* MEASURE *H*,

AND LET,

AS MANY TIMES AS *C* MEASURES *G*,
SO MANY TIMES, ALSO, *D* MEASURE *K*.

Now,

E EITHER MEASURES OR DOES NOT MEASURE *K*.

First, let,

IT MEASURE IT.

And let,

as many times as E measures K,
so many times F measure L also.

Now, since,
A measures H, the same number of times
that B measures G,

[vii. Def. 20, vii. 13] therefore,
as A is to B,
so is H to G.

For the same reason also,
as C is to D,
so is G to K,

and further,
as E is to F,
so is K to L;

therefore,
H, G, K, L are continuously proportional in
the ratio, of A to B, in
the ratio, of C to D, and in
the ratio, of E to F.

I say next that;
they are, also, the least that have this property.

For,
if H, G, K, L are not
the least numbers continuously proportional in
the ratios
of A to B,
of C to D, and
of E to F,

let,
them be N, O, M, P.

Then since,
as A is to B,
so is N to O,

while,
A, B are least,

and,
the least numbers measure those which have
the same ratio the same number of times,
the greater the greater and,
the less the less,

[VII. 20] THAT IS,
THE ANTECEDENT THE ANTECEDENT AND,
THE CONSEQUENT THE CONSEQUENT;

THEREFORE,
B MEASURES *O*.

FOR THE SAME REASON,
C, ALSO, MEASURES *O*;

THEREFORE,
B, *C* MEASURE *O*;

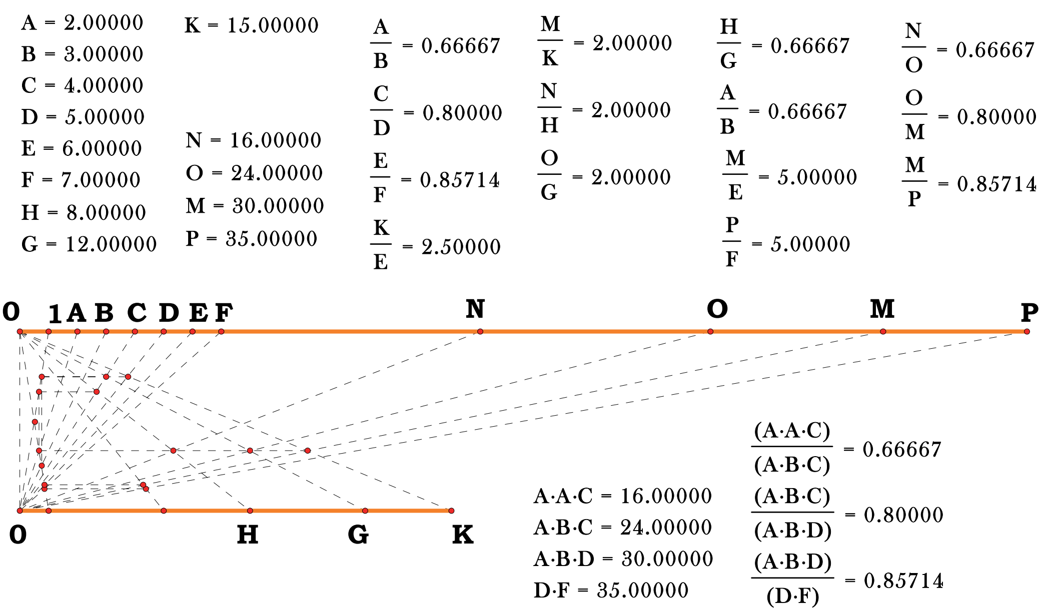
[VII. 35] THEREFORE,
THE LEAST NUMBER MEASURED
BY *B*, *C*, WILL, ALSO, MEASURE *O*.

BUT,
G IS THE LEAST NUMBER MEASURED BY *B*, *C*;
THEREFORE *G* MEASURES *O*,
THE GREATER THE LESS:

WHICH,
IS IMPOSSIBLE.

THEREFORE,
THERE WILL BE NO NUMBERS LESS THAN *H*, *G*, *K*, *L*
WHICH ARE CONTINUOUSLY IN THE RATIO
OF *A* TO *B*,
OF *C* TO D, AND
OF *E* TO *F*.

NEXT, LET,
E NOT MEASURE *K*.



LET,

M, THE LEAST NUMBER MEASURED BY E, K, BE TAKEN.

AND LET,

AS MANY TIMES AS K MEASURES M,
SO MANY TIMES H, G MEASURE N, O, RESPECTIVELY, AND
AS MANY TIMES AS E MEASURES M,
SO MANY TIMES LET F MEASURE P, ALSO.

SINCE,

H MEASURES N, THE SAME NUMBER OF TIMES
THAT G MEASURES O,

[VII. 13 AND DEF. 20] THEREFORE,

AS H IS TO G,
SO IS N TO O.

BUT,

AS H IS TO G,
SO IS A TO B;

THEREFORE ALSO,

AS A IS TO B,
SO IS N TO O.

FOR THE SAME REASON ALSO,

AS C IS TO D,
SO IS O TO M.

AGAIN, SINCE,

E MEASURES M,
THE SAME NUMBER OF TIMES THAT F MEASURES P,

[VII. 13 AND DEF. 20] THEREFORE,

AS E IS TO F,
SO IS M TO P;

THEREFORE,

N, O, M, P ARE CONTINUOUSLY PROPORTIONAL IN THE RATIOS
OF A TO B,
OF C TO D, AND
OF E TO F.

I SAY NEXT THAT;

THEY ARE, ALSO, THE LEAST THAT ARE IN THE RATIOS
$A : B$, $C : D$, $E : F$.

FOR,

IF NOT, THERE WILL BE SOME NUMBERS LESS THAN
N, O, M, P, CONTINUOUSLY PROPORTIONAL IN
THE RATIOS $A : B$, $C : D$, $E : F$.

LET,
THEM BE Q, R, S, T.

NOW SINCE,
AS Q IS TO R,
SO IS A TO B.

WHILE,
A, B ARC LEAST,

[VII. 20] AND,
THE LEAST NUMBERS MEASURE THOSE WHICH HAVE
THE SAME RATIO WITH THEM
THE SAME NUMBER OF TIMES,
THE ANTECEDENT THE ANTECEDENT AND,
THE CONSEQUENT THE CONSEQUENT,

THEREFORE,
B MEASURES R.

FOR THE SAME REASON,
C, ALSO, MEASURES R;

THEREFORE,
B, C MEASURE R.

[VII. 35] THEREFORE,
THE LEAST NUMBER MEASURED BY
B, C WILL, ALSO, MEASURE R.

BUT,
G IS THE LEAST NUMBER MEASURED BY B, C;

THEREFORE,
G MEASURES R.

[VII. 13] AND,
AS G IS TO R,
SO IS K TO S:

THEREFORE,
K, ALSO, MEASURES S.

BUT,
E, ALSO, MEASURES S;

THEREFORE,
E, K MEASURE S.

[VII. 35] THEREFORE,
THE LEAST NUMBER MEASURED BY E, K,
WILL, ALSO, MEASURE S.

BUT,

M IS THE LEAST NUMBER MEASURED BY E, K;

THEREFORE,

M MEASURES S,
THE GREATER THE LESS: WHICH,
IS IMPOSSIBLE.

THEREFORE,

THERE WILL NOT BE ANY NUMBERS LESS THAN
N, O, M, P CONTINUOUSLY PROPORTIONAL IN THE RATIOS
OF A TO B,
OF C TO D, AND
OF E TO F;

THEREFORE,

N, O, M, P ARE
THE LEAST NUMBERS CONTINUOUSLY PROPORTIONAL IN
THE RATIOS $A : B$, $C : D$, $E : F$.

Q. E. D.

PROPOSITION 5.

PLANE NUMBERS HAVE TO ONE ANOTHER THE RATIO COMPOUNDED OF THE RATIOS OF THEIR SIDES.

$F = 2.00000$ $\quad \frac{A}{B} = 1.87500 \quad \frac{D \cdot C}{D \cdot E} = 1.25000$

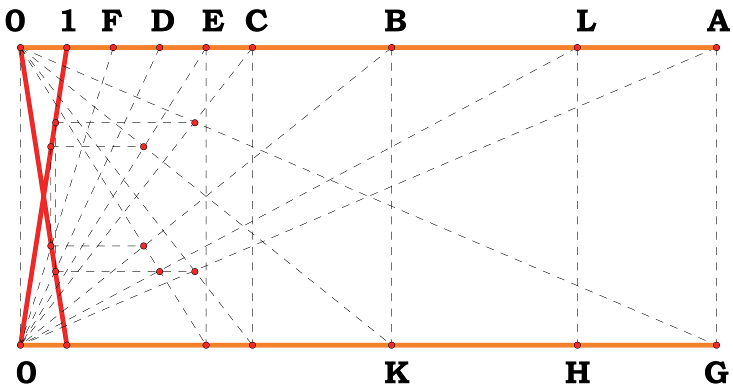
$D = 3.00000$

$E = 4.00000$ $\quad \frac{C}{E} = 1.25000 \quad \frac{E \cdot D}{E \cdot F} = 1.50000$

$C = 5.00000$

$B = 8.00000$ $\quad \frac{D}{F} = 1.50000 \quad \frac{C}{E} \cdot \frac{D}{F} = 1.87500$

$A = 15.00000$



$G = 15.00000$ $\quad \frac{G}{H} = 1.25000 \quad \frac{G}{K} = 1.87500$

$H = 12.00000$

$K = 8.00000$ $\quad \frac{H}{K} = 1.50000 \quad \frac{A}{H} = 1.25000$

LET,

A, *B* BE PLANE NUMBERS,

AND LET,

THE NUMBERS, *C*, *D*, BE THE SIDES, OF *A*, AND
E, *F* OF *B*;

I SAY THAT;

A HAS TO *B*,
THE RATIO COMPOUNDED OF THE RATIOS OF THE SIDES.

FOR,

THE RATIOS BEING GIVEN WHICH *C* HAS TO *E*, AND
D TO *F*,

LET,

THE LEAST NUMBERS, *G*, *H*, *K*, THAT ARE CONTINUOUSLY IN
THE RATIOS *C* : *E*, *D* : *F*, BE TAKEN,

[VIII. 4] SO THAT,

AS *C* IS TO *E* ,
SO IS *G* TO *H*, AND
AS *D* IS TO *F*,
SO IS *H* TO *K*.

AND LET,

D, BY MULTIPLYING E, MAKE L.

NOW, SINCE,

D, BY MULTIPLYING C, HAS MADE A, AND
BY MULTIPLYING E, HAS MADE L,

[VII. 17] THEREFORE,

AS C IS TO E,
SO IS A TO L.

BUT,

AS C IS TO E,
SO IS G TO H;

THEREFORE ALSO,

AS G IS TO H,
SO IS A TO L.

AGAIN, SINCE,

E, BY MULTIPLYING D, HAS MADE L, AND FURTHER
BY MULTIPLYING F, HAS MADE B,

[VII. 17] THEREFORE,

AS D IS TO F,
SO IS L TO B.

BUT,

AS D IS TO F,
SO IS H TO K;

THEREFORE ALSO,

AS H IS TO K,
SO IS L TO B.

BUT,

IT WAS, ALSO, PROVED THAT;
AS G IS TO H,
SO IS A TO L;

[VII. 14] THEREFORE, *EX AEQUALIY*,

AS G IS TO K,
SO IS A TO B.

BUT,

G HAS TO K,
THE RATIO COMPOUNDED OF THE RATIOS OF THE SIDES;

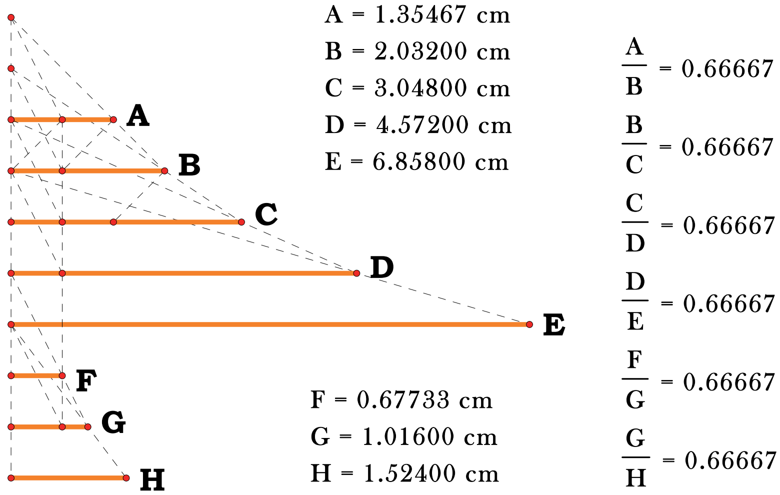
THEREFORE,

A, ALSO, HAS TO B,
THE RATIO COMPOUNDED OF THE RATIOS OF THE SIDES.

Q. E. D.

Proposition 6.

If there be as many numbers as we please in continued proportion, and the first do not measure the second, neither will any other measure any other.



Let,

there be as many numbers as we please,

A, *B*, *C*, *D*, *E*, in continued proportion,

and let,

A not measure *B*;

I say that;

neither will any other measure any other.

Now,

it is manifest that

A, *B*, *C*, *D*, *E*, do not measure one another, in order;

for,

A does not even measure *B*.

I say, then, that;

neither will any other measure any other.

For, if possible, let,

A measure *C*.

[vii. 33] And let,

however many, *A*, *B*, *C*, are, as many numbers, *E*, *G*, *H*,

the least of those which have

the same ratio with *A*, *B*, *C*, be taken.

Now, since,

F, *G*, *H* are in the same ratio with *A*, *B*, *C*, and

the multitude of the numbers, *A*, *B*, *C*, =

the multitude of the numbers, *F*, *G*, *H*,

[VII. 14] THEREFORE, *EX AEQUALI*,

as A is to C,
so is F to H.

And since,

as A is to B,
so is F to G,

while,

A does not measure B,

[VII. Def. 20] therefore,

neither does F measure G;

therefore,

F is not an unit,

for,

the unit measures any number.

[VIII. 3] Now,

F, H are prime to one another.

And,

as F is to H,
so is A to C;

therefore,

neither does A measure C.

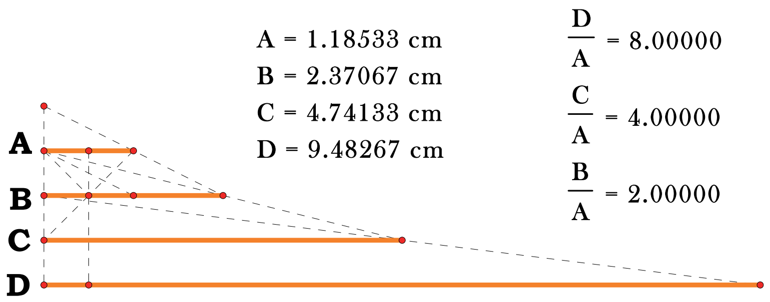
Similarly we can prove that,

neither will any other measure any other.

Q. E. D.

Proposition 7.

If there be as many numbers as we please in continued proportion and the first measure the last, it will measure the second also.



Let,

there be as many numbers as we please,

A, *B*, *C*, *D*, in continued proportion;

and let,

A measure *D*;

I say that;

A, also, measures *B*.

[viii. 6] For,

if *A* does not measure *B*,

neither will any other of

the numbers measure any other.

But *A* measures *D*.

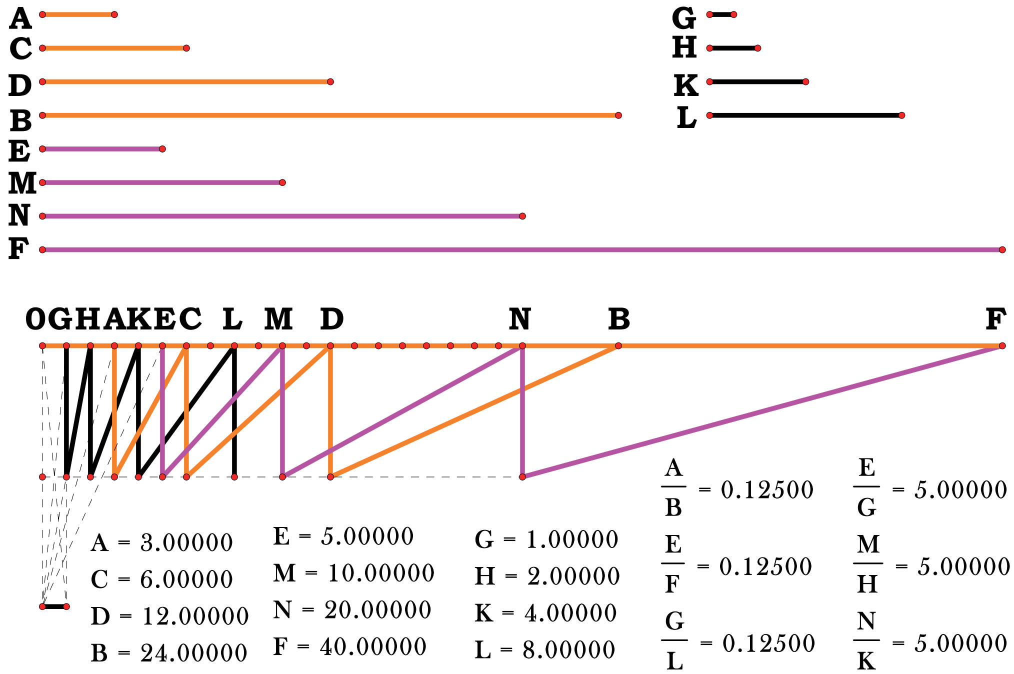
Therefore,

A, also, measures *B*.

Q. E. D.

Proposition 8.

If between two numbers there fall numbers in continued proportion with them, then, however many numbers fall between them in continued proportion, so many will, also, *fall in continued proportion between the numbers which have the same ratio with the original numbers.*



Let,

the numbers, C, D, fall between the two numbers, A, B in continued proportion with them,

and let,

E be made in the same ratio to F,
as A is to B;

I say that;

as many numbers as have fallen
between A, B in continued proportion,
so many will, also, fall
between E, F in continued proportion.

For,

as many as A, B C, D, are in multitude,

[VII. 33] let,

so many numbers, G, H, K, L, the least of those
which have the same ratio with A, C, D, B, be taken;

[VIII. 3] therefore,

the extremes of them, G, L, are prime to one another.

Now, since,

A, *C*, *D*, *B* ARE IN THE SAME RATIO WITH *G*, *H*, *K*, *L*, AND
THE MULTITUDE, OF THE NUMBERS, *A*, *C*, *D*, *B*, =
THE MULTITUDE, OF THE NUMBERS, *G*, *H*, *K*, *L*,

[VII. 14] THEREFORE, *EX AEQUALI*,
AS *A* IS TO *B*,
SO IS *G* TO *L*.

BUT,
AS *A* IS TO *B*,
SO IS *E* TO *F*;

THEREFORE ALSO,
AS *G* IS TO *L*,
SO IS *E* TO *F*,

[VII. 21] BUT,
G, *L* ARE PRIME, PRIMES ARE, ALSO, LEAST,

AND,
THE LEAST NUMBERS MEASURE THOSE WHICH HAVE
THE SAME RATIO THE SAME NUMBER OF TIMES,
THE GREATER THE GREATER AND,
THE LESS THE LESS,

[VII. 20] THAT IS,
THE ANTECEDENT THE ANTECEDENT AND,
THE CONSEQUENT THE CONSEQUENT.

THEREFORE,
G MEASURES *E*,
THE SAME NUMBER OF TIMES AS *L* MEASURES *F*.

NEXT LET,
AS MANY TIMES AS *G* MEASURES *E*,
SO MANY TIMES *H*, *K*, ALSO, MEASURE *M*, *N*, RESPECTIVELY;

THEREFORE,
G, *H*, *K*, *L* MEASURE *E*, *M*, *N*, *F*, THE SAME NUMBER OF TIMES.

[VII. DEF. 20] THEREFORE,
G, *H*, *K*, *L* ARE IN THE SAME RATIO WITH *E*, *M*, *N*, *F*.

BUT,
G, *H*, *K*, *L* ARE IN THE SAME RATIO WITH *A*, *C*, *D*, *B*;

THEREFORE,
A, *C*, *D*, *B* ARE, ALSO, IN THE SAME RATIO WITH *E*, *M*, *N*, *F*.

BUT,
A, *C*, *D*, *B* ARE IN CONTINUED PROPORTION;

THEREFORE,

E, M, N, F ARE, ALSO, IN CONTINUED PROPORTION.

THEREFORE,

AS MANY NUMBERS AS HAVE FALLEN BETWEEN A, B,
IN CONTINUED PROPORTION WITH THEM,
SO MANY NUMBERS HAVE, ALSO, FALLEN BETWEEN E, F
IN CONTINUED PROPORTION.

Q. E. D.

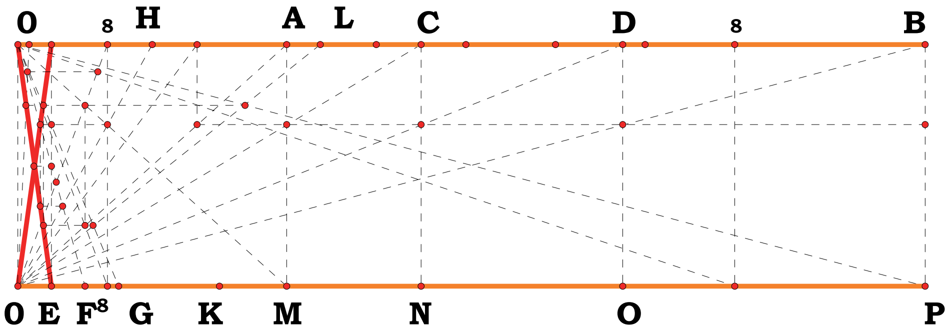
Proposition 9.

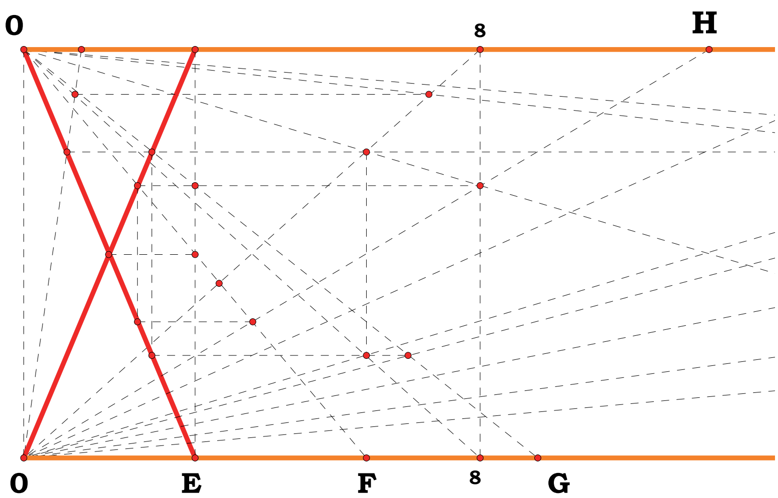
If two numbers be prime to one another, and numbers fall between them in continued proportion then however many numbers fall between them in continued proportion, so many will, also, *fall between each, of them and an unit in continued proportion.*

1 = 0.44450 cm
A = 8.00000
C = 12.00000
D = 18.00000
B = 27.00000
$\frac{1}{E} = 1.00000$

F = 2.00000
G = 3.00000
H = 4.00000
K = 6.00000
L = 9.00000

M = 8.00000
N = 12.00000
O = 18.00000
P = 27.00000





Let,

A, *B* be two numbers prime to one another,

and let,

C, *D* fall between them in continued proportion,

and let,

the unit *E* be set out;

I say that;

as many numbers as fall between *A*, *B*,
in continued proportion,
so many will, also, fall between

EITHER OF THE NUMBERS, A, B, AND
THE UNIT IN CONTINUED PROPORTION.

[VIII. 2] FOR LET,
TWO NUMBERS, F, G,
THE LEAST THAT ARE IN THE RATIO, OF A, C, D, B, BE TAKEN,
THREE NUMBERS, H, K, L, WITH THE SAME PROPERTY, AND
OTHERS MORE BY ONE CONTINUALLY, UNTIL
THEIR MULTITUDE EQUALS THE MULTITUDE OF A, C, D, B.

LET,
THEM BE TAKEN,

AND LET,
THEM BE M, N, O, P.

[VIII. 2, POR.] IT IS NOW MANIFEST THAT,
F, BY MULTIPLYING ITSELF, HAS MADE H, AND
BY MULTIPLYING H, HAS MADE M, WHILE
G, BY MULTIPLYING ITSELF, HAS MADE L, AND
BY MULTIPLYING L, HAS MADE P.

[VIII. 1] AND, SINCE,
M, N, O, P, ARE THE LEAST OF THOSE
WHICH HAVE THE SAME RATIO WITH F, G, AND
A, C, D, B ARE, ALSO, THE LEAST OF THOSE
WHICH HAVE THE SAME RATIO WITH F, G,

WHILE,
THE MULTITUDE OF THE NUMBERS, M, N, O, P, EQUALS
THE MULTITUDE OF THE NUMBERS, A, C, D, B,

THEREFORE,
M, N, O, P ARE EQUAL TO A, C, D, B, RESPECTIVELY;

THEREFORE,
$M = A$, AND
$P = B$.

NOW, SINCE,
F, BY MULTIPLYING ITSELF, HAS MADE H,

THEREFORE,
F MEASURES H, ACCORDING TO THE UNITS IN F.

BUT,
THE UNIT, E, ALSO, MEASURES F ACCORDING TO
THE UNITS IN IT;

THEREFORE,
THE UNIT, E, MEASURES THE NUMBER, F,
THE SAME NUMBER OF TIMES AS F MEASURES H.

[vii. Def. 20] Therefore,
as the unit, E, is to the number, F,
so is F to H.

Again, since,
F, by multiplying H, has made M,

therefore,
H measures M, according to the units in F.

But,
the unit, E, also, measures the number, F,
according to the units in it;

therefore,
the unit, E, measures the number, F,
the same number of times as H measures M.

Therefore,
as the unit, E, is to the number, F,
so is H to M.

But,
it was, also, proved that, as
the unit, E, is to the number, F,
so is F to H;

therefore also,
as the unit, E, is to the number, F,
so is F to H,
and H to M.

But,
$M = A$;

therefore,
as the unit, E, is to the number, F,
so is F to H,
and H to A.

For the same reason also,
as the unit, E, is to the number, G,
so is G to L,
and L to B.

Therefore,
as many numbers as have fallen
between A, B in continued proportion,
so many numbers, also, have fallen
between each, of the numbers, A, B, and
the unit, E, in continued proportion.

Q. E. D.

Proposition 10.

If numbers fall between each, of two numbers and an unit in continued proportion, however many numbers fall between each, of them and an unit in continued proportion, so many, also, *will fall between the numbers themselves in continued proportion.*

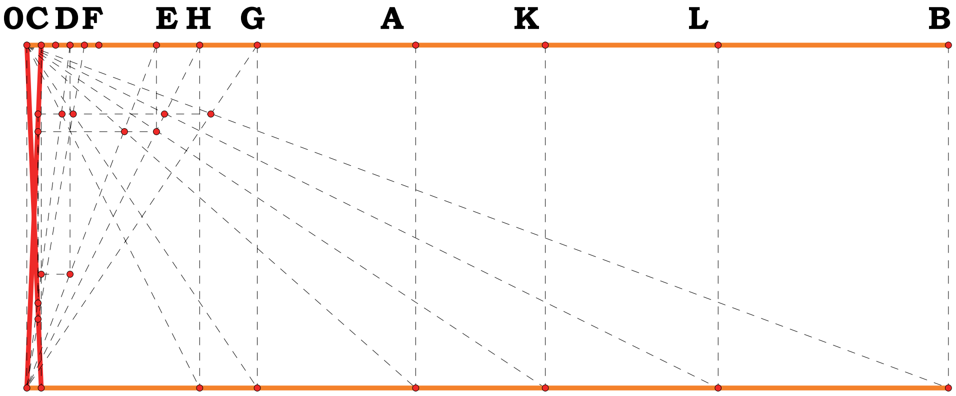
A = 27.00000
B = 64.00000
D = 3.00000
E = 9.00000
F = 4.00000

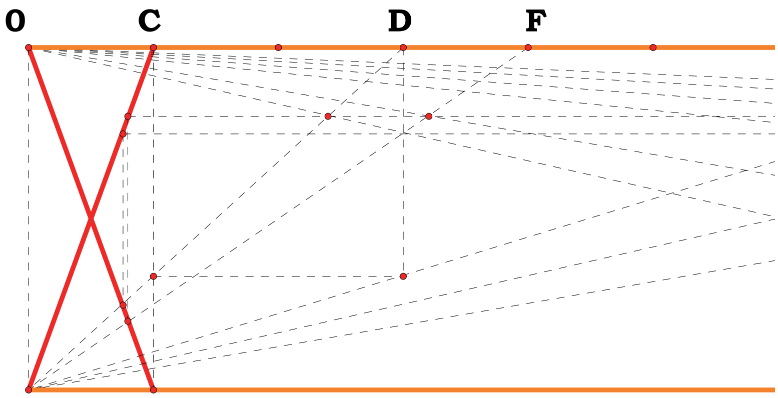
G = 16.00000
H = 12.00000
K = 36.00000
L = 48.00000

$\frac{0A}{0E} = 3.00000$

$\frac{0E}{0D} = 3.00000$

D^2-E = 0.00000
F^2-G = 0.00000
F·G = 64.00000
D·F-H = 0.00000
D·H-K = 0.00000
F·H-L = 0.00000





For let,

the numbers, *D*, *E*, and *F*, *G*, respectively, fall between
the two numbers, *A*, *B*, and,
the unit, *C*, in continued proportion;

I say that;

as many numbers as have fallen between each, of
the numbers, *A*, *B*, and
the unit, *C*, in continued proportion,
so many numbers will, also, fall between
A, *B* in continued proportion.

For let,

D, BY MULTIPLYING F, MAKE H,

AND LET,

THE NUMBERS, D, F, BY MULTIPLYING H,
MAKE K, L, RESPECTIVELY.

NOW, SINCE,

AS THE UNIT, C, IS TO THE NUMBER, D,
SO IS D TO E,

[VII. DEF. 20] THEREFORE,

THE UNIT, C, MEASURES THE NUMBER, D,
THE SAME NUMBER OF TIMES AS D MEASURES E.

BUT,

THE UNIT, C, MEASURES THE NUMBER, D,
ACCORDING TO THE UNITS IN D;

THEREFORE,

THE NUMBER, D, ALSO, MEASURES E,
ACCORDING TO THE UNITS IN D;

THEREFORE,

D, BY MULTIPLYING ITSELF, HAS MADE E.

AGAIN, SINCE,

AS C IS TO THE NUMBER D,
SO IS E TO A,

THEREFORE,

THE UNIT, C, MEASURES THE NUMBER, D,
THE SAME NUMBER OF TIMES AS E MEASURES A.

BUT,

THE UNIT, C, MEASURES THE NUMBER, D,
ACCORDING TO THE UNITS IN D;

THEREFORE,

E, ALSO, MEASURES A,
ACCORDING TO THE UNITS IN D;

THEREFORE,

D, BY MULTIPLYING, HAS MADE A.

FOR THE SAME REASON ALSO,

F, BY MULTIPLYING ITSELF, HAS MADE G, AND
BY MULTIPLYING G, HAS MADE B.

AND, SINCE,

D, BY MULTIPLYING ITSELF, HAS MADE E, AND
BY MULTIPLYING F, HAS MADE H,

[VII. 17] THEREFORE,

as D is to F,
so is E to H.

[VII. 18] For the same reason also,
as D is to F,
so is H to G.

Therefore also,
as E is to H,
so is H to G.

Again, since,
D, by multiplying the numbers, E, H,
has made A, K, respectively,

[VII. 17] therefore,
as E is to H,
so is A to K.

But,
as E is to H,
so is D to F;

therefore also,
as D is to F,
so is A to K.

Again, since,
the numbers, D, F, by multiplying H,
have made K, L, respectively,

[VII. 18] therefore,
as D is to F,
so is K to L.

But,
as D is to F,
so is A to K;

therefore also,
as A is to K,
so is K to L.

Further, since,
F, by multiplying the numbers, H, G,
has made L, B, respectively,

[VII. 17] therefore,
as H is to G,
so is L to B.

But,
as H is to G,

so is D to F;

therefore also,
as D is to F,
so is L to B.

But it was, also, proved that,
as D is to F,
so is A to K,
and K to L,

therefore also,
as A is to K,
so is K to L,
and L to B.

Therefore,
A, K, L, B are in continued proportion.

Therefore,
as many numbers as fall between each, of
the numbers, A, B, and,
the unit, C, in continued proportion,
so many, also, will fall between
A, B in continued proportion.

Q. E. D.

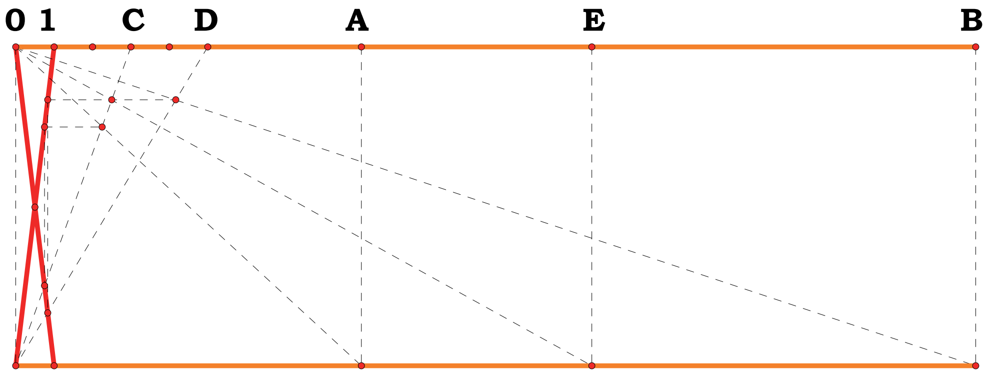
Proposition 11.

Between two square numbers there is one mean proportional number, and the square has to the square the ratio duplicate of that which the side has to the side.

A = 9.00000
B = 25.00000
C = 3.00000
D = 5.00000
E = 15.00000

$\frac{C}{D} = 0.60000$ $\frac{E}{B} = 0.60000$ $\frac{A^2}{E} = 0.36000$

$\frac{A}{E} = 0.60000$ $\frac{A}{B} = 0.36000$



Let,

A, *B* be square numbers,

and let,

C be the side of *A*, and,
D of *B*;

I say that; between

A, *B*, there is one mean proportional number, and
A has to *B* the ratio duplicate of
that which *C* has to *D*.

For let,

C, by multiplying *D*, make *E*.

Now, since,

A is a square and *C* is its side,

therefore,

C, by multiplying itself, has made *A*.

For the same reason also,

D, by multiplying itself, has made *B*.

Since then,

C, by multiplying the numbers
C, *D*, has made *A*, *E*, respectively,

[vii. 17] therefore,

as *C* is to *D*,
so is *A* to *E*.

[VII. 18] FOR THE SAME REASON ALSO,
AS C IS TO D,
SO IS E TO B.

THEREFORE ALSO,
AS A IS TO E,
SO IS E TO B.

THEREFORE,
BETWEEN A, B, THERE IS ONE MEAN PROPORTIONAL NUMBER.

I SAY NEXT THAT;
A, ALSO, HAS TO B, THE RATIO DUPLICATE OF
THAT WHICH C HAS TO D.

FOR, SINCE,
A, E, B ARE THREE NUMBERS IN PROPORTION,

[V. DEF. 9] THEREFORE,
A HAS TO B,
THE RATIO DUPLICATE OF THAT WHICH A HAS TO E.

BUT,
AS A IS TO E,
SO IS C TO D.

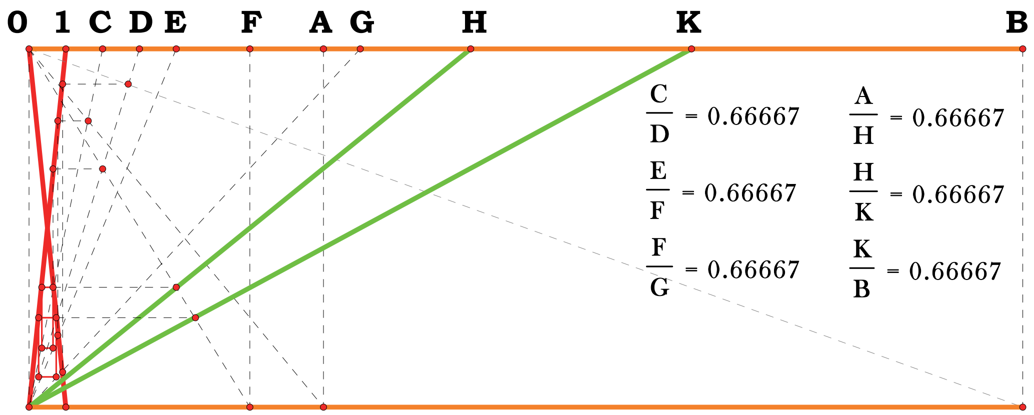
THEREFORE,
A HAS TO B THE RATIO DUPLICATE OF
THAT WHICH THE SIDE C HAS TO D.

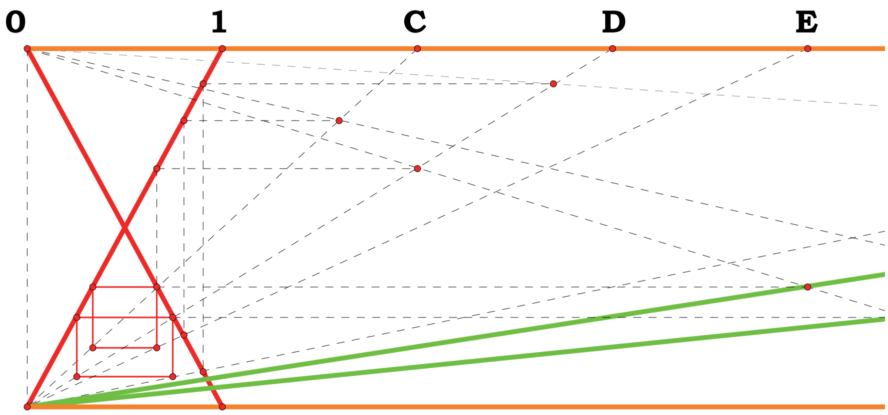
Q. E. D.

Proposition 12.

Between two cube numbers there are two mean proportional numbers, and the cube has to the cube the ratio triplicate of that which the side has to the side.

A = 8.00000	E = 4.00000	K = 18.00000	C·D-F = 0.00000
B = 27.00000	F = 6.00000	C^3-A = 0.00000	D^2-G = 0.00000
C = 2.00000	G = 9.00000	D^3-B = 0.00000	C·F-H = 0.00000
D = 3.00000	H = 12.00000	C^2-E = 0.00000	D·F-K = 0.00000





Let,

A, *B* be cube numbers,

and let,

C be the side of *A*, and,

D of *B*;

I say that;

between *A*, *B*,

there are two mean proportional numbers, and

A has to *B*,

the ratio triplicate of that which *C* has to *D*.

For let,

C, by multiplying itself, make *E*,

and let,

by multiplying *D*, it make *F*;

LET,
  D, BY MULTIPLYING ITSELF, MAKE G,

AND LET,
  THE NUMBERS, C, D, BY MULTIPLYING F, MAKE H, K, RESPECTIVELY.

NOW, SINCE,
  A IS A CUBE, AND,
  C ITS SIDE, AND,
  C, BY MULTIPLYING ITSELF, HAS MADE E,

THEREFORE,
  C, BY MULTIPLYING ITSELF, HAS MADE E, AND
  BY MULTIPLYING E, HAS MADE A.

FOR THE SAME REASON ALSO,
  D, BY MULTIPLYING ITSELF, HAS MADE G, AND
  BY MULTIPLYING G, HAS MADE B.

AND, SINCE,
  C, BY MULTIPLYING
  THE NUMBERS, C, D, HAS MADE E, F, RESPECTIVELY,

[VII. 17] THEREFORE,
  AS C IS TO D,
  SO IS E TO F.

[VII. 18] FOR THE SAME REASON ALSO,
  AS C IS TO D,
  SO IS F TO G.

AGAIN, SINCE,
  C, BY MULTIPLYING
  THE NUMBERS, E, F, HAS MADE A, H, RESPECTIVELY,

[VII. 17] THEREFORE,
  AS E IS TO F,
  SO IS A TO H.

BUT,
  AS E IS TO F,
  SO IS C TO D.

THEREFORE ALSO,
  AS C IS TO D,
  SO IS A TO H.

AGAIN, SINCE,
  THE NUMBERS, C, D, BY MULTIPLYING F, HAVE MADE H, K, RESPECTIVELY,

[VII. 18] THEREFORE,

as C is to D,
so is H to K.

Again, since,
D, by multiplying each, of the numbers, F, G,
has made K, B, respectively,

[vii. 17] therefore,
as F is to G,
so is K to B.

But,
as F is to G,
so is C to D;

therefore also,
as C is to D,
so is A to H,
H to K, and
K to B.

Therefore,
H, K are two mean proportionals between A, B.

I say next that;
A, also, has to B, the ratio triplicate
of that which C has to D.

For, since,
A, H, K, B are four numbers in proportion,

[v. Def. 10] therefore,
A has to B, the ratio triplicate of
that which A has to H.

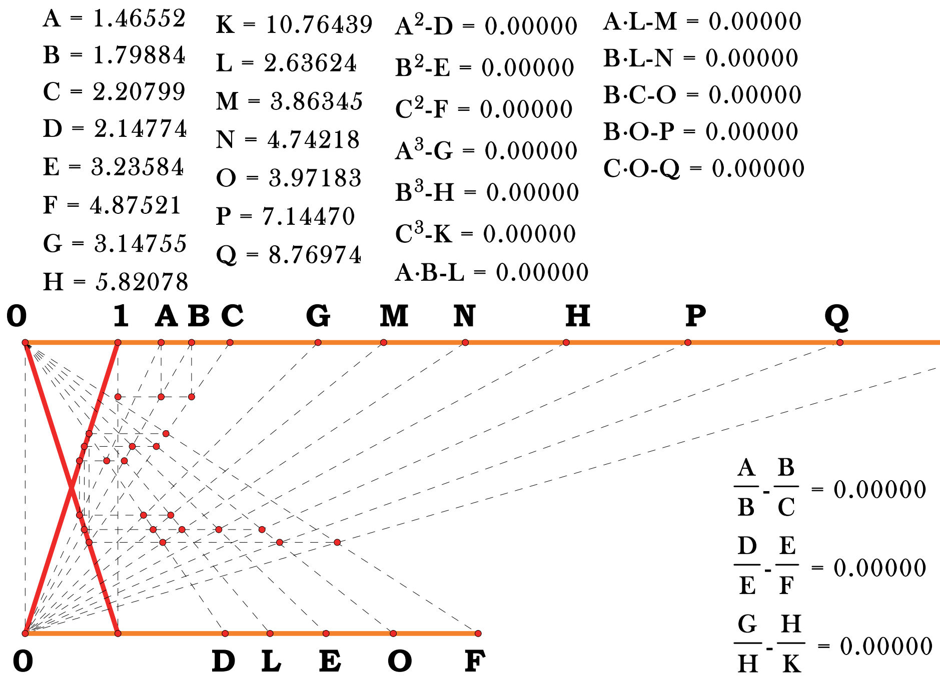
But,
as A is to H,
so is C to D;

therefore,
A, also, has to B, the ratio triplicate of that
which C has to D.

Q. E. D.

PROPOSITION 13.

IF THERE BE AS MANY NUMBERS AS WE PLEASE IN CONTINUED PROPORTION, AND EACH, BY MULTIPLYING ITSELF, MAKE SOME NUMBER, THE PRODUCTS WILL BE PROPORTIONAL; AND, IF THE ORIGINAL NUMBERS, BY MULTIPLYING THE PRODUCTS, MAKE CERTAIN NUMBERS, THE LATTER WILL, ALSO, *BE PROPORTIONAL.*



LET,

THERE BE AS MANY NUMBERS AS WE PLEASE,

A, *B*, *C*, IN CONTINUED PROPORTION,

SO THAT,

AS *A* IS TO *B*,

SO IS *B* TO *C*;

LET,

A, *B*, *C*, BY MULTIPLYING THEMSELVES, MAKE *D*, *E*, *F*,

AND LET,

BY MULTIPLYING *D*, *E*, *F*, THEM MAKE *G*, *H*, *K*;

I SAY THAT;

D, *E*, *F* AND *G*, *H*, *K* ARE IN CONTINUED PROPORTION.

FOR LET,

A, BY MULTIPLYING *B*, MAKE *L*,

AND LET,

THE NUMBERS, *A*, *B*, BY MULTIPLYING *L*, MAKE *M*, *N*, RESPECTIVELY.

AND AGAIN LET,

B, BY MULTIPLYING *C*, MAKE *O*,

AND LET,

THE NUMBERS, B, C, BY MULTIPLYING O, MAKE P, Q, RESPECTIVELY.

THEN,

IN MANNER SIMILAR TO THE FOREGOING,

WE CAN PROVE THAT,

D, L, E AND G, M, N, H ARE
CONTINUOUSLY PROPORTIONAL IN THE RATIO, OF A TO B,

AND FURTHER,

E, O, F AND H, P, Q, K ARE
CONTINUOUSLY PROPORTIONAL IN THE RATIO, OF B TO C.

NOW,

AS A IS TO B,
SO IS B TO C;

THEREFORE,

D, L, E ARE, ALSO, IN THE SAME RATIO WITH E, O, F,

AND FURTHER,

G, M, N, H, IN THE SAME RATIO WITH H, P, Q, K.

AND,

THE MULTITUDE, OF D, L, E, EQUALS
THE MULTITUDE, OF E, O, F, AND
THAT, OF G, M, N, H TO THAT, OF H, P, Q, K

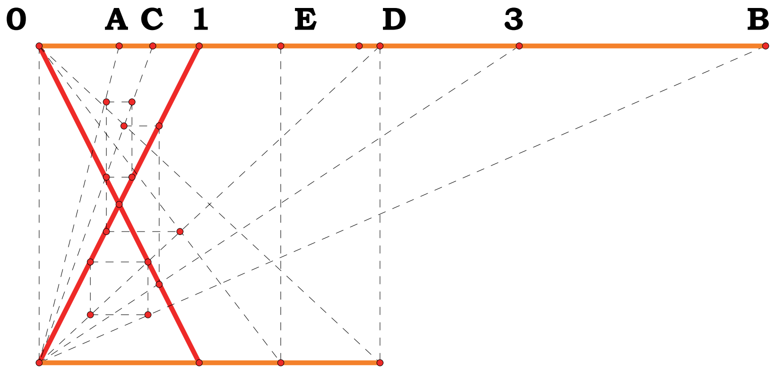
[VII. 14] THEREFORE, *EX AEQUALI*,

AS D IS TO E,
SO IS E TO F, AND,
AS IS G TO H,
SO IS H TO K.

PROPOSITION 14.

IF A SQUARE MEASURE A SQUARE, THE SIDE WILL, ALSO, *MEASURE THE SIDE; AND, IF THE SIDE MEASURE THE SIDE, THE SQUARE WILL,* ALSO, *MEASURE THE SQUARE.*

01 = 2.11667 cm	A = 0.50410	C^2-A = 0.00000	$\frac{A}{E}$ = 0.33333
0C = 1.50283 cm	B = 4.53690	$\frac{D}{C}$ = 3.00000	$\frac{C}{D}$ = 0.33333
0A = 1.06701 cm	C = 0.71000	D^2-B = 0.00000	$\frac{E}{B}$ = 0.33333
0D = 4.50850 cm	D = 2.13000	C·D-E = 0.00000	
0B = 9.60310 cm	E = 1.51230		
0E = 3.20103 cm			



LET,

A, *B* BE SQUARE NUMBERS,

LET,

C, *D* BE THEIR SIDES,

AND LET,

A MEASURE *B*;

I SAY THAT;

C, ALSO, MEASURES *D*.

FOR LET,

C, BY MULTIPLYING *D*, MAKE *E*;

[VIII. 11] THEREFORE,

A, *E*, *B*, ARE CONTINUOUSLY PROPORTIONAL IN THE RATIO, OF *C* TO *D*.

AND, SINCE,

A, *E*, *B*, ARE CONTINUOUSLY PROPORTIONAL, AND
A MEASURES *B*,

[VIII. 7] THEREFORE,

A, ALSO, MEASURES *E*.

AND,

AS *A* IS TO *E*,
SO IS *C* TO *D*;

[VII. DEF. 20] THEREFORE ALSO,
C MEASURES *D*.

AGAIN, LET,
C MEASURE *D*;

I SAY THAT;
A, ALSO, MEASURES *B*.

FOR,

WITH THE SAME CONSTRUCTION,

WE CAN IN A SIMILAR MANNER PROVE THAT;
A, *E*, *B* ARE CONTINUOUSLY PROPORTIONAL
IN THE RATIO, OF *C* TO *D*.

AND SINCE,
AS *C* IS TO *D*,
SO IS *A* TO *E*, AND
C MEASURES *D*,

[VII. DEF. 20] THEREFORE,
A, ALSO, MEASURES *E*.

AND,
A, *E*, *B* ARE CONTINUOUSLY PROPORTIONAL;

THEREFORE,
A, ALSO, MEASURES *B*.

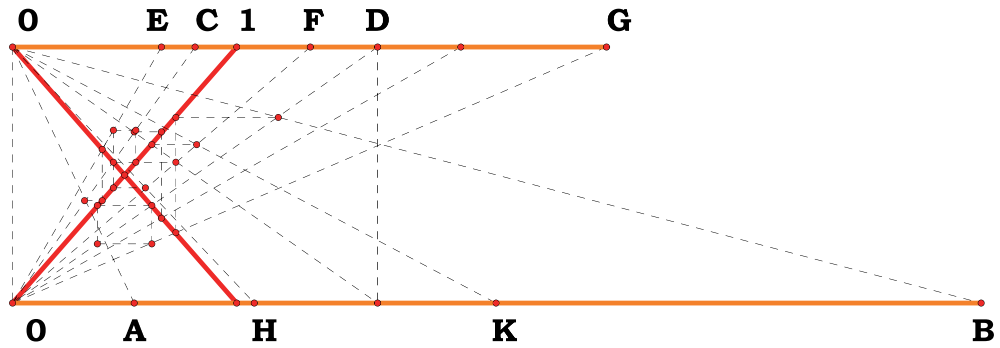
THEREFORE ETC.

Q. E. D.

Proposition 15.

If a cube number measure a cube number, the side will, also, *measure the side; and, if the side measure the side, the cube will,* also, *measure the cube.*

$A = 0.53992$	$F = 1.32612$	$\frac{B}{A} = 8.00000$	$D^2\text{-}G = 0.00000$
$B = 4.31937$	$G = 2.65224$		$C{\cdot}F\text{-}H = 0.00000$
$C = 0.81429$	$H = 1.07984$	$C^3\text{-}A = 0.00000$	$D{\cdot}F\text{-}K = 0.00000$
$D = 1.62857$	$K = 2.15969$	$D^3\text{-}B = 0.00000$	$\frac{A}{H}\text{-}\frac{C}{D} = 0.00000$
$E = 0.66306$		$C^2\text{-}E = 0.00000$	



For let,

the cube number, *A*, measure the cube, *B*,

and let,

C be the side, of *A*, and
D of *B*;

I say that;

C measures *D*.

For let,

C, by multiplying itself, make *E*,

and let,

D, by multiplying itself, make *G*;

further, let,

C, by multiplying *D*, make *F*,

and let,

C, *D*, by multiplying *F*, make *H*, *K*, respectively.

[viii. *N*, 12] Now,

it is manifest that *E*, *F*, *G*, and
A, *H*, *K*, *B*, are continuously proportional in the ratio, of *C* to *D*.

And, since,

A, *H*, *K*, *B*, are continuously proportional, and
A measures *B*,

[viii. 7] therefore,

it, also, measures H.

And,

as A is to H,
so is C to D;

[vii. Def. 20] therefore,
C, also, measures D.

Next, let,
C measure D;

I say that;
A will, also, measure B.

For,

with the same construction,

we can prove in a similar manner that;
A, H, K, B, are continuously proportional in the ratio, of C to D.

And, since,
C measures D, and
as C is to D,
so is A to H,

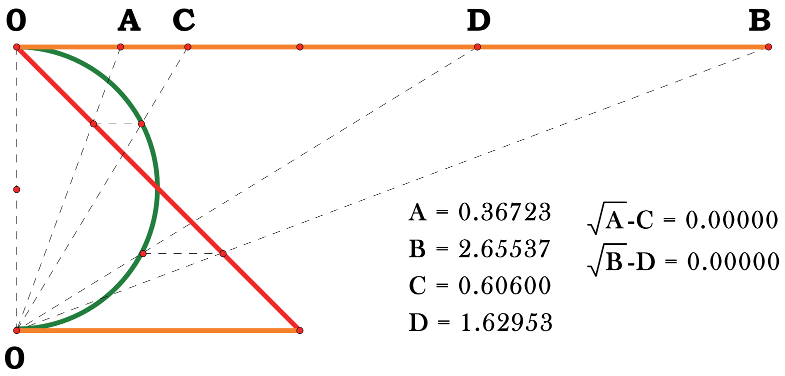
[vii. Def. 20] therefore,
A, also, measures H,

so that,
A measures B, also.

Q. E. D.

PROPOSITION 16.

IF A SQUARE NUMBER DO NOT MEASURE A SQUARE NUMBER, NEITHER WILL THE SIDE MEASURE THE SIDE; AND, IF THE SIDE DO NOT MEASURE THE SIDE, NEITHER WILL THE SQUARE MEASURE THE SQUARE.



LET,

A, *B* BE SQUARE NUMBERS, AND LET,

C, *D* BE THEIR SIDES;

AND LET,

A NOT MEASURE *B*;

I SAY THAT;

NEITHER DOES *C* MEASURE *D*.

[VIII. 14] FOR,

IF *C* MEASURES *D*,

A WILL, ALSO, MEASURE *B*.

BUT,

A DOES NOT MEASURE *B*;

THEREFORE,

NEITHER WILL *C* MEASURE *D*.

AGAIN, LET,

C NOT MEASURE *D*;

I SAY THAT;

NEITHER WILL *A* MEASURE *B*.

[VIII. 14] FOR,

IF *A* MEASURES *B*,

C WILL, ALSO, MEASURE *D*.

BUT,

C DOES NOT MEASURE *D*;

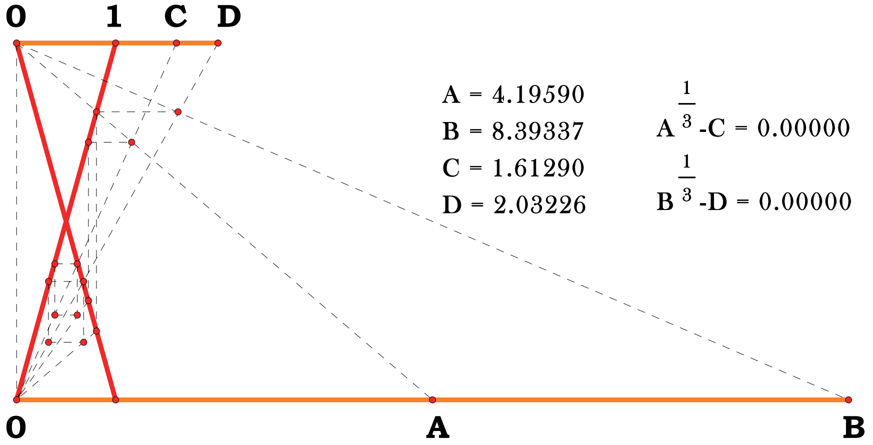
THEREFORE,

NEITHER WILL *A* MEASURE *B*.

Q. E. D.

Proposition 17.

If a cube number do not measure a cube number, neither will the side measure the side; and, if the side do not measure the side, neither will the cube measure the cube.



For let,

the cube number, A, not measure the cube number, B,

and let,

C be the side, of A, and
D of B;

I say that;

C will not measure D.

[viii. 15] For,

if C measures D,
A will, also, measure B.

But,

A does not measure B;

therefore,

neither does C measure D.

Again, let,

C not measure D;

I say that;

neither will A measure B.

[viii. 15] For,

if A measures B,
C will, also, measure D.

But,

C does not measure D; therefore,
neither will A measure B.

Q. E. D.

PROPOSITION 18.

BETWEEN TWO SIMILAR PLANE NUMBERS THERE IS ONE MEAN PROPORTIONAL NUMBER; AND THE PLANE NUMBER HAS TO THE PLANE NUMBER THE RATIO DUPLICATE OF THAT WHICH THE CORRESPONDING SIDE HAS TO THE CORRESPONDING SIDE.

A = 3.83040
B = 5.88962
C = 1.71000
D = 2.24000
E = 2.12040
F = 2.77760
G = 4.74970

C·D-A = 0.00000
E·F-B = 0.00000

$\frac{C}{D} - \frac{E}{F} = 0.00000$

$\frac{A}{B} - \frac{C}{E}^2 = 0.00000$

D·E-G = 0.00000

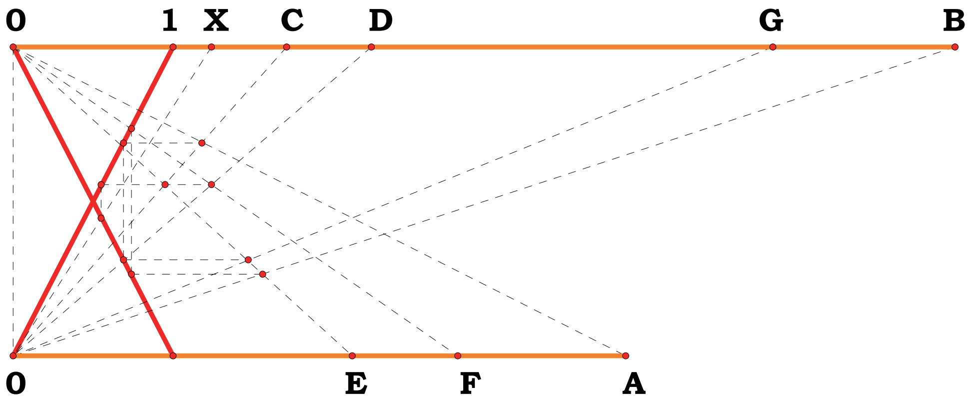
$\frac{C}{E} - \frac{A}{G} = 0.00000$

$\frac{C}{E} - \frac{D}{F} = 0.00000$

$\frac{D}{F} - \frac{G}{B} = 0.00000$

$\frac{D}{F} - \frac{A}{G} = 0.00000$

$\frac{A}{G} - \frac{G}{B} = 0.00000$



LET,

A, *B* BE TWO SIMILAR PLANE NUMBERS,

AND LET,

THE NUMBERS, *C*, *D*, BE THE SIDES, OF *A*, AND
E, *F* OF *B*.

[VII. DEF. 21] NOW, SINCE,

SIMILAR PLANE NUMBERS ARE THOSE
WHICH HAVE THEIR SIDES PROPORTIONAL,

THEREFORE,

AS *C* IS TO *D*
SO IS *E* TO *F*.

I SAY THEN THAT;

BETWEEN *A*, *B*,
THERE IS ONE MEAN PROPORTIONAL NUMBER, AND
A HAS TO *B* THE RATIO DUPLICATE OF THAT WHICH
C HAS TO *E*, OR
D TO *F*,

THAT IS,

OF THAT WHICH THE CORRESPONDING SIDE HAS TO
THE CORRESPONDING SIDE.

NOW SINCE,
as C is to D
so is E to F,

[VII. 13] therefore, alternately,
as C is to E,
so is D to F.

AND, SINCE,
A is plane, and
C, D are its sides,

therefore,
D, by multiplying C, has made A.

FOR THE SAME REASON ALSO,
E, by multiplying F, has made B.

NOW LET,
D, by multiplying E, make G.

THEN, SINCE,
D, by multiplying C, has made A, and,
by multiplying E, has made G,

[VII. 17] therefore,
as C is to E,
so is A to G.

BUT,
as C is to E,
so is D to F;

therefore also,
as D is to F,
so is A to G.

AGAIN, SINCE,
E, by multiplying D, has made G, and,
by multiplying F, has made B,

[VII. 17] therefore,
as D is to F,
so is G to B.

BUT IT WAS, ALSO, PROVED THAT,
as D is to F,
so is A to G;

therefore also,
as A is to G,
so is G to B.

THEREFORE,

A, *G*, *B*, ARE IN CONTINUED PROPORTION.

THEREFORE,

BETWEEN *A*, *B*, THERE IS ONE MEAN PROPORTIONAL NUMBER.

I SAY NEXT THAT;

A, ALSO, HAS TO *B*, THE RATIO DUPLICATE OF THAT
WHICH THE CORRESPONDING SIDE HAS
TO THE CORRESPONDING SIDE,

THAT IS,

OF THAT WHICH *C* HAS TO *E*, OR
D TO *F*.

[V. DEF. 9] FOR, SINCE,

A, *G*, *B*, ARE IN CONTINUED PROPORTION,
A HAS TO *B*,
THE RATIO DUPLICATE OF THAT WHICH IT HAS TO *G*.

AND,

AS *A* IS TO *G*,
SO IS *C* TO *E*, AND
SO IS *D* TO *F*.

THEREFORE,

A, ALSO, HAS TO *B*, THE RATIO DUPLICATE OF THAT WHICH
C HAS TO *E*, OR
D TO *F*.

Q. E. D.

Proposition 19.

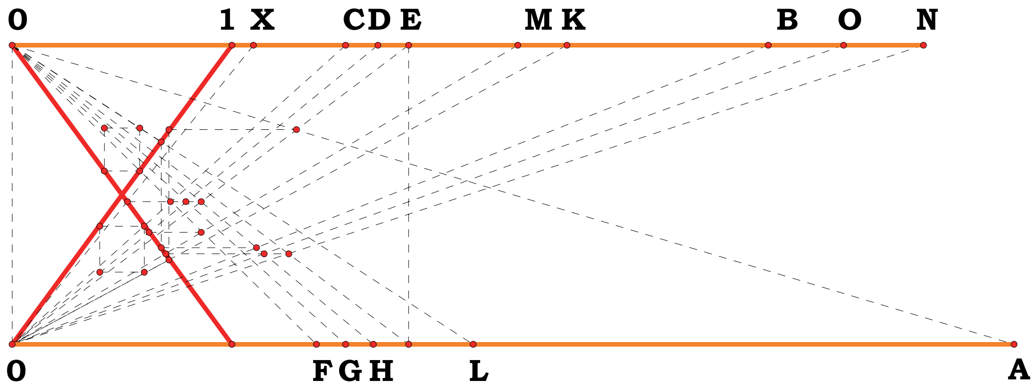
Between two similar solid numbers there fall two mean proportional numbers; and the solid number has to the similar solid number the ratio triplicate of that which the corresponding side has to the corresponding side.

A = 4.55668
B = 3.44316
C = 1.51748
D = 1.66434
E = 1.80420
F = 1.38217
G = 1.51592

H = 1.64331
K = 2.52560
L = 2.09526
M = 2.30039
N = 4.15035
O = 3.78025

C·D·E-A = 0.00000
F·G·H-B = 0.00000
C·D-K = 0.00000
F·G-L = 0.00000
D·F-M = 0.00000

E·M-N = 0.00000
H·M-O = 0.00000
$\frac{A}{B}$ = 1.32340



$\frac{C}{D}$ = 0.91176 $\frac{G}{H}$ = 0.92248 $\frac{C}{F}$ = 1.09790 $\frac{E}{H}$ = 1.09790 $\frac{M}{L}$ = 1.09790 $\frac{N}{O}$ = 1.09790

$\frac{F}{G}$ = 0.91176 $\frac{D}{E}$ = 0.92248 $\frac{D}{G}$ = 1.09790 $\frac{K}{M}$ = 1.09790 $\frac{A}{N}$ = 1.09790 $\frac{O}{B}$ = 1.09790

Let,

A, *B*, be two similar solid numbers,

and let,

C, *D*, *E*, be the sides, of *A*, and
F, *G*, *H* of *B*.

[vii. Def. 21] Now, since,

similar solid numbers are those
which have their sides proportional,

therefore,

as *C* is to *D*,
so is *F* to *G*, and
as *D* is to *E*,
so is *G* to *H*.

I say that;

between *A*, *B*,
there fall two mean proportional numbers, and
A has to *B* the ratio triplicate of that which
C has to *F*,

D TO G, AND, ALSO,
E TO H.

FOR LET,
C, BY MULTIPLYING D, MAKE K,

AND LET,
F, BY MULTIPLYING G, MAKE L.

[VII. DEF. 21] NOW, SINCE,
C, D ARE IN THE SAME RATIO WITH F, G, AND
K IS THE PRODUCT, OF C, D, AND
L THE PRODUCT, OF F, G,

THEN,
K, L ARE SIMILAR PLANE NUMBERS;

[VIII. 18] THEREFORE,
BETWEEN K, L THERE IS ONE MEAN PROPORTIONAL NUMBER.

LET,
IT BE M.

[VIII. 18] THEREFORE,
M IS THE PRODUCT, OF D, F,
AS WAS PROVED IN THE THEOREM PRECEDING THIS.

NOW, SINCE,
D, BY MULTIPLYING C, HAS MADE K, AND
BY MULTIPLYING F, HAS MADE M,

[VII. 17] THEREFORE,
AS C IS TO F,
SO IS K TO M.

BUT,
AS K IS TO M,
SO IS M TO L.

THEREFORE,
K, M, L ARE CONTINUOUSLY PROPORTIONAL IN
THE RATIO, OF C TO F.

AND SINCE,
AS C IS TO D,
SO IS F TO G,

[VII. 13] ALTERNATELY THEREFORE,
AS C IS TO F,
SO IS D TO G.

FOR THE SAME REASON ALSO,
AS D IS TO G,

so is E to H.

Therefore,
K, M, L are continuously proportional
in the ratio, of C to F,
in the ratio, of D to G, and, also,
in the ratio, of E to H.

Next, let,
E, H, by multiplying M, make N, O, respectively.

Now, since,
A is a solid number, and C, D, E are its sides,

therefore,
E, by multiplying the product, of C, D, has made A.

But,
the product, of C, D, is K;

therefore,
E, by multiplying K, has made A.

For the same reason also,
H, by multiplying L, has made B.

Now, since,
E, by multiplying K, has made A,

and further also,
by multiplying M, has made N,

[vii. 17] therefore,
as K is to M,
so is A to N.

But,
as K is to M,
so is C to F,
D to G, and, also,
E to H;

therefore also,
as C is to F,
D to G and
E to H so is
A to N.

Again, since,
E, H, by multiplying M, have made N, O, respectively,

[vii. 18] therefore,
as E is to H,

so is N to O.

But,

as E is to H,
so is C to F, and
D to G;

therefore also,

as C is to F,
D to G, and
E to H,
so is A to N, and
N to O.

Again, since,

H, by multiplying M, has made O,

and further also,

by multiplying L, has made B,

[vii. 17] therefore,

as M is to L,
so is O to B.

But,

as M is to L,
so is C to F,
D to G, and
E to H.

Therefore also,

as C is to F,
D to G, and
E to H,

so,

not only is O to B,

but also,

A to N, and,
N to O.

Therefore,

A, N, O, B, are continuously proportional in
the aforesaid ratios of the sides.

I say that;

A, also, has to B, the ratio triplicate of that which
the corresponding side has to the corresponding side,

that is,

of the ratio which the number, C, has to F, or

D to *G*, and also,
E to *H*.

For, since,
A, *N*, *O*, *B*, are four numbers in continued proportion,

[v. Def. 10] therefore,
A has to *B*, the ratio triplicate of
that which *A* has to *N*.

But,
as *A* is to *N*, so it was proved that
C is to *F*,
D to *G*, and also,
E to *H*.

Therefore,
A, also, has to *B*, the ratio triplicate of that which
the corresponding side has to the corresponding side,

that is,
of the ratio which the number, *C*, has to *F*,
D to *G*, and also,
E to *H*.

Q. E. D.

PROPOSITION 20.

IF ONE MEAN PROPORTIONAL NUMBER FALL BETWEEN TWO NUMBERS, THE NUMBERS WILL BE SIMILAR PLANE NUMBERS.

$A = 1.54167$
$B = 2.82292$
$C = 2.08614$
$D = 1.35317$
$E = 1.83108$
$F = 1.13930$
$G = 1.54167$

$\frac{A}{D} - \frac{C}{E} = 0.00000$

$\frac{A}{D} - F = 0.00000$

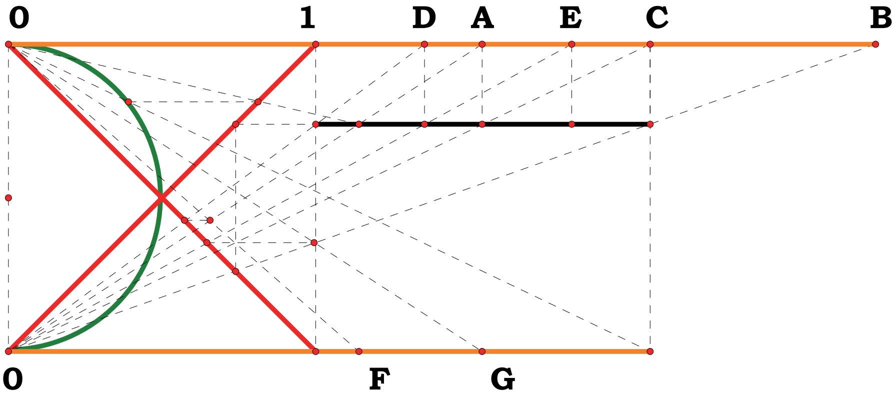
$D \cdot F - A = 0.00000$

$\sqrt{A \cdot B} - C = 0.00000$

$\frac{C}{D} - \frac{B}{E} = 0.00000$

$E \cdot G - B = 0.00000$

$\frac{D}{E} - \frac{C}{B} = 0.00000$



FOR LET,

ONE MEAN PROPORTIONAL NUMBER, *C*, FALL BETWEEN THE TWO NUMBERS, *A*, *B*;

I SAY THAT;

A, *B* ARE SIMILAR PLANE NUMBERS.

[VII. 33] LET,

D, *E*, THE LEAST NUMBERS OF THOSE WHICH HAVE THE SAME RATIO WITH *A*, *C*, BE TAKEN;

[VII. 20] THEREFORE,

D MEASURES *A*, THE SAME NUMBER OF TIMES THAT *E* MEASURES *C*.

NOW LET,

AS MANY TIMES AS *D* MEASURES *A*,
SO MANY UNITS THERE BE IN *F*; THEREFORE,
F, BY MULTIPLYING *D*, HAS MADE *A*,
SO THAT *A* IS PLANE, AND
D, *F* ARE ITS SIDES.

AGAIN, SINCE,

D, *E* ARE THE LEAST OF THE NUMBERS WHICH HAVE THE SAME RATIO WITH *C*, *B*,

[VII. 20] THEREFORE,

D MEASURES *C*.
THE SAME NUMBER OF TIMES THAT *E* MEASURES *B*.

AS MANY TIMES, THEN LET,
as E measures B,
so many units there be in G;

therefore,
E measures B, according to the units in G;

therefore,
G, by multiplying E, has made B.

Therefore,
B is plane, and E, G are its sides.

Therefore,
A, B are plane numbers.

I say next that;
they are, also, similar.

For, since,
F, by multiplying D, has made A, and
by multiplying E, has made C,

[vii. 17] therefore,
as D is to E,
so is A to C, that is,
C to B.

Again, since,
E, by multiplying F, G, has made C, B, respectively,

[vii. 17] therefore,
as E is to G,
so is C to B.

But,
as C is to B,
so is D to E;

therefore also,
as D is to E,
so is F to G.

[vii. 13] And alternately,
as D is to F,
so E is to G.

Therefore,
A, B are similar plane numbers;
for their sides are proportional.

Q. E. D.

Proposition 21.

If two mean proportional numbers fall between two numbers, the numbers are similar solid numbers.

L = 2.02516
M = 0.72335
K = 0.39002
H = 2.91195
N = 1.62810
0 = 1.84906

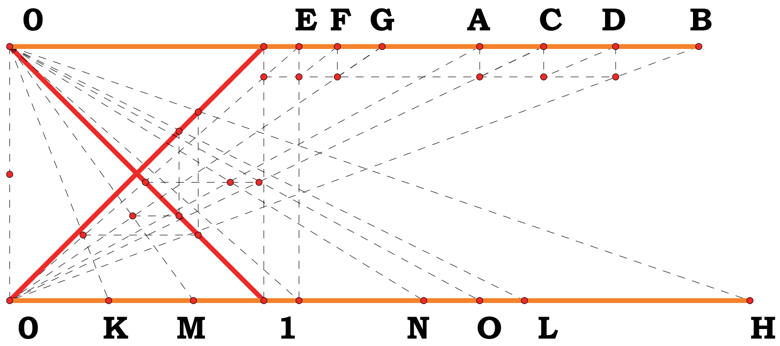
$\frac{H}{L} = 1.43789$

$\frac{K}{M} = 0.53918$

$\sqrt{\frac{H}{L} \cdot \frac{K}{M}} = 0.88050$

E = 1.13571
F = 1.28985
G = 1.46490

A = 1.84906
C = 2.10000
D = 2.38500
B = 2.70868
N·E-A = 0.00000



$\frac{E}{F} = 0.88050$

$\frac{F}{G} = 0.88050$

$\frac{E}{G} = 0.77529$

$\frac{A}{D} = 0.77529$

$\frac{A}{E} - \frac{D}{G} = 0.00000$

$\frac{C}{E} - \frac{B}{G} = 0.00000$

$\frac{B}{G} - 0 = 0.00000$

For let,

two mean proportional numbers, *C*, *D*, fall
between the two numbers, *A*, *B*;

I say that;

A, *B* are similar solid numbers.

[VII. 33 or VIII. 2] For let,

three numbers, *E*, *F*, *G*, the least of those
which have the same ratio with *A*, *C*, *D*, be taken;

[VIII. 3] Therefore,

the extremes of them, *E*, *G*, are prime to one another.

Now, since,

one mean proportional number,
F, has fallen between *E*, *G*,

[VIII. 20] Therefore,

E, *G* are similar plane numbers.

Let, then,

H, *K* be the sides of *E*, and
L, *M* of *G*.

THEREFORE, IT IS MANIFEST FROM THE THEOREM BEFORE THIS
THAT E, F, G ARE CONTINUOUSLY PROPORTIONAL IN
THE RATIO, OF H TO L, AND
THAT OF K TO M.

NOW, SINCE,
E, F, G ARE THE LEAST OF THE NUMBERS
WHICH HAVE THE SAME RATIO WITH A, C, D, AND
THE MULTITUDE OF THE NUMBERS, E, F, G, =
THE MULTITUDE OF THE NUMBERS, A, C, D,

[VII. 14] THEREFORE, *EX AEQUALI*,
AS E IS TO G,
SO IS A TO D.

[VII. 21] BUT,
E, G ARE PRIME, PRIMES ARE, ALSO, LEAST, AND
THE LEAST MEASURE THOSE WHICH HAVE
THE SAME RATIO WITH THEM THE SAME NUMBER OF TIMES,
THE GREATER THE GREATER AND,
THE LESS THE LESS,

[VII. 20] THAT IS,
THE ANTECEDENT THE ANTECEDENT AND
THE CONSEQUENT THE CONSEQUENT;

THEREFORE,
E MEASURES A
THE SAME NUMBER OF TIMES THAT G MEASURES D.

NOW LET,
AS MANY TIMES AS E MEASURES A,
SO MANY UNITS THERE BE IN N.

THEREFORE,
N, BY MULTIPLYING E, HAS MADE A.

BUT,
E IS THE PRODUCT OF H, K;

THEREFORE,
N, BY MULTIPLYING THE PRODUCT OF H, K, HAS MADE A.

THEREFORE,
A IS SOLID, AND
H, K, N ARE ITS SIDES.

AGAIN, SINCE,
E, F, G ARE THE LEAST OF THE NUMBERS WHICH HAVE
THE SAME RATIO AS C, D, B,

THEREFORE,

E measures C,
the same number of times that G measures B.

Now,
as many times as E measures C,
so many units let there be in O.

Therefore,
G measures B, according to the units in O;

therefore,
O, by multiplying G, has made B.

But,
G is the product of L, M;

therefore,
O, by multiplying the product of L, M, has made B.

Therefore,
B is solid, and
L, M, O are its sides; therefore,
A, B are solid.

I say that;
they are, also, similar.

For since,
N, O, by multiplying E, have made A, C,

[vii. 18] therefore,
as N is to O,
so is A to C, that is,
E to F.

But,
as E is to F,
so is H to L, and
K to M

therefore also,
as H is to L,
so is K to M and
N to O.

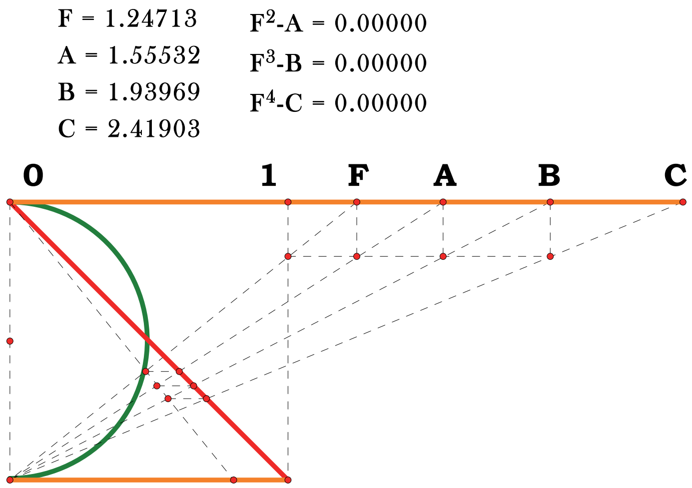
And,
H, K, N are the sides, of A, and,
O, L, M the sides, of B.

Therefore,
A, B are similar solid numbers.

Q. E. D.

PROPOSITION 22.

IF THREE NUMBERS BE IN CONTINUED PROPORTION, AND THE FIRST BE SQUARE, THE THIRD WILL, ALSO, *BE SQUARE.*



LET,

A, *B*, *C*, BE THREE NUMBERS IN CONTINUED PROPORTION,

AND LET,

A THE FIRST BE SQUARE;

I SAY THAT;

C, THE THIRD, IS, ALSO, SQUARE.

FOR, SINCE,

BETWEEN *A*, *C*,

THERE IS ONE MEAN PROPORTIONAL NUMBER, *B*,

[VIII. 20]

THEREFORE,

A, *C* ARE SIMILAR PLANE NUMBERS.

BUT,

A IS SQUARE;

THEREFORE,

C IS, ALSO, SQUARE.

Q. E. D.

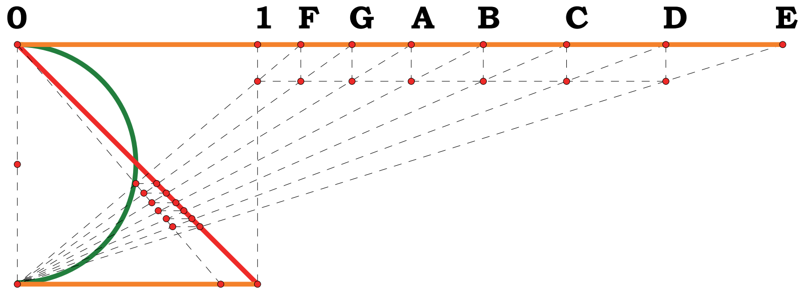
Proposition 23.

If four numbers be in continued proportion, and the first be cube, the fourth will, also, be cube.

F = 1.18000
G = 1.39240

A = 1.64303
B = 1.93878
C = 2.28776
D = 2.69955
E = 3.18547

F^2-G = 0.00000
F^3-A = 0.00000
F^4-B = 0.00000
F^5-C = 0.00000
F^6-D = 0.00000
F^7-E = 0.00000



Let,

A, *B*, *C*, *D*, be four numbers in continued proportion,

and let,

A be cube;

I say that;

D is, also, cube.

For, since,

between *A*, *D*,

there are two mean proportional numbers,

[viii. 21] therefore,

A, *D* are similar solid numbers.

But,

A is cube;

therefore,

D is, also, cube.

Q. E. D.

PROPOSITION 24.

IF TWO NUMBERS HAVE TO ONE ANOTHER THE RATIO WHICH A SQUARE NUMBER HAS TO A SQUARE NUMBER, AND THE FIRST BE SQUARE, THE SECOND WILL, ALSO, *BE SQUARE.*

$Z = 1.60462$ $\quad \frac{Z}{X} = 1.37061 \quad \frac{B}{A} - \frac{Z}{X} = 0.00000$

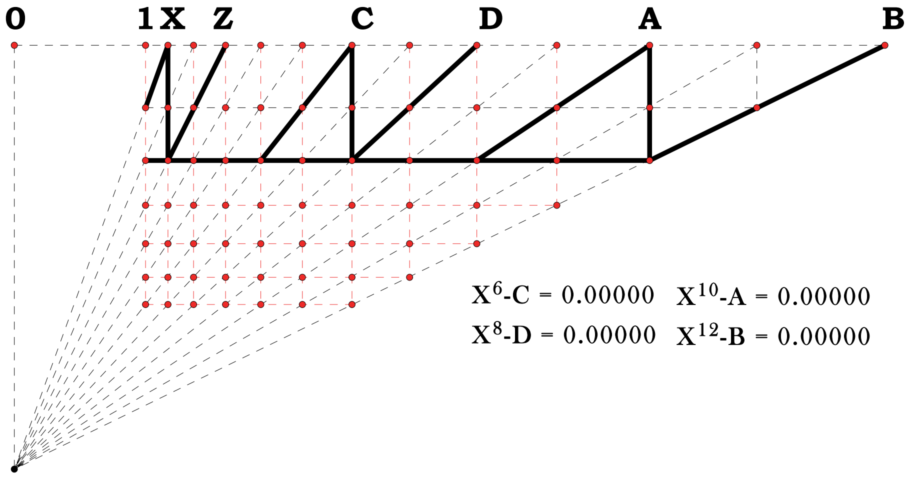
$X = 1.17073$

$A = 4.83697$ $\quad \frac{B}{A} = 1.37061 \quad \frac{D}{C} - \frac{Z}{X} = 0.00000$

$B = 6.62962$

$C = 2.57480$ $\quad \frac{D}{C} = 1.37061 \quad \frac{B}{A} - \frac{D}{C} = 0.00000$

$D = 3.52906$



FOR LET,

THE TWO NUMBERS, *A*, *B*, HAVE TO ONE ANOTHER
THE RATIO WHICH THE SQUARE NUMBER, *C*, HAS TO
THE SQUARE NUMBER, *D*, AND LET,
A BE A SQUARE;

I SAY THAT;

B IS, ALSO, SQUARE.

FOR, SINCE,

C, *D* ARE SQUARE, *C*, *D* ARE SIMILAR PLANE NUMBERS.

[VIII. 18] THEREFORE,

ONE MEAN PROPORTIONAL NUMBER FALLS BETWEEN *C*, *D*.

AND,

AS *C* IS TO *D*,
SO IS *A* TO *B*;

[VIII. 8] THEREFORE,

ONE MEAN PROPORTIONAL NUMBER FALLS
BETWEEN *A*, *B*, ALSO. AND,
A IS SQUARE;

[VIII. 22] THEREFORE,

B IS, ALSO, SQUARE.

Q. E. D.

Proposition 25.

If two numbers have to one another the ratio which a cube number has to a cube number, and the first be cube, the second will, also, be cube.

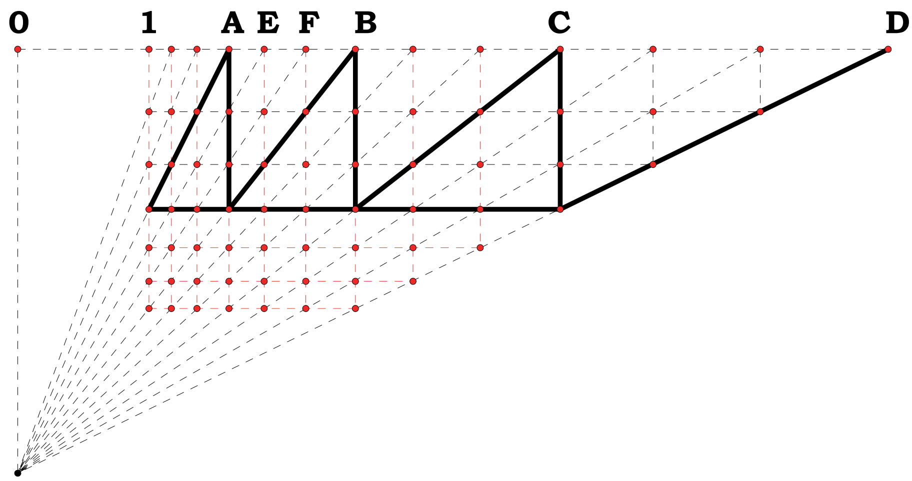
$A = 1.60462$
$B = 2.57480$
$C = 4.13158$
$D = 6.62962$
$E = 1.87858$
$F = 2.19931$

$\frac{C}{B} = 1.60462$

$\frac{D}{C} = 1.60462$

$\frac{F}{E} = 1.17073$

$\frac{F^3}{E} = 1.60462$



For let,

The two numbers, *A*, *B*, have to one another the ratio which the cube number, *C*, has to the cube number, *D*,

and let,

A be cube;

I say that;

B is, also, cube.

For, since,

C, *D* are cube,

C, *D* are similar solid numbers.

[VIII. 19] Therefore,

two mean proportional numbers fall between *C*, *D*.

[VIII. 8] And,

as many numbers as fall between *C*, *D*, in continued proportion, so many will, also, fall between those which have the same ratio with them;

so that,

TWO MEAN PROPORTIONAL NUMBERS FALL
BETWEEN A, B, ALSO.

LET,
E, F SO FALL.

SINCE, THEN,
THE FOUR NUMBERS,
A, E, F, B, ARE IN CONTINUED PROPORTION,

AND,
A IS CUBE,

[VIII. 23] THEREFORE,
B IS, ALSO, CUBE.

Q. E. D.

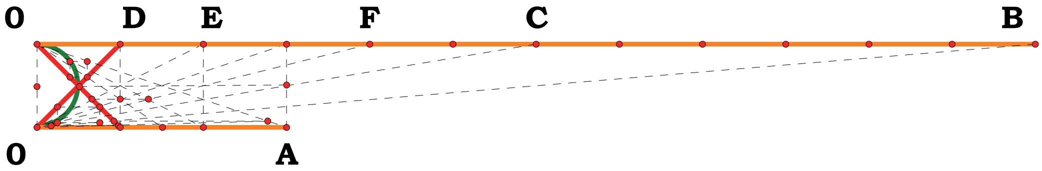
PROPOSITION 26.

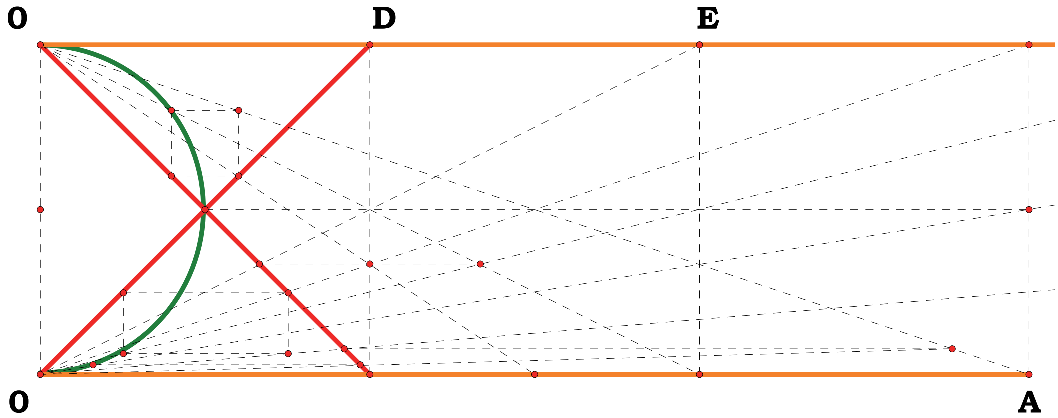
SIMILAR PLANE NUMBERS HAVE TO ONE ANOTHER THE RATIO WHICH A SQUARE NUMBER HAS TO A SQUARE NUMBER.

A = 3.00000 B = 12.00000 F = 4.00000

C = 6.00000 E = 2.00000

$\frac{B}{A}$-F = 0.00000





LET,

A, *B* BE SIMILAR PLANE NUMBERS;

I SAY THAT;

A HAS TO *B*, THE RATIO WHICH

A SQUARE NUMBER HAS TO A SQUARE NUMBER.

FOR, SINCE,

A, *B* ARE SIMILAR PLANE NUMBERS,

[VIII. 18] THEREFORE,

ONE MEAN PROPORTIONAL NUMBER FALLS BETWEEN *A*, *B*.

LET,

IT SO FALL,

AND LET,

IT BE *C*;

[VII. 33 OR VIII. 2]

AND LET,

D, *E*, *F*, THE LEAST NUMBERS OF THOSE WHICH HAVE

THE SAME RATIO WITH *A*, *C*, *B*, BE TAKEN;

[VIII. 2, POR.] THEREFORE,

THE EXTREMES OF THEM, *D*, *F*, ARE SQUARE,

AND SINCE,

AS D IS TO F,
SO IS A TO B, AND
D, F ARE SQUARE,

THEREFORE,

A HAS TO B, THE RATIO WHICH
A SQUARE NUMBER HAS TO A SQUARE NUMBER.

Q. E. D.

Proposition 27.

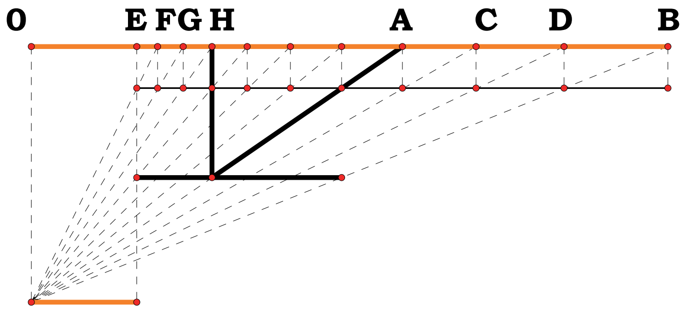
Similar solid numbers have to one another the ratio which a cube number has to a cube number.

Unit = 1.39700 cm

H = 1.71494 $\quad \frac{B}{A}$-H = 0.00000

A = 3.52032

B = 6.03714



Let,

A, *B* be similar solid numbers;

I say that;

A has to *B*, the ratio which
a cube number has to a cube number.

For, since,

A, *B* are similar solid numbers,

[viii. 19] therefore,

two mean proportional numbers fall between *A*, *B*.

Let,

C, *D* so fall,

and let,

E, *F*, *G*, *H*, the least numbers of those which have
the same ratio with *A*, *C*, *D*, *B*,

[vii. 33 or viii. 2] and,

equal with them in multitude, be taken;

[viii. 2, Por.] therefore,

the extremes of them, *E*, *H*, are cube.

And,

as *E* is to *H*,
so is *A* to *B*;

therefore,

A, also, has to *B*, the ratio which
a cube number has to a cube number.

Q. E. D.

BOOK IX.

OF

EUCLID'S ELEMENTS

TRANSLATED FROM THE TEXT OF HEIBERG

BY

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K. C. B., K. C. V. O., F. R. S.,

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HONORARY FELLOW (SOMETIME FELLOW) OF TRINITY COLLEGE CAMBRIDGE

2013 *EDITION*

REVISED WITH SUBTRACTIONS

REFORMATTED AND ABRIDGED FOR STUDY OF *THE ELEMENTS*.

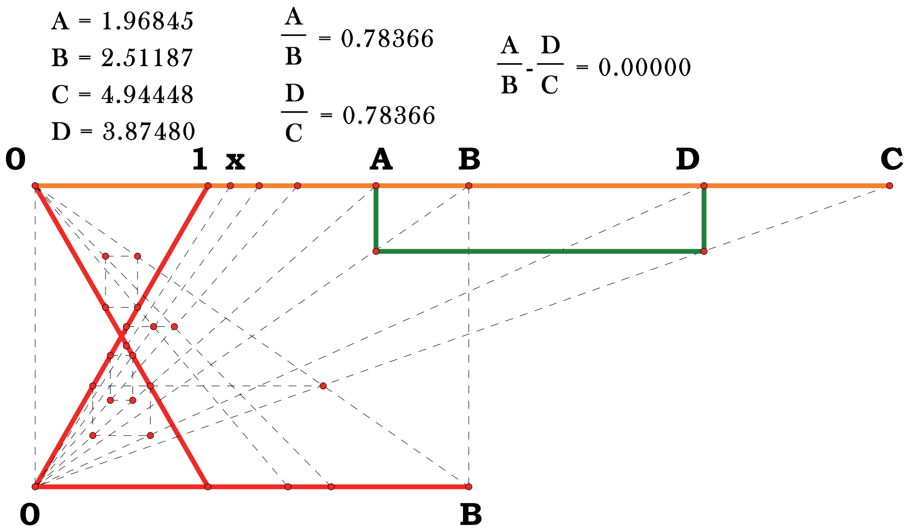
BY JOHN CLARK.

BOOK IX.

PROPOSITIONS.

PROPOSITION 1.

IF TWO SIMILAR PLANE NUMBERS, BY MULTIPLYING ONE ANOTHER, MAKE SOME NUMBER, THE PRODUCT WILL BE SQUARE.



LET,

A, *B* BE TWO SIMILAR PLANE NUMBERS,

AND LET,

A, BY MULTIPLYING *B*, MAKE *C*;

I SAY THAT;

C IS SQUARE.

FOR LET,

A, BY MULTIPLYING ITSELF, MAKE *D*.

THEREFORE,

D IS SQUARE.

SINCE THEN,

A, BY MULTIPLYING ITSELF, HAS MADE *D*, AND
BY MULTIPLYING *B*, HAS MADE *C*,

[VII. 17]

THEREFORE,

AS *A* IS TO *B*,
SO IS *D* TO *C*.

AND, SINCE,

A, *B* ARE SIMILAR PLANE NUMBERS,

[VIII. 18]

THEREFORE,

ONE MEAN PROPORTIONAL NUMBER FALLS
BETWEEN *A*, *B*.

[VIII. 8]

BUT,

IF NUMBERS FALL BETWEEN TWO NUMBERS IN CONTINUED PROPORTION, AS MANY AS FALL BETWEEN THEM, SO MANY, ALSO, FALL BETWEEN THOSE WHICH HAVE THE SAME RATIO;

SO THAT,

ONE MEAN PROPORTIONAL NUMBER FALLS BETWEEN *D*, *C*, ALSO.

AND,

D IS SQUARE;

[VIII. 22]

THEREFORE,

C IS, ALSO, SQUARE.

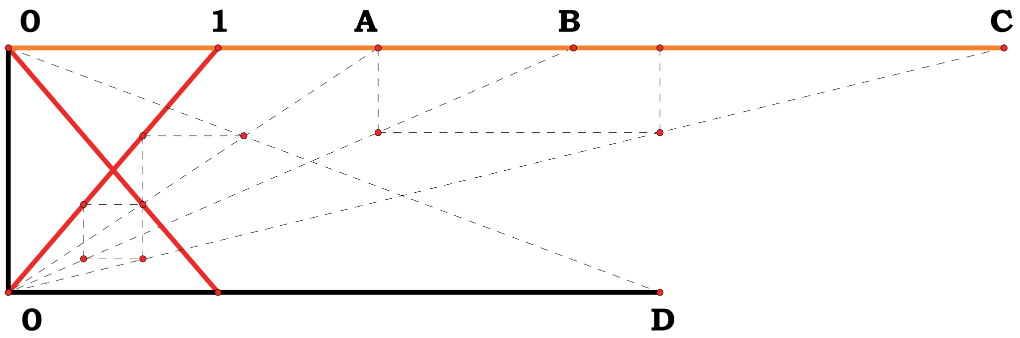
Q. E. D.

$A = 1.76336$ $\qquad \frac{A}{B} = 0.65439 \qquad A \cdot B = 4.75165$

$B = 2.69466$

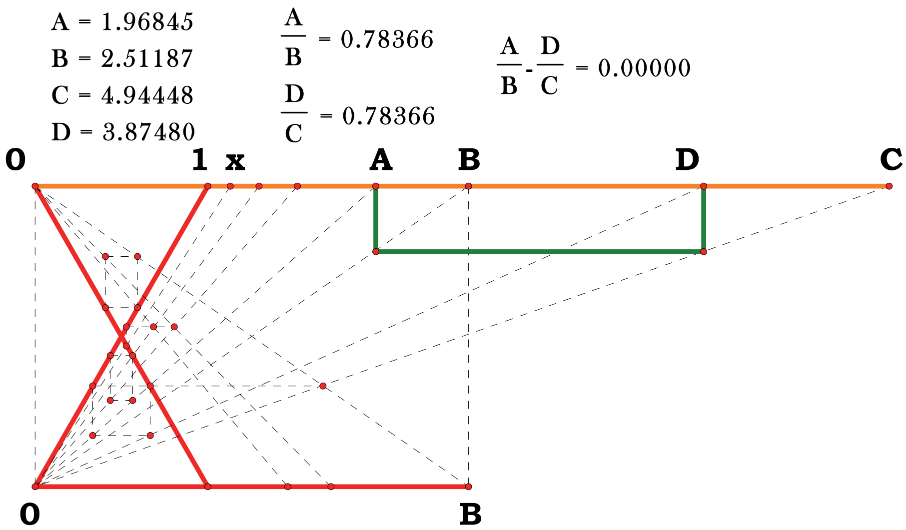
$C = 4.75165$ $\qquad \frac{A}{B} \cdot (A \cdot B) - A^2 = 0.00000$

$D = 3.10943$



PROPOSITION 2.

IF TWO NUMBERS, BY MULTIPLYING ONE ANOTHER, MAKE A SQUARE NUMBER, THEY ARE SIMILAR PLANE NUMBERS.



LET,

A, *B* BE TWO NUMBERS,

AND LET,

A, BY MULTIPLYING *B*, MAKE THE SQUARE NUMBER *C*;

I SAY THAT;

A, *B* ARE SIMILAR PLANE NUMBERS.

FOR LET,

A, BY MULTIPLYING ITSELF, MAKE *D*;

THEREFORE,

D IS SQUARE.

NOW, SINCE,

A, BY MULTIPLYING ITSELF, HAS MADE *D*, AND
BY MULTIPLYING *B*, HAS MADE *C*,

[VII. 17]

THEREFORE,

AS *A* IS TO *B*,
SO IS *D* TO *C*.

AND, SINCE,

D IS SQUARE, AND
C IS SO ALSO,

THEREFORE,

D, *C* ARE SIMILAR PLANE NUMBERS.

[VIII. 18]

THEREFORE,

ONE MEAN PROPORTIONAL NUMBER FALLS
BETWEEN *D*, *C*.

AND,

AS D IS TO C,
SO IS TO B;

[VIII. 8]
THEREFORE,
ONE MEAN PROPORTIONAL NUMBER FALLS
BETWEEN A, B, ALSO.

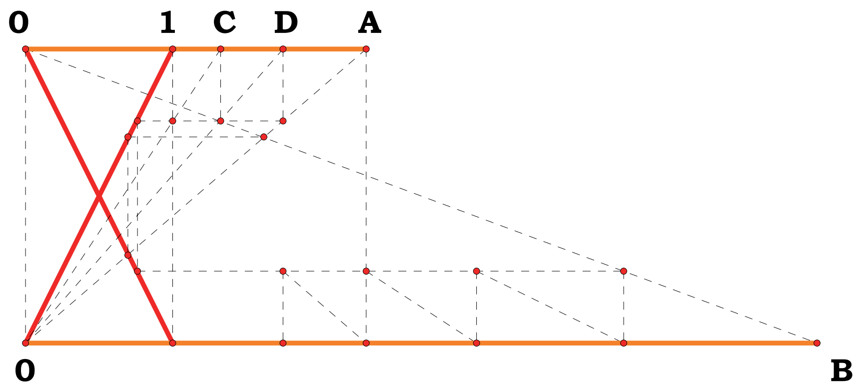
[VIII. 20]
BUT,
IF ONE MEAN PROPORTIONAL NUMBER FALL
BETWEEN TWO NUMBERS,
THEY ARE SIMILAR PLANE NUMBERS;
THEREFORE,
A, B ARE SIMILAR PLANE NUMBERS.

Q. E. D.

PROPOSITION 3.

IF A CUBE NUMBER, BY MULTIPLYING ITSELF, MAKE SOME NUMBER, THE PRODUCT WILL BE CUBE.

A = 2.31958	C^3-A = 0.00000
B = 5.38044	C^2-D = 0.00000
C = 1.32374	A^2-B = 0.00000
D = 1.75229	



FOR LET,

THE CUBE NUMBER *A*, BY MULTIPLYING ITSELF, MAKE *B*;

I SAY THAT;

B IS CUBE.

FOR LET,

C, THE SIDE OF *A*, BE TAKEN,

AND LET,

C, BY MULTIPLYING ITSELF, MAKE *D*.

IT IS THEN MANIFEST THAT;

C, BY MULTIPLYING *D*, HAS MADE *A*.

NOW, SINCE,

C, BY MULTIPLYING ITSELF, HAS MADE *D*,

THEREFORE,

C MEASURES *D*, ACCORDING TO THE UNITS IN ITSELF.

BUT FURTHER,

THE UNIT, ALSO, MEASURES *C*, ACCORDING TO THE UNITS IN IT;

[VII. DEF. 20]

THEREFORE,

AS THE UNIT IS TO *C*,

SO IS *C* TO *D*.

AGAIN, SINCE,

C, BY MULTIPLYING *D*, HAS MADE *A*,

THEREFORE,

D MEASURES *A*, ACCORDING TO THE UNITS IN *C*.

BUT,

THE UNIT, ALSO, MEASURES *C*, ACCORDING TO THE UNITS IN IT;

THEREFORE,

AS THE UNIT IS TO C,
SO IS D TO A.
BUT,
AS THE UNIT IS TO C,
SO IS C TO D;
THEREFORE ALSO,
AS THE UNIT IS TO C,
SO IS C TO D, AND
D TO A.
THEREFORE,
BETWEEN THE UNIT AND THE NUMBER A,
TWO MEAN PROPORTIONAL NUMBERS,
C, D, HAVE FALLEN IN CONTINUED PROPORTION.
AGAIN, SINCE,
A, BY MULTIPLYING ITSELF, HAS MADE B,
THEREFORE,
A MEASURES B, ACCORDING TO THE UNITS IN ITSELF.
BUT,
THE UNIT, ALSO, MEASURES A, ACCORDING TO THE UNITS IN IT;

[VII. DEF. 20]
THEREFORE,
AS THE UNIT IS TO A,
SO IS A TO B.
BUT,
BETWEEN THE UNIT AND A,
TWO MEAN PROPORTIONAL NUMBERS HAVE FALLEN;

[VIII. 8]
THEREFORE,
TWO MEAN PROPORTIONAL NUMBERS WILL, ALSO, FALL
BETWEEN A, B.

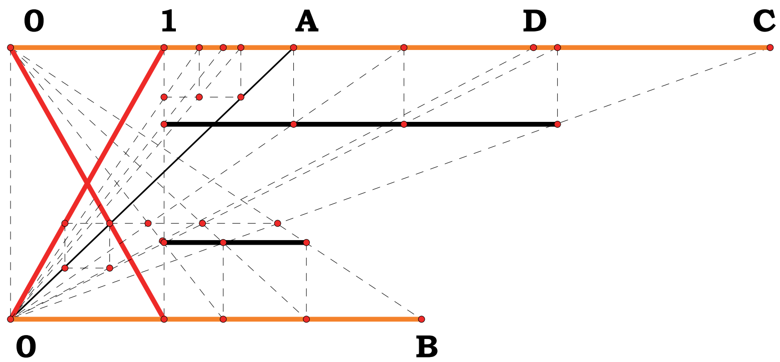
[VIII. 23]
BUT,
IF TWO MEAN PROPORTIONAL NUMBERS FALL
BETWEEN TWO NUMBERS, AND
THE FIRST BE CUBE,
THE SECOND WILL, ALSO, BE CUBE.
AND,
A IS CUBE;
THEREFORE,
B IS, ALSO, CUBE.

Q. E. D.

PROPOSITION 4.

IF A CUBE NUMBER, BY MULTIPLYING A CUBE NUMBER, MAKE SOME NUMBER, THE PRODUCT WILL BE CUBE.

A = 1.84537 A·B-C = 0.00000
B = 2.68125 A²-D = 0.00000
C = 4.94789
D = 3.40539



FOR LET,
THE CUBE NUMBER, *A*, BY MULTIPLYING
THE CUBE NUMBER, *B*, MAKE *C*;
I SAY THAT;
C IS CUBE.

[IX. 3]
FOR LET,
A, BY MULTIPLYING ITSELF, MAKE *D*;
THEREFORE,
D IS CUBE.
AND, SINCE,
A, BY MULTIPLYING ITSELF, HAS MADE *D*, AND
BY MULTIPLYING *B*, HAS MADE *C*,

[VII. 17]
THEREFORE,
AS *A* IS TO *B*,
SO IS *D* TO *C*.
AND, SINCE,
A, *B* ARE CUBE NUMBERS,
A, *B* ARE SIMILAR SOLID NUMBERS.

[VIII. 19]
THEREFORE,
TWO MEAN PROPORTIONAL NUMBERS FALL
BETWEEN *A*, *B*;

[VIII. 8]
SO THAT,
TWO MEAN PROPORTIONAL NUMBERS WILL FALL

BETWEEN *D*, *C* ALSO.

AND,

D IS CUBE;

[VIII. 23]

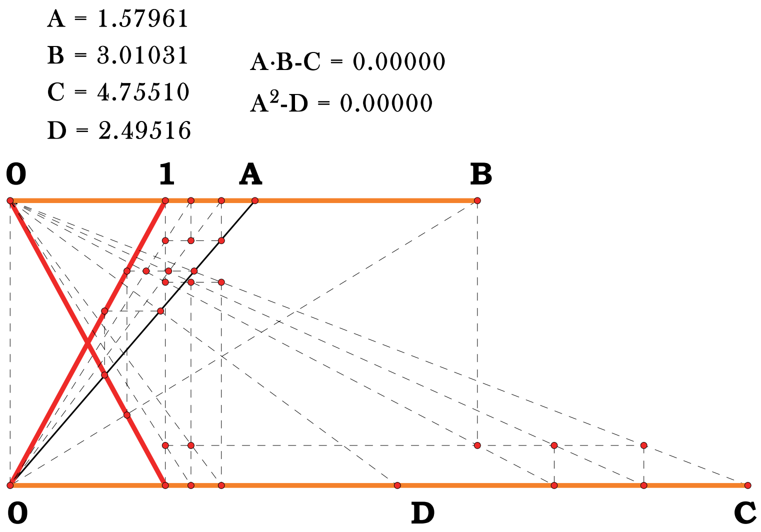
THEREFORE,

C IS, ALSO, CUBE

Q. E. D.

Proposition 5.

If a cube number, by multiplying any number, make a cube number, the multiplied number will, also, *be cube.*



For let,

the cube number, *A*, by multiplying any number, *B*,

make the cube number, *C*;

I say that;

B is cube.

[IX. 3]

For let,

A, by multiplying itself, make *D*;

therefore,

D is cube.

Now, since,

A, by multiplying itself, has made *D*, and

by multiplying *B*, has made *C*,

[VII. 17]

therefore,

as *A* is to *B*,

so is *D* to *C*.

And since,

D, *C* are cube,

they are similar solid numbers.

[VIII. 19]

Therefore,

two mean proportional numbers fall between *D*, *C*.

And,

as *D* is to *C*,

so is *A* to *B*;

[VIII. 8]
THEREFORE,
TWO MEAN PROPORTIONAL NUMBERS FALL
BETWEEN A, B, ALSO.
AND,
A IS CUBE;
[VIII. 23]
THEREFORE,
B IS, ALSO, CUBE.

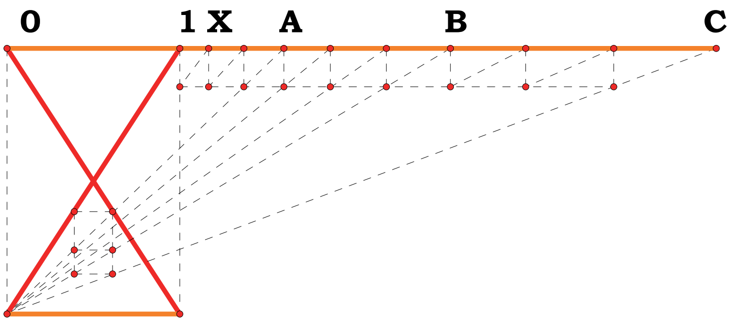
PROPOSITION 6.

IF A NUMBER, BY MULTIPLYING ITSELF, MAKE A CUBE NUMBER, IT WILL ITSELF, ALSO, *BE CUBE.*

A = 1.60125
B = 2.56400
C = 4.10560
X = 1.16991

$A^{\frac{1}{3}}$-X = 0.00000
A^2-B = 0.00000
A·B-C = 0.00000

X^{3^2}-C = 0.00000
X^9-C = 0.00000



FOR LET,

THE NUMBER, *A*, BY MULTIPLYING ITSELF,

MAKE THE CUBE NUMBER, *B*;

I SAY THAT,

A IS, ALSO, CUBE.

FOR LET,

A, BY MULTIPLYING *B*, MAKE *C*.

SINCE, THEN,

A, BY MULTIPLYING ITSELF, HAS MADE *B*, AND

BY MULTIPLYING *B*, HAS MADE *C*,

THEREFORE,

C IS CUBE.

AND, SINCE,

A, BY MULTIPLYING ITSELF, HAS MADE *B*,

THEREFORE,

A MEASURES *B*, ACCORDING TO THE UNITS IN ITSELF.

BUT,

THE UNIT, ALSO, MEASURES *A*, ACCORDING TO THE UNITS IN IT.

[VII. DEF. 20]

THEREFORE,

AS THE UNIT IS TO *A*,

SO IS *A* TO *B*.

AND, SINCE,

A, BY MULTIPLYING *B*, HAS MADE *C*,

THEREFORE,

B MEASURES *C*, ACCORDING TO THE UNITS IN *A*.

BUT,

THE UNIT, ALSO, MEASURES *A*, ACCORDING TO THE UNITS IN IT.

[VII. DEF. 20]
THEREFORE,
AS THE UNIT IS TO A,
SO IS B TO C.
BUT,
AS THE UNIT IS TO A,
SO IS A TO B;
THEREFORE ALSO,
AS A IS TO B,
SO IS B TO C.
AND, SINCE,
B, C ARE CUBE,
THEY ARE SIMILAR SOLID NUMBERS.

[VIII. 19]
THEREFORE,
THERE ARE TWO MEAN PROPORTIONAL NUMBERS
BETWEEN B, C.
AND,
AS B IS TO C,
SO IS A TO B.

[VIII. 8]
THEREFORE,
THERE ARE TWO MEAN PROPORTIONAL NUMBERS
BETWEEN A, B, ALSO.
AND,
B IS CUBE;

[CF. VIII. 23]
THEREFORE,
A IS, ALSO, CUBE.

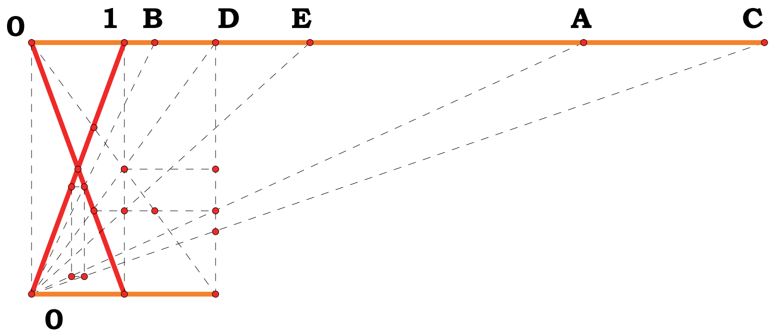
Q. E. D.

Proposition 7.

If a composite number, by multiplying any number, make some number, the product will be solid.

A = 5.94828
B = 1.32759
C = 7.89685
D = 1.98276
E = 3.00000

$\frac{A}{D}$ = 3.00000
D·E-A = 0.00000
A·B-C = 0.00000
D·E·B-C = 0.00000



For let,
    the composite number, *A*,
    by multiplying any number, *B*, make *C*;
I say that;
    C is solid.

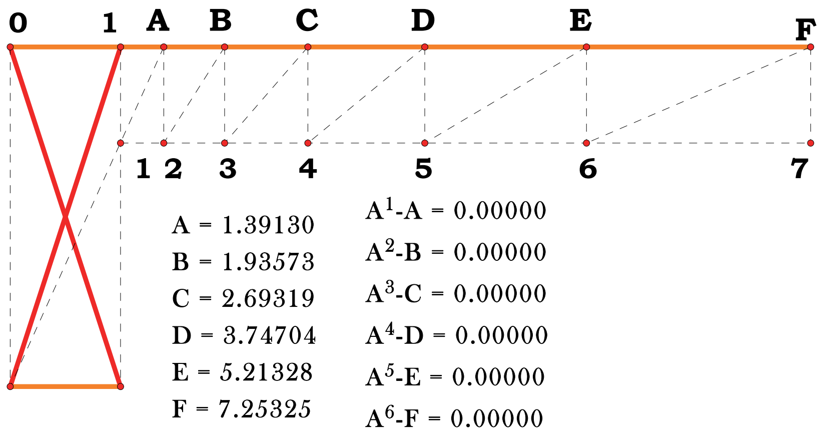
[vii. Def. 13]
For, since,
    A is composite, it will be measured by some number.
Let it,
    be measured by *D*; and
    as many times as *D* measures *A*,
    so many units let there be in *B*.
Since then,
    D measures *A*, according to the units in *E*,

[vii. Def. 15]
therefore,
    E, by multiplying *D*, has made *A*.
And, since,
    A, by multiplying *B*, has made *C*,
and,
    A is the product of *D*, *E*,
therefore,
    the product of *D*, *E*, by multiplying *B*, has made *C*.
Therefore,
    C is solid, and
    D, *E*, *B* are its sides.

Q. E. D.

Proposition 8.

If as many numbers as we please beginning from an unit be in continued proportion, the third from the unit will be square, as will, also, *those which successively leave out one; the fourth will be cube, as will,* also, *all those which leave out two; and the seventh will be at once cube and square, as will,* also, *those which leave out five.*



LET,

THERE BE AS MANY NUMBERS AS WE PLEASE,
A, *B*, *C*, *D*, *E*, *F*, BEGINNING FROM AN UNIT AND
IN CONTINUED PROPORTION;

I SAY THAT;

B, THE THIRD FROM THE UNIT, IS SQUARE,
AS ARE, ALSO, ALL THOSE WHICH LEAVE OUT ONE;
C, THE FOURTH, IS CUBE,
AS ARE, ALSO, ALL THOSE WHICH LEAVE OUT TWO; AND
F, THE SEVENTH, IS AT ONCE CUBE AND SQUARE,
AS ARE, ALSO, ALL THOSE WHICH LEAVE OUT FIVE.

FOR SINCE,

AS THE UNIT IS TO *A*,
SO IS *A* TO *B*,

[VII. DEF. 20]

THEREFORE,

THE UNIT MEASURES THE NUMBER, *A*,
THE SAME NUMBER OF TIMES THAT *A* MEASURES *B*.

BUT,

THE UNIT MEASURES THE NUMBER, *A*,
ACCORDING TO THE UNITS IN IT;

THEREFORE,

A, ALSO, MEASURES *B*, ACCORDING TO THE UNITS IN *A*.

THEREFORE,

A, BY MULTIPLYING ITSELF, HAS MADE *B*;

THEREFORE,

B is square.
And, since,
B, *C*, *D* are in continued proportion, and
B is square,

[viii. 22]
therefore,
D is, also, square.
For the same reason,
F is, also, square.
Similarly we can prove that,
all those which leave out one are square.
I say next that;
C, the fourth from the unit, is cube,
as, are, also, all those which leave out two.
For since,
as the unit is to *A*,
so is *B* to *C*,
therefore,
the unit measures the number, *A*,
the same number of times that, *B* measures *C*.
But,
the unit measures the number, *A*,
according to the units in *A*;
therefore,
B, also, measures *C*
according to the units in *A*.
Therefore,
A, by multiplying *B*, has made *C*.
Since then,
A, by multiplying itself, has made *B*, and
by multiplying *B*, has made *C*,
therefore,
C is cube.
And, since,
C, *D*, *E*, *F* are in continued proportion, and
C is cube,

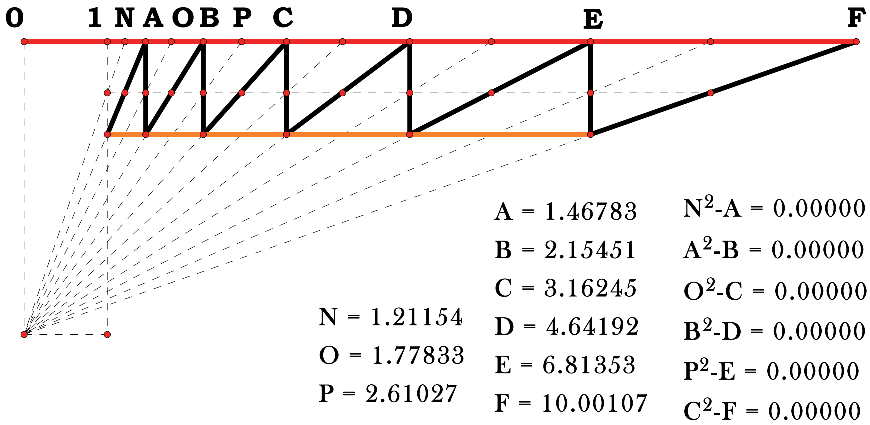
[viii. 23]
therefore,
F is, also, cube.
But,
it was, also, proved square;
therefore,
the seventh from the unit is both cube and square.
Similarly we can prove that,

ALL THE NUMBERS WHICH LEAVE OUT FIVE ARE, ALSO, BOTH CUBE AND SQUARE.

Q. E. D.

Proposition 9.

If as many numbers as we please, beginning from an unit, be in continued proportion, and the number after the unit be square, all the rest will, also, *be square. And, if the number after the unit be cube, all the rest will,* also, *be cube.*



Let,

there be as many numbers as we please,

A, *B*, *C*, *D*, *E*, *F*, beginning from an unit, and

in continued proportion,

and let,

A, the number after the unit, be square;

I say that;

all the rest will, also, be square.

[ix. 8]

Now,

it has been proved that *B*,

the third from the unit, is square,

as are also, all those which leave out one;

I say that;

all the rest are, also, square.

[viii. 22]

For, since,

A, *B*, *C*, are in continued proportion, and

A is square,

therefore,

C is, also, square.

[viii. 22]

Again, since,

B, *C*, *D* are in continued proportion, and

B is square,

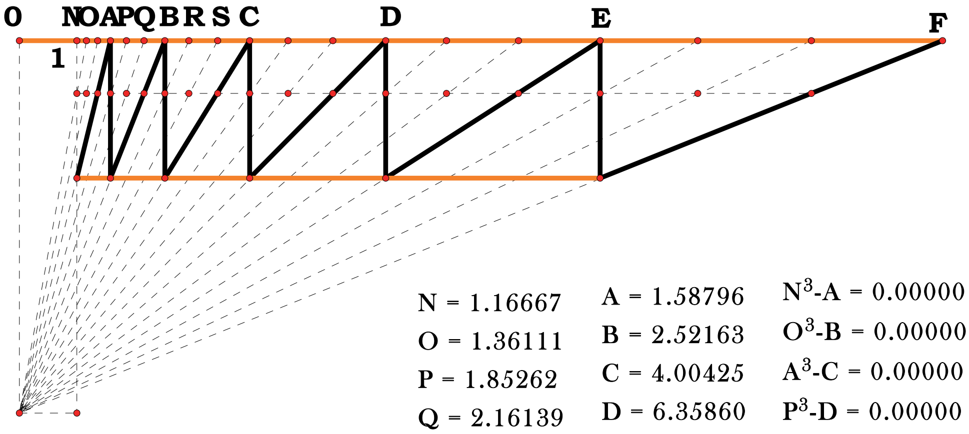
D is, also, square.

Similarly we can prove that;

all the rest are, also, square.

Next, let,

A be cube;



N = 1.16667	A = 1.58796	N^3-A = 0.00000
O = 1.36111	B = 2.52163	O^3-B = 0.00000
P = 1.85262	C = 4.00425	A^3-C = 0.00000
Q = 2.16139	D = 6.35860	P^3-D = 0.00000
R = 2.94190	E = 10.09722	Q^3-E = 0.00000
S = 3.43221	F = 16.03401	B^3-F = 0.00000

I say that;

all the rest are, also, cube.

[ix. 8]

Now it has been proved that;

C, the fourth from the unit, is cube,

as, also, are all, those which leave out two;

I say that;

all the rest are, also, cube.

For, since,

as the unit is to A,

so is A to B,

therefore,

the unit measures A,

the same number of times as A measures B.

But,

the unit measures A, according to the units in it;

therefore,

A, also, measures B ,according to the units in itself;

therefore,

A, by multiplying itself, has made B.

And,

A is cube.

[ix. 3]

But,

if a cube number, by multiplying itself,

make some number, the product is cube.

Therefore,

B IS, ALSO, CUBE.

[VIII. 23]

AND, SINCE,

THE FOUR NUMBERS,

A, *B*, *C* *D*, ARE IN CONTINUED PROPORTION, AND

A IS CUBE,

D, ALSO, IS CUBE.

FOR THE SAME REASON,

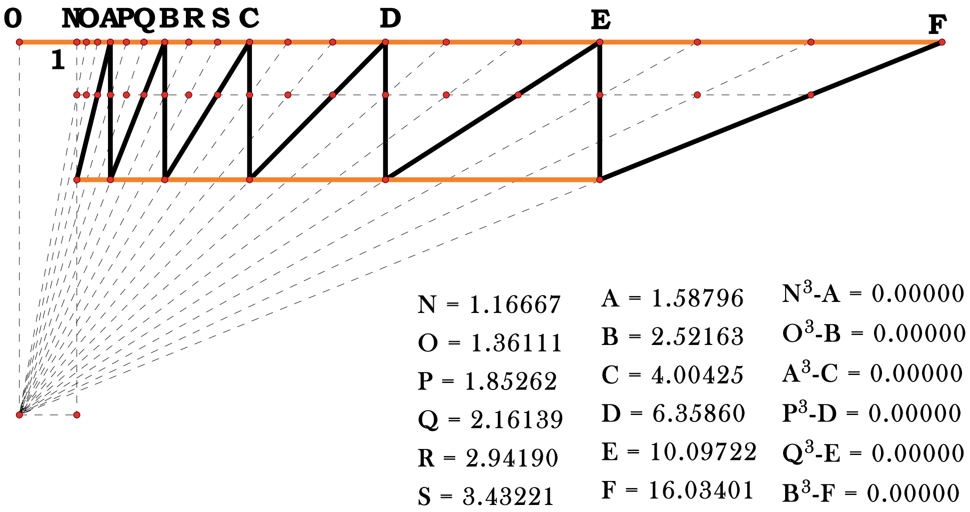
E IS, ALSO, CUBE, AND

SIMILARLY ALL THE REST ARE CUBE.

Q. E. D.

PROPOSITION 10.

IF AS MANY NUMBERS AS WE PLEASE BEGINNING FROM AN UNIT BE IN CONTINUED PROPORTION, AND THE NUMBER AFTER THE UNIT BE NOT SQUARE, NEITHER WILL ANY OTHER BE SQUARE EXCEPT THE THIRD FROM THE UNIT AND ALL THOSE WHICH LEAVE OUT ONE. AND IF THE NUMBER AFTER THE UNIT BE NOT CUBE, NEITHER WILL ANY OTHER BE CUBE EXCEPT THE FOURTH FROM THE UNIT AND ALL THOSE WHICH LEAVE OUT TWO.



LET,
THERE BE AS MANY NUMBERS AS WE PLEASE,
A, *B*, *C*, *D*, *E*, *F*, BEGINNING FROM AN UNIT AND
IN CONTINUED PROPORTION,
AND LET,
A, THE NUMBER AFTER THE UNIT, NOT BE SQUARE;
I SAY THAT;
NEITHER WILL ANY OTHER BE SQUARE EXCEPT
THE THIRD FROM THE UNIT
< AND THOSE WHICH LEAVE OUT ONE. >

[IX. 8]
FOR, IF POSSIBLE, LET,
C BE SQUARE.
BUT,
B IS, ALSO, SQUARE;
[THEREFORE,
B, *C* HAVE TO ONE ANOTHER THE RATIO WHICH
A SQUARE NUMBER HAS TO A SQUARE NUMBER.]
AND,
AS *B* IS TO *C*,
SO IS *A* TO *B*;
THEREFORE,
A, *B* HAVE TO ONE ANOTHER THE RATIO WHICH

A SQUARE NUMBER HAS TO A SQUARE NUMBER;

[VIII. 26, CONVERSE]

[SO THAT,

A, B ARE SIMILAR PLANE NUMBERS].

AND,

B IS SQUARE;

THEREFORE,

A IS, ALSO, SQUARE:

WHICH,

IS CONTRARY TO THE HYPOTHESIS.

THEREFORE,

C IS NOT SQUARE.

SIMILARLY WE CAN PROVE THAT;

NEITHER IS ANY OTHER OF THE NUMBERS SQUARE EXCEPT
THE THIRD FROM THE UNIT AND THOSE WHICH LEAVE OUT ONE.

NEXT, LET,

A NOT BE CUBE.

I SAY THAT;

NEITHER WILL ANY OTHER BE CUBE, EXCEPT
THE FOURTH FROM THE UNIT, AND
THOSE WHICH LEAVE OUT TWO.

FOR, IF POSSIBLE, LET,

D BE CUBE.

NOW,

C IS, ALSO, CUBE;

[IX. 8]

FOR,

IT IS FOURTH FROM THE UNIT.

AND,

AS C IS TO D,
SO IS B TO C;

THEREFORE,

B, ALSO, HAS TO C, THE RATIO WHICH A CUBE HAS TO A CUBE.

AND,

C IS CUBE;

[VIII. 25]

THEREFORE,

B IS, ALSO, CUBE.

AND SINCE,

AS THE UNIT IS TO A,
SO IS A TO B, AND
THE UNIT MEASURES A, ACCORDING TO THE UNITS IN IT,

THEREFORE,

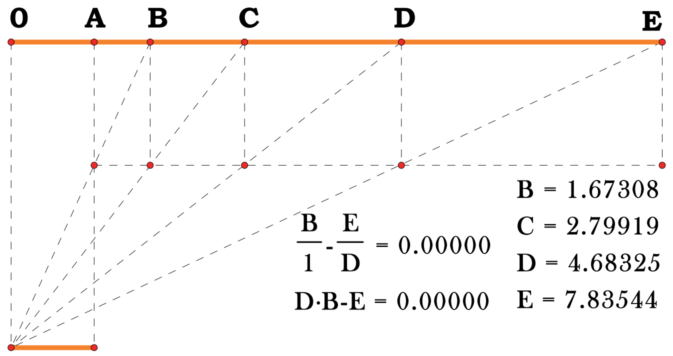
A, ALSO, MEASURES *B*, ACCORDING TO THE UNITS IN ITSELF;
THEREFORE,
A, BY MULTIPLYING ITSELF, HAS MADE THE CUBE NUMBER *B*.

[IX. 6]
BUT,
IF A NUMBER, BY MULTIPLYING ITSELF, MAKE A CUBE NUMBER,
IT IS, ALSO, ITSELF CUBE.
THEREFORE,
A IS, ALSO, CUBE:
WHICH,
IS CONTRARY TO THE HYPOTHESIS.
THEREFORE,
D IS NOT CUBE.
SIMILARLY WE CAN PROVE THAT;
NEITHER IS ANY OTHER OF THE NUMBERS CUBE EXCEPT
THE FOURTH FROM THE UNIT AND
THOSE WHICH LEAVE OUT TWO.

Q. E. D.

PROPOSITION 11.

IF AS MANY NUMBERS AS WE PLEASE BEGINNING FROM AN UNIT BE IN CONTINUED PROPORTION, THE LESS MEASURES THE GREATER ACCORDING TO SOME ONE OF THE NUMBERS WHICH HAVE PLACE AMONG THE PROPORTIONAL NUMBERS.



LET,

THERE BE AS MANY NUMBERS AS WE PLEASE,
B, *C*, *D*, *E*, BEGINNING FROM THE UNIT, *A*, AND
IN CONTINUED PROPORTION;

I SAY THAT;

B, THE LEAST OF THE NUMBERS, *B*, *C*, *D*, *E*, MEASURES
E, ACCORDING TO SOME ONE OF THE NUMBERS, *C*, *D*.

FOR SINCE,

AS THE UNIT *A* IS TO *B*,
SO IS *D* TO *E*,

THEREFORE,

THE UNIT, *A*, MEASURES THE NUMBER, *B*,
THE SAME NUMBER OF TIMES AS *D* MEASURES *E*;

[VII. 15]

THEREFORE, ALTERNATELY,

THE UNIT, *A*, MEASURES *D*,
THE SAME NUMBER OF TIMES AS *B* MEASURES *E*.

BUT,

THE UNIT, *A*, MEASURES, *D*, ACCORDING TO THE UNITS IN IT;

THEREFORE,

B, ALSO, MEASURES *E*, ACCORDING TO THE UNITS IN *D*;

SO THAT,

B THE LESS MEASURES THE GREATER ACCORDING TO
SOME NUMBER OF THOSE
WHICH HAVE PLACE AMONG THE PROPORTIONAL NUMBERS.—

PORISM.

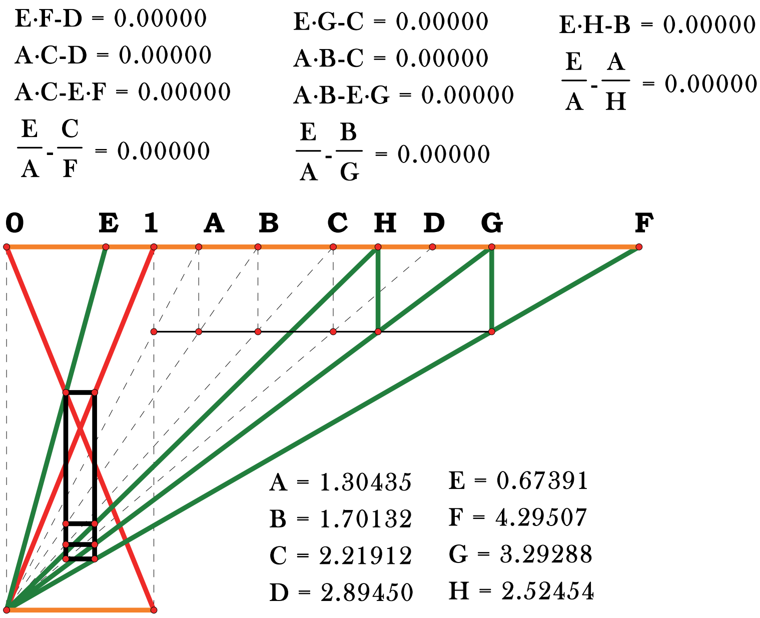
AND IT IS MANIFEST THAT, WHATEVER PLACE THE MEASURING NUMBER HAS, RECKONED FROM THE UNIT, THE SAME PLACE, ALSO, HAS THE NUMBER ACCORDING TO WHICH IT MEASURES, RECKONED

FROM THE NUMBER MEASURED, IN THE DIRECTION OF THE NUMBER BEFORE IT.—

Q. E. D.

Proposition 12.

If as many numbers as we please beginning from an unit be in continued proportion, by however many prime numbers the last is measured, the next to the unit will, also, be measured by the same.



Let,

there be as many numbers as we please,

A, *B*, *C*, *D*, beginning from an unit, and

in continued proportion;

I say that;

by however many prime numbers, *D*, is measured,

A will, also, be measured by the same.

For let,

D be measured by any prime number, *E*;

I say that;

E measures *A*.

For,

suppose it does not;

[vii. 29]

now,

E is prime, and

any prime number is prime to

any which it does not measure;

therefore,

E, *A* are prime to one another.

And, since,

E measures *D*,

let,

it measure it according to *F*,

THEREFORE,

E, BY MULTIPLYING *F*, HAS MADE *D*.

[IX. 11 AND POR.]

AGAIN, SINCE,

A MEASURES *D*, ACCORDING TO THE UNITS IN *C*,

THEREFORE,

A, BY MULTIPLYING *C*, HAS MADE *D*.

BUT, FURTHER ALSO,

E HAS, BY MULTIPLYING *F*, MADE *D*;

THEREFORE,

THE PRODUCT, OF *A*, *C*, =
THE PRODUCT, OF *E*, *F*.

[VII. 19]

THEREFORE,

AS *A* IS TO *E*,
SO IS *F* TO *C*.

[VII. 21]

BUT,

A, *E* ARE PRIME, PRIMES ARE, ALSO, LEAST,

[VII. 20]

AND,

THE LEAST MEASURE THOSE WHICH HAVE THE SAME RATIO
THE SAME NUMBER OF TIMES,
THE ANTECEDENT THE ANTECEDENT AND
THE CONSEQUENT THE CONSEQUENT;

THEREFORE,

E MEASURES *C*.

LET,

IT MEASURE IT ACCORDING TO *G*;

THEREFORE,

BY MULTIPLYING *G*, HAS MADE *C*.

[IX. 11 AND POR.]

BUT, FURTHER, BY THE THEOREM BEFORE THIS,

A HAS ALSO, BY MULTIPLYING *B*, MADE *C*.

THEREFORE,

THE PRODUCT, OF *A*, *B*, EQUALS
THE PRODUCT, OF *E*, *G*.

[VII. 19]

THEREFORE,

AS *A* IS TO *E*,
SO IS *G* TO *B*.

[VII. 21]

BUT,

A, E ARE PRIME, PRIMES ARE, ALSO, LEAST,

[VII. 20]

AND,

THE LEAST NUMBERS MEASURE THOSE WHICH HAVE
THE SAME RATIO WITH THEM THE SAME NUMBER OF TIMES,
THE ANTECEDENT THE ANTECEDENT AND
THE CONSEQUENT THE CONSEQUENT:

THEREFORE,

E MEASURES B.

LET,

IT MEASURE IT ACCORDING TO H,

THEREFORE,

BY MULTIPLYING H, HAS MADE B.

[IX. 8]

BUT FURTHER,

A, HAS ALSO, BY MULTIPLYING ITSELF, MADE B;

THEREFORE,

THE PRODUCT, OF E, H, EQUALS
THE SQUARE, ON A.

[VII. 19]

THEREFORE,

AS E IS TO A,
SO IS A TO H.

[VII. 21]

BUT A, E ARE PRIME, PRIMES ARE, ALSO, LEAST,

[VII. 20]

AND,

THE LEAST MEASURE THOSE WHICH HAVE THE SAME RATIO
THE SAME NUMBER OF TIMES,
THE ANTECEDENT THE ANTECEDENT AND
THE CONSEQUENT THE CONSEQUENT;

THEREFORE,

E MEASURES A, AS ANTECEDENT ANTECEDENT.

BUT, AGAIN,

IT, ALSO, DOES NOT MEASURE IT:

WHICH,

IS IMPOSSIBLE.

THEREFORE,

E, A ARE NOT PRIME TO ONE ANOTHER.

THEREFORE,

THEY ARE COMPOSITE TO ONE ANOTHER.

[vii. Def. 14]
But,
    numbers composite to one another
    are measured by some number.
And, since,
    E is by hypothesis prime, and
    the prime is not measured by any number
    other than itself,
therefore,
    E measures A, E,
so that,
    E measures A.
[But,
    it, also, measures D;
therefore,
    E measures A, D.]
Similarly we can prove that;
    by however many prime numbers, D, is measured,
    A will, also, be measured by the same.

Q. E. D.

PROPOSITION 13.

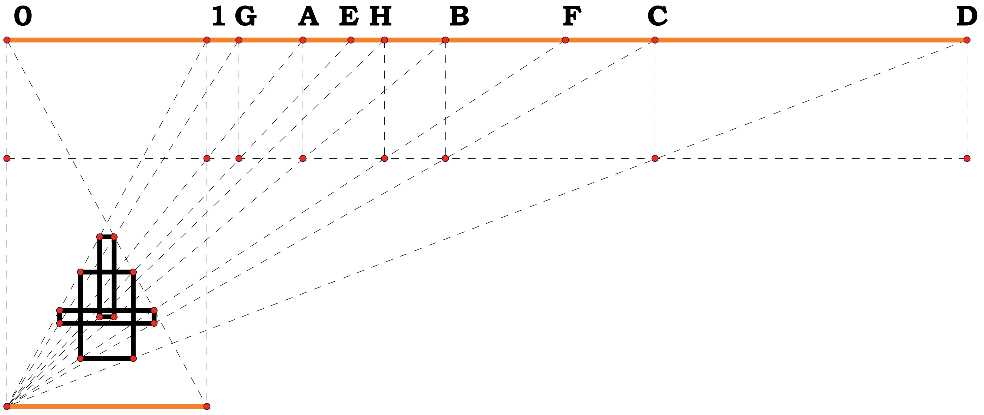
IF AS MANY NUMBERS AS WE PLEASE BEGINNING FROM AN UNIT BE IN CONTINUED PROPORTION, AND THE NUMBER AFTER THE UNIT BE PRIME, THE GREATEST WILL NOT BE MEASURED BY ANY EXCEPT THOSE WHICH, HAVE A PLACE AMONG THE PROPORTIONAL NUMBERS.

$A = 1.48000$
$B = 2.19040$
$C = 3.24179$
$D = 4.79785$
$E = 1.72000$
$F = 2.78945$
$G = 1.16216$
$H = 1.88476$

$\frac{D}{E} - F = 0.00000$
$E \cdot F - D = 0.00000$
$A \cdot C - E \cdot F = 0.00000$
$\frac{E}{A} - \frac{C}{F} = 0.00000$
$\frac{C}{F} - G = 0.00000$

$F \cdot G - C = 0.00000$
$A \cdot B - F \cdot G = 0.00000$
$\frac{F}{A} - \frac{B}{G} = 0.00000$
$\frac{B}{G} - H = 0.00000$
$G \cdot H - B = 0.00000$
$A^2 - B = 0.00000$

$G \cdot H - A^2 = 0.00000$
$\frac{H}{A} - \frac{A}{G} = 0.00000$



LET,

THERE BE AS MANY NUMBERS AS WE PLEASE,
A, *B*, *C*, *D*, BEGINNING FROM AN UNIT AND
IN CONTINUED PROPORTION,

AND LET,

A, THE NUMBER AFTER THE UNIT, BE PRIME;

I SAY THAT;

D, THE GREATEST OF THEM, WILL NOT BE MEASURED BY
ANY OTHER NUMBER EXCEPT *A*, *B*, *C*.

FOR, IF POSSIBLE, LET,

IT BE MEASURED BY *E*,

AND LET,

E NOT BE THE SAME WITH ANY OF THE NUMBERS, *A*, *B*, *C*.

IT IS THEN MANIFEST THAT,

E IS NOT PRIME.

[IX. 12]

FOR,

IF *E* IS PRIME, AND
MEASURES *D*, IT WILL, ALSO, MEASURE *A*,
WHICH IS PRIME, THOUGH IT IS NOT THE SAME WITH IT:

WHICH,

IS IMPOSSIBLE.

THEREFORE,

E IS NOT PRIME.

THEREFORE,

IT IS COMPOSITE.

[VII. 31]

BUT,

ANY COMPOSITE NUMBER IS MEASURED

BY SOME PRIME NUMBER;

THEREFORE,

E IS MEASURED BY SOME PRIME NUMBER.

I SAY NEXT THAT;

IT WILL NOT BE MEASURED BY ANY OTHER PRIME EXCEPT A.

FOR,

IF E IS MEASURED BY ANOTHER, AND

E MEASURES D,

THAT OTHER WILL, ALSO, MEASURE D;

[IX. 12]

SO THAT,

IT WILL, ALSO, MEASURE A, WHICH IS PRIME,

THOUGH IT IS NOT THE SAME WITH IT:

WHICH,

IS IMPOSSIBLE.

THEREFORE,

A MEASURES E.

AND, SINCE,

E MEASURES D,

LET,

IT MEASURE IT ACCORDING TO F.

I SAY THAT;

F IS NOT THE SAME WITH ANY OF THE NUMBERS, A, B, C.

FOR,

IF F IS THE SAME WITH ONE OF THE NUMBERS, A, B, C, AND

MEASURES D, ACCORDING TO E,

THEREFORE,

ONE OF THE NUMBERS, A, B, C, ALSO,

MEASURES D, ACCORDING TO E.

[IX. 11]

BUT,

ONE OF THE NUMBERS, A, B, C, MEASURES D,

ACCORDING TO SOME ONE OF THE NUMBERS, A, B, C;

THEREFORE,

E IS, ALSO, THE SAME WITH ONE OF THE NUMBERS, A, B, C:

WHICH IS CONTRARY TO THE HYPOTHESIS.

THEREFORE,

F IS NOT THE SAME AS ANY ONE OF THE NUMBERS, A, B, C.

SIMILARLY WE CAN PROVE THAT;

F IS MEASURED BY A,

BY PROVING AGAIN THAT F IS NOT PRIME.

[IX. 12]

FOR,

IF IT IS, AND MEASURES D,

IT WILL, ALSO, MEASURE A, WHICH IS PRIME,

THOUGH IT IS NOT THE SAME WITH IT:

WHICH,

IS IMPOSSIBLE;

THEREFORE F IS NOT PRIME.

[VII. 31]

THEREFORE,

IT IS COMPOSITE.

BUT,

ANY COMPOSITE NUMBER IS MEASURED BY

SOME PRIME NUMBER;

THEREFORE,

F IS MEASURED BY SOME PRIME NUMBER.

I SAY NEXT THAT;

IT WILL NOT BE MEASURED BY ANY OTHER PRIME EXCEPT A.

[IX. 12]

FOR,

IF ANY OTHER PRIME NUMBER MEASURES F,

AND,

F MEASURES D, THAT OTHER WILL, ALSO, MEASURE D;

SO THAT,

IT WILL, ALSO, MEASURE A, WHICH IS PRIME,

THOUGH IT IS NOT THE SAME WITH IT:

WHICH,

IS IMPOSSIBLE.

THEREFORE,

A MEASURES F.

AND, SINCE,

E MEASURES D ACCORDING TO F,

THEREFORE,

E, BY MULTIPLYING F, HAS MADE D.

[IX. 11]

BUT, FURTHER,

A, HAS ALSO, BY MULTIPLYING C, MADE D;
THEREFORE,
THE PRODUCT, OF A, C, =
THE PRODUCT, OF E, F.

[VII. 19]
THEREFORE, PROPORTIONALLY,
AS A IS TO E,
SO IS F TO C.
BUT,
A MEASURES E;
THEREFORE,
F, ALSO, MEASURES C.
LET,
IT MEASURE IT ACCORDING TO G.
SIMILARLY, THEN, WE CAN PROVE THAT;
G IS NOT THE SAME WITH ANY OF THE NUMBERS, A, B,
AND THAT,
IT IS MEASURED BY A.
AND, SINCE,
F MEASURES C, ACCORDING TO G,
THEREFORE,
F, BY MULTIPLYING G, HAS MADE C.

[IX. 11]
BUT, FURTHER,
A, HAS ALSO, BY MULTIPLYING B, MADE C;
THEREFORE,
THE PRODUCT, OF A, B, =
THE PRODUCT, OF F, G.

[VII. 19]
THEREFORE, PROPORTIONALLY,
AS A IS TO F,
SO IS G TO B.
BUT,
A MEASURES F;
THEREFORE,
G, ALSO, MEASURES B.
LET,
IT MEASURE IT ACCORDING TO H.
SIMILARLY THEN WE CAN PROVE THAT;
H IS NOT THE SAME WITH A.
AND, SINCE,
G MEASURES B, ACCORDING TO H,
THEREFORE,

G, BY MULTIPLYING H, HAS MADE B.

[IX. 8]

BUT FURTHER,

A, HAS ALSO, BY MULTIPLYING ITSELF, MADE B;

THEREFORE,

THE PRODUCT, OF H, G, =

THE SQUARE, ON A.

[VII. 19]

THEREFORE,

AS H IS TO A,

SO IS A TO G.

BUT,

A MEASURES G;

THEREFORE,

H, ALSO, MEASURES A, WHICH IS PRIME,

THOUGH IT IS NOT THE SAME WITH IT:

WHICH,

IS ABSURD.

THEREFORE,

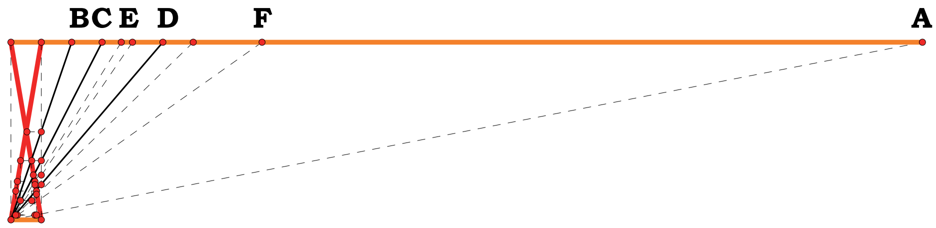
D, THE GREATEST WILL NOT BE MEASURED

BY ANY OTHER NUMBER EXCEPT A, B, C.

Q. E. D.

PROPOSITION 14.

IF A NUMBER BE THE LEAST THAT IS MEASURED BY PRIME NUMBERS, IT WILL NOT BE MEASURED BY ANY OTHER PRIME NUMBER EXCEPT THOSE ORIGINALLY MEASURING IT.



FOR LET,

THE NUMBER, *A*, BE THE LEAST THAT IS MEASURED BY
THE PRIME NUMBERS, *B*, *C*, *D*;

I SAY THAT;

A WILL NOT BE MEASURED BY
ANY OTHER PRIME NUMBER EXCEPT, *B*, *C*, *D*.

FOR, IF POSSIBLE, LET,

IT BE MEASURED BY THE PRIME NUMBER *E*,

AND LET,

E NOT BE THE SAME WITH ANY ONE OF
THE NUMBERS, *B*, *C*, *D*.

NOW, SINCE,

E MEASURES *A*,

LET,

IT MEASURE IT ACCORDING TO *F*;

THEREFORE,

E, BY MULTIPLYING *F*, HAS MADE *A*.

AND,

A IS MEASURED BY THE PRIME NUMBERS, *B*, *C*, *D*.

BUT,

IF TWO NUMBERS, BY MULTIPLYING ONE ANOTHER,
MAKE SOME NUMBER,

[VII. 30]

AND,

ANY PRIME NUMBER MEASURE THE PRODUCT,
IT WILL, ALSO, MEASURE ONE OF THE ORIGINAL NUMBERS;

THEREFORE,

B, *C*, *D* WILL MEASURE ONE OF THE NUMBERS, *E*, *F*.

NOW,

THEY WILL NOT MEASURE *E*;
FOR *E* IS PRIME, AND
NOT THE SAME WITH ANY ONE OF THE NUMBERS, *B*, *C*, *D*.

THEREFORE,

THEY WILL MEASURE F, WHICH IS LESS THAN A:

WHICH,

IS IMPOSSIBLE,

FOR A IS, BY HYPOTHESIS,

THE LEAST NUMBER MEASURED BY B, C, D.

THEREFORE,

NO PRIME NUMBER WILL MEASURE A, EXCEPT, B, C, D.

Q. E. D.

Proposition 15.

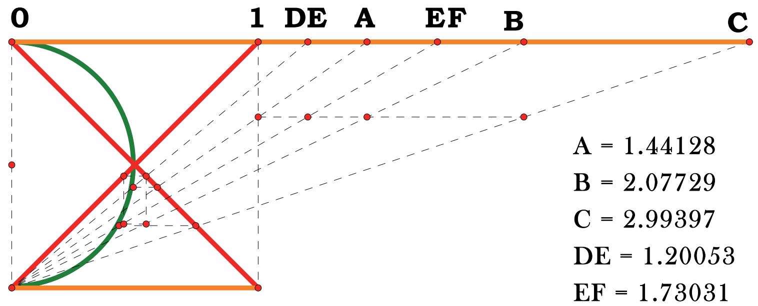
If three numbers in continued proportion be the least of those which have the same ratio with them, any two whatever added together will be prime to the remaining number.

$\frac{A}{B} = 0.69383$ $\frac{DE}{EF} = 0.69383$

$DE^2\text{-}A = 0.00000$

$DE \cdot EF\text{-}B = 0.00000$

$EF^2\text{-}C = 0.00000$



A = 1.44128
B = 2.07729
C = 2.99397
DE = 1.20053
EF = 1.73031

LET,

A, *B*, *C*, THREE NUMBERS IN CONTINUED PROPORTION,
BE THE LEAST OF THOSE WHICH HAVE
THE SAME RATIO WITH THEM;

I SAY THAT;

ANY TWO OF THE NUMBERS, *A*, *B*, *C*, WHATEVER,
ADDED TOGETHER, ARE PRIME TO THE REMAINING NUMBER,

NAMELY,

A, *B* TO *C*;
B, *C* TO *A*; AND FURTHER
A, *C* TO *B*.

[VIII. 2]

FOR LET,

TWO NUMBERS, *DE*, *EF*, THE LEAST OF THOSE WHICH HAVE
THE SAME RATIO WITH *A*, *B*, *C*, BE TAKEN.

[VIII. 2]

IT IS THEN MANIFEST THAT;

DE, BY MULTIPLYING ITSELF, HAS MADE *A*, AND
BY MULTIPLYING *EF*, HAS MADE *B*, AND, FURTHER
EF, BY MULTIPLYING ITSELF, HAS MADE *C*.

[VII. 22]

NOW, SINCE,

DE, *EF* ARE LEAST, THEY ARE PRIME TO ONE ANOTHER.

[VII. 28]

BUT,

IF TWO NUMBERS BE PRIME TO ONE ANOTHER,
THEIR SUM IS, ALSO, PRIME TO EACH;

THEREFORE,

DF IS, ALSO, PRIME TO EACH, OF THE NUMBERS, *DE*, *EF*.

BUT FURTHER,

DE IS, ALSO, PRIME TO *EF*;

THEREFORE,

DF, *DE* ARE PRIME TO *EF*.

[VII. 24]

BUT,

IF TWO NUMBERS BE PRIME TO ANY NUMBER,
THEIR PRODUCT IS, ALSO, PRIME TO THE OTHER;

SO THAT,

THE PRODUCT, OF *FD*, *DE*, IS PRIME TO *EF*;

[VII. 25]

HENCE,

THE PRODUCT, OF *FD*, *DE*, IS, ALSO, PRIME TO
THE SQUARE, ON *EF*.

[II. 3]

BUT,

THE PRODUCT, OF *FD*, *DE*, IS THE SQUARE, ON *DE*,
TOGETHER WITH THE PRODUCT, OF *DE*, *EF*;

THEREFORE,

THE SQUARE, ON *DE*, TOGETHER WITH
THE PRODUCT, OF *DE*, *EF*, IS PRIME TO THE SQUARE, ON *EF*.

AND,

THE SQUARE, ON *DE*, IS *A*,
THE PRODUCT, OF *DE*, *EF*, IS *B*, AND
THE SQUARE, ON *EF*, IS *C*;

THEREFORE,

A, *B* ADDED TOGETHER ARE PRIME TO *C*.

SIMILARLY WE CAN PROVE THAT;

B, *C* ADDED TOGETHER ARE PRIME TO *A*.

I SAY NEXT THAT;

A, *C* ADDED TOGETHER ARE, ALSO, PRIME TO *B*.

[VII. 24, 25]

FOR, SINCE,

DF IS PRIME TO EACH, OF THE NUMBERS, *DE*, *EF*,
THE SQUARE, ON *DF*, IS, ALSO, PRIME TO
THE PRODUCT, OF *DE*, *EF*.

[II. 4]

BUT,

THE SQUARES, ON *DE*, *EF*, TOGETHER WITH TWICE
THE PRODUCT, OF *DE*, *EF*, ARE EQUAL TO

THE SQUARE, ON DF;
THEREFORE,
THE SQUARES, ON DE, EF, TOGETHER WITH TWICE THE PRODUCT, OF DE, EF, ARE PRIME TO THE PRODUCT, OF DE, EF.

SEPARANDO,
THE SQUARES, ON DE, EF, TOGETHER WITH ONCE THE PRODUCT, OF DE, EF, ARE PRIME TO THE PRODUCT, OF DE, EF.

THEREFORE, SEPARANDO AGAIN,
THE SQUARES, ON DE, EF, ARE PRIME TO THE PRODUCT, OF DE, EF.

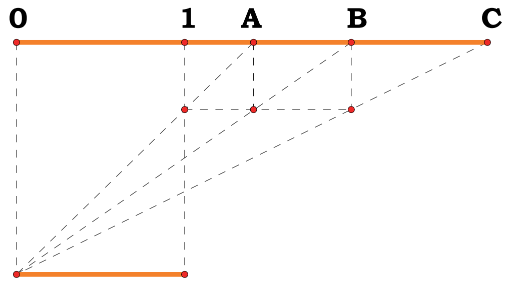
AND,
THE SQUARE, ON DE, IS A,
THE PRODUCT, OF DE, EF, IS B, AND
THE SQUARE, ON EF, IS C.

THEREFORE,
A, C ADDED TOGETHER ARE PRIME TO B.

Q. E. D.

Proposition 16.

If two numbers be prime to one another, the second will not be to any other number as the first is to the second.



For let,

the two numbers, A, B, be prime to one another;

I say that;

B is not to any other number as A is to B.

For, if possible, let,

as A is to B,

so B be to C.

[vii. 21]

Now,

A, B are prime, primes are, also, least,

[vii. 20]

and,

the least numbers measure those which have the same ratio the same number of times, the antecedent the antecedent and the consequent the consequent;

therefore,

A measures B, as antecedent antecedent.

But,

it, also, measures itself;

therefore,

A measures A, B, which are prime to one another:

which,

is absurd.

Therefore,

B will not be to C, as A is to B.

Q. E. D.

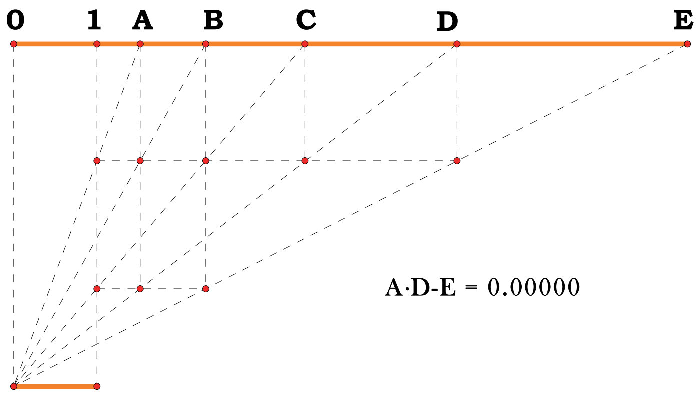
Proposition 17.

If there be as many numbers as we please in continued proportion, and the extremes of them be prime to one another, the last will not be to any other number as the first to the second.

A = 1.51923
B = 2.30806
C = 3.50648
D = 5.32715
E = 8.09317

$\frac{A}{B} = 0.65823$ $\frac{A}{D} = 0.28519$ $\frac{B}{C} = 0.65823$

$\frac{D}{E} = 0.65823$ $\frac{B}{E} = 0.28519$ $\frac{C}{D} = 0.65823$



For let,

there be as many numbers as we please,

A, *B*, *C*, *D*, in continued proportion,

and let,

the extremes of them, *A*, *D*, be prime to one another;

I say that;

D is not to any other number as *A* is to *B*.

For, if possible let,

as *A* is to *B*,

so *D* be to *E*

[viii. 13]

therefore, alternately,

as *A* is to *D*,

so is *B* to *E*.

[vii. 21]

But,

A, *D* are prime, primes are, also, least,

[vii. 20]

and,

the least numbers measure those which have
the same ratio the same number of times,
the antecedent the antecedent and
the consequent the consequent.

Therefore,

A MEASURES *B*.

AND,

AS *A* IS TO *B*,

SO IS *B* TO *C*.

THEREFORE,

B, ALSO, MEASURES *C*;

SO THAT,

A, ALSO, MEASURES *C*.

AND SINCE,

AS *B* IS TO *C*,

SO IS *C* TO *D*, AND

B MEASURES *C*,

THEREFORE,

C, ALSO, MEASURES *D*.

BUT,

A MEASURED *C*;

SO THAT,

A, ALSO, MEASURES *D*.

BUT,

IT, ALSO, MEASURES ITSELF;

THEREFORE,

A MEASURES *A*, *D*, WHICH ARE PRIME TO ONE ANOTHER:

WHICH,

IS IMPOSSIBLE.

THEREFORE,

D WILL NOT BE TO ANY OTHER NUMBER AS *A* IS TO *B*.

Q. E. D.

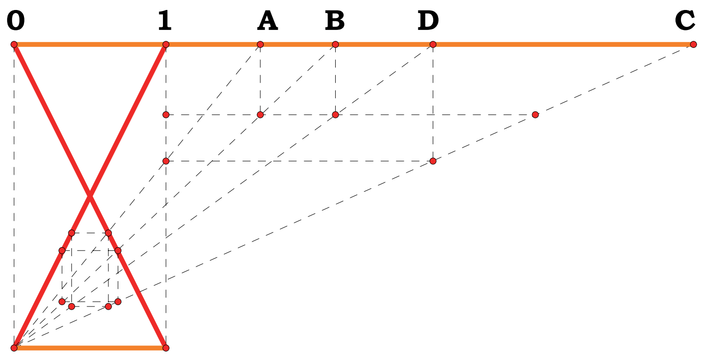
PROPOSITION 18.

GIVEN TWO NUMBERS, TO INVESTIGATE WHETHER IT IS POSSIBLE TO FIND A THIRD PROPORTIONAL TO THEM.

A = 1.62105
B = 2.11579
C = 4.47657
D = 2.76152

B^2-C = 0.00000
A·D-C = 0.00000
A·D-B^2 = 0.00000

$\frac{A}{B} - \frac{B}{D} = 0.00000$



LET,

A, *B* BE THE GIVEN TWO NUMBERS,

AND LET IT BE REQUIRED,

TO INVESTIGATE WHETHER

IT IS POSSIBLE TO FIND A THIRD PROPORTIONAL TO THEM.

NOW,

A, *B* ARE EITHER PRIME TO ONE ANOTHER OR NOT.

[IX. 16]

AND,

IF THEY ARE PRIME TO ONE ANOTHER,

IT HAS BEEN PROVED THAT

IT IS IMPOSSIBLE TO FIND A THIRD PROPORTIONAL TO THEM.

NEXT, LET,

A, *B* NOT BE PRIME TO ONE ANOTHER,

AND LET,

B, BY MULTIPLYING ITSELF, MAKE *C*.

THEN,

A EITHER MEASURES *C*, OR DOES NOT MEASURE IT.

FIRST, LET,

IT MEASURE IT ACCORDING TO *D*;

THEREFORE,

A, BY MULTIPLYING *D*, HAS MADE *C*.

BUT, FURTHER,

B, HAS ALSO, BY MULTIPLYING ITSELF, MADE *C*;

THEREFORE,

THE PRODUCT, OF *A*, *D*, =

THE SQUARE, ON *B*.

[VII. 19]
THEREFORE,
AS A IS TO B,
SO IS B TO D;
THEREFORE,
A THIRD PROPORTIONAL NUMBER, D,
HAS BEEN FOUND TO A, B.
NEXT, LET,
A NOT MEASURE C;
I SAY THAT;
IT IS IMPOSSIBLE TO FIND
A THIRD PROPORTIONAL NUMBER, TO A, B.
FOR, IF POSSIBLE, LET,
D, SUCH THIRD PROPORTIONAL, HAVE BEEN FOUND.
THEREFORE,
THE PRODUCT, OF A, D, =
THE SQUARE, ON B.
BUT,
THE SQUARE, ON B, IS C;
THEREFORE,
THE PRODUCT, OF A, D, = C.
HENCE,
A, BY MULTIPLYING D, HAS MADE C;
THEREFORE,
A MEASURES C, ACCORDING TO D.
BUT, BY HYPOTHESIS,
IT, ALSO, DOES NOT MEASURE IT:
WHICH,
IS ABSURD.
THEREFORE,
IT IS NOT POSSIBLE TO FIND
A THIRD PROPORTIONAL NUMBER, TO A, B, WHEN
A DOES NOT MEASURE C.

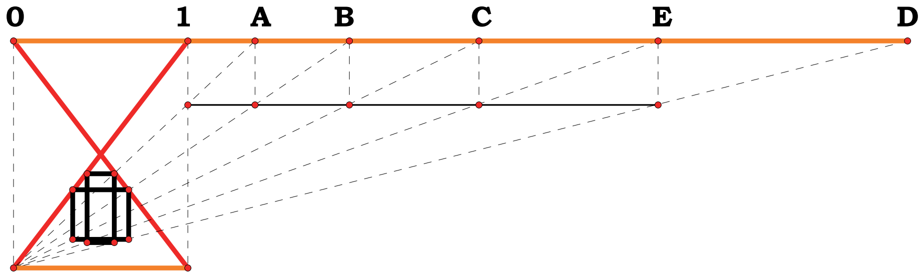
Q. E. D.

Proposition 19.

Given three numbers, to investigate when it is possible to find a fourth proportional to them.

A = 1.38694
B = 1.92360
C = 2.66792
E = 3.70024
D = 5.13200

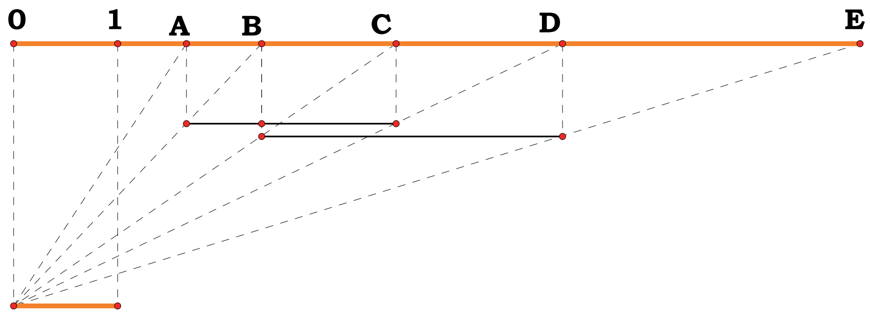
$B \cdot C - D = 0.00000$
$A \cdot E - B \cdot C = 0.00000$
$\frac{A}{B} - \frac{C}{E} = 0.00000$



A = 1.66154
B = 2.38462
C = 3.67692
D = 5.27707
E = 8.13689

$\frac{A}{B} = 0.69677$ $\quad \frac{B}{C} = 0.64854$ $\quad \frac{A}{C} = 0.45188$

$\frac{C}{D} = 0.69677$ $\quad \frac{D}{E} = 0.64854$ $\quad \frac{C}{E} = 0.45188$



Let,

A, *B*, *C*, be the given three numbers,

and let it be required,

to investigate when it is possible to find
a fourth proportional to them.

Now,

either they are not in continued proportion, and
the extremes of them are prime to one another;

or

they are in continued proportion, and
the extremes of them are not prime to one another;

or

they are not in continued proportion,
nor are the extremes of them prime to one another;

or

they are in continued proportion, and
the extremes of them are prime to one another.

IF THEN,

A, B C ARE IN CONTINUED PROPORTION,

[IX. 17]

AND,

THE EXTREMES OF THEM, A, C, ARE PRIME TO ONE ANOTHER,
IT HAS BEEN PROVED THAT IT IS IMPOSSIBLE
TO FIND A FOURTH PROPORTIONAL NUMBER TO THEM.

NEXT, LET,

A, B C NOT BE IN CONTINUED PROPORTION,
THE EXTREMES BEING AGAIN PRIME TO ONE ANOTHER;

I SAY THAT;

IN THIS CASE ALSO,
IT IS IMPOSSIBLE TO FIND A FOURTH PROPORTIONAL TO THEM.

FOR, IF POSSIBLE, LET,

D HAVE BEEN FOUND, SO THAT,
AS A IS TO B,
SO IS C TO D,

AND LET,

IT BE CONTRIVED THAT,
AS B IS TO C,
SO IS D TO E.

NOW, SINCE,

AS A IS TO B,
SO IS C TO D, AND
AS B IS TO C,
SO IS D TO E

[VII. 14]

THEREFORE, *EX AEQUALI*,

AS A IS TO C,
SO IS C TO E.

[VII. 21]

BUT,

A, C ARE PRIME, PRIMES ARE, ALSO, LEAST,

[VII. 20]

AND,

THE LEAST NUMBERS MEASURE THOSE WHICH HAVE
THE SAME RATIO,
THE ANTECEDENT THE ANTECEDENT AND
THE CONSEQUENT THE CONSEQUENT.

THEREFORE,

A MEASURES C AS ANTECEDENT, ANTECEDENT.

BUT IT, ALSO, MEASURES ITSELF;

THEREFORE,

A MEASURES *A*, *C*,

WHICH ARE PRIME TO ONE ANOTHER:

WHICH,

IS IMPOSSIBLE.

THEREFORE,

IT IS NOT POSSIBLE

TO FIND A FOURTH PROPORTIONAL, TO *A*, *B*, *C*.

NEXT, LET,

A, *B*, *C* BE AGAIN IN CONTINUED PROPORTION,

BUT LET,

A, *C* NOT BE PRIME TO ONE ANOTHER.

I SAY THAT;

IT IS POSSIBLE TO FIND A FOURTH PROPORTIONAL TO THEM.

FOR LET,

B, BY MULTIPLYING *C*, MAKE *D*;

THEREFORE,

A EITHER MEASURES *D*, OR DOES NOT MEASURE IT.

FIRST, LET,

IT MEASURE IT ACCORDING TO *E*;

THEREFORE,

A, BY MULTIPLYING *E*, HAS MADE *D*.

BUT, FURTHER,

B, HAS ALSO, BY MULTIPLYING *C*, MADE *D*;

THEREFORE,

THE PRODUCT, OF *A*, *E*, =

THE PRODUCT, OF *B*, *C*;

[VII. 19]

THEREFORE, PROPORTIONALLY,

AS *A* IS TO *B*,

SO IS *C* TO *E*;

THEREFORE,

E HAS BEEN FOUND A FOURTH PROPORTIONAL, TO *A*, *B*, *C*.

NEXT, LET,

A NOT MEASURE *D*;

I SAY THAT;

IT IS IMPOSSIBLE TO FIND

A FOURTH PROPORTIONAL NUMBER, TO *A*, *B*, *C*.

FOR, IF POSSIBLE, LET,

E HAVE BEEN FOUND;

[VII. 19]

THEREFORE,

THE PRODUCT, OF *A*, *E*, =

THE PRODUCT, OF *B*, *C*.

BUT,

THE PRODUCT, OF B, C, IS D;

THEREFORE,

THE PRODUCT, OF A, E, = D.

THEREFORE,

A, BY MULTIPLYING E, HAS MADE D;

THEREFORE,

A MEASURES D ACCORDING TO E,

SO THAT,

A MEASURES D.

BUT,

IT, ALSO, DOES NOT MEASURE IT:

WHICH,

IS ABSURD.

THEREFORE,

IT IS NOT POSSIBLE TO FIND
A FOURTH PROPORTIONAL NUMBER, TO A, B, C, WHEN
A DOES NOT MEASURE D.

NEXT, LET,

A, B, C NOT BE IN CONTINUED PROPORTION,

NOR,

THE EXTREMES PRIME TO ONE ANOTHER.

AND LET,

B, BY MULTIPLYING C, MAKE D.

SIMILARLY THEN, IT CAN BE PROVED THAT;

IF A MEASURES D,
IT IS POSSIBLE TO FIND A FOURTH PROPORTIONAL TO THEM,

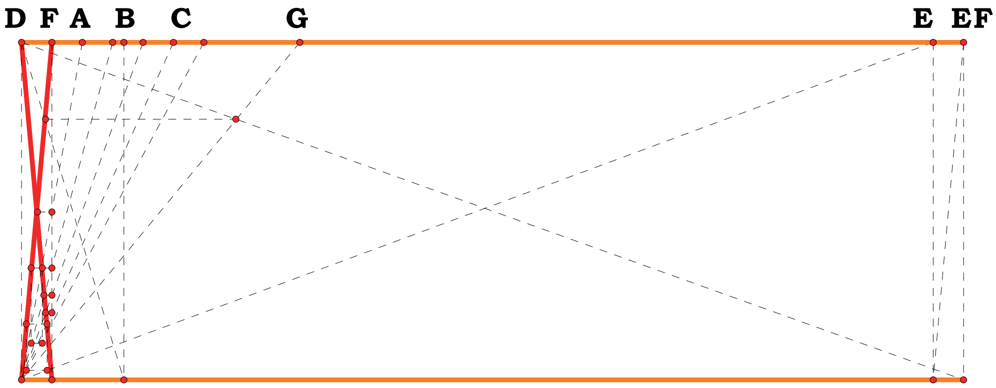
BUT,

IF IT DOES NOT MEASURE IT, IMPOSSIBLE.

Q. E. D.

Proposition 20.

Prime numbers are more than any assigned multitude of prime numbers.



Let,

A, *B*, *C*, be the assigned prime numbers;

I say that;

there are more prime numbers, than *A*, *B*, *C*.

For let,

the least number, measured by *A*, *B*, *C*, be taken,

and let,

it be *DE*;

let,

the unit, *DF*, be added, to *DE*.

Then,

EF is either prime or not.

First, let,

it be prime;

then,

the prime numbers, *A*, *B*, *C*, *EF*, have been found which are more than *A*, *B*, *C*.

Next, let,

EF not be prime;

[vii. 31]

therefore,

it is measured by some prime number.

Let,

it be measured by the prime number, *G*.

I say that;

G is not the same with any of the numbers, *A*, *B*, *C*.

For, if possible, let,

it be so.

Now,

A, *B*, *C* measure *DE*;

therefore,

G, ALSO, WILL MEASURE *DE*.

BUT,

IT, ALSO, MEASURES *EF*.

THEREFORE,

G BEING A NUMBER, WILL MEASURE THE REMAINDER,
THE UNIT, *DF*:

WHICH,

IS ABSURD.

THEREFORE,

G IS NOT THE SAME WITH ANY ONE OF
THE NUMBERS, *A*, *B*, *C*.

AND,

BY HYPOTHESIS IT IS PRIME.

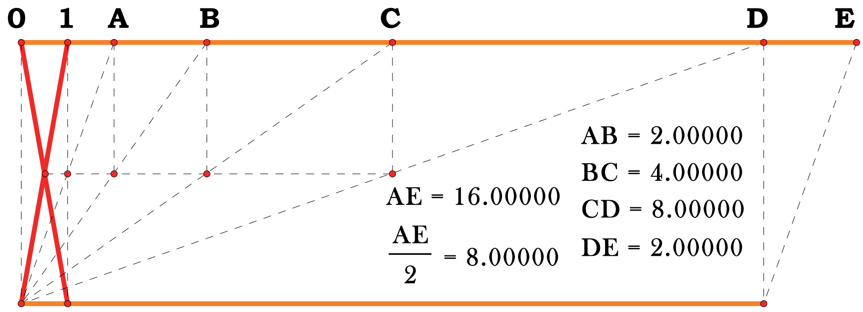
THEREFORE,

THE PRIME NUMBERS, *A*, *B*, *C*, *G*, HAVE BEEN FOUND
WHICH ARE MORE THAN THE ASSIGNED MULTITUDE OF *A*, *B*, *C*.

Q. E. D.

Proposition 21.

If as many even numbers as we please be added together, the whole is even.



For let,

as many even numbers as we please,

AB, *BC*, *CD*, *DE*, be added together;

I say that;

the whole, *AE*, is even.

[VII. Def. 6]

For, since,

each, of the numbers, *AB*, *BC*, *CD*, *DE*, is even,

it has a half part;

so that,

the whole, *AE*, also, has a half part.

[*Id.*]

But,

an even number is that

which is divisible into two equal parts;

therefore,

AE is even.

Q. E. D.

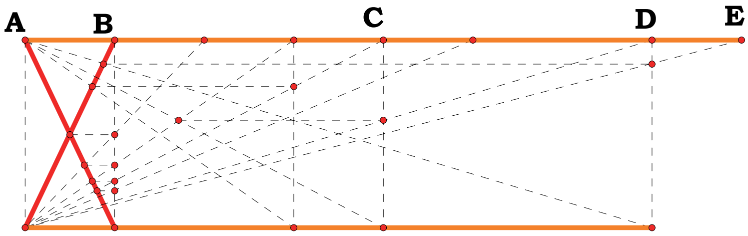
PROPOSITION 22.

IF AS MANY ODD NUMBERS AS WE PLEASE BE ADDED TOGETHER, AND THEIR MULTITUDE BE EVEN, THE WHOLE WILL BE EVEN.

AB = 1.00000
BC = 3.00000
CD = 3.00000
DE = 1.00000

$$\frac{AB+BC+CD+DE}{2} = 4.00000$$

$$(AB+BC+CD+DE)-4 = 4.00000$$



FOR LET,

AS MANY ODD NUMBERS AS WE PLEASE,
AB, *BC*, *CD*, *DE*, EVEN IN MULTITUDE,
BE ADDED TOGETHER;

I SAY THAT;

THE WHOLE, *AE*, IS EVEN.

[VII. DEF. 7]

FOR, SINCE,

EACH, OF THE NUMBERS, *AB*, *BC*, *CD*, *DE*, IS ODD,
IF AN UNIT BE SUBTRACTED FROM EACH,
EACH, OF THE REMAINDERS WILL BE EVEN;

[IX. 21]

SO THAT,

THE SUM OF THEM WILL BE EVEN.

BUT,

THE MULTITUDE OF THE UNITS IS, ALSO, EVEN.

[IX. 21]

THEREFORE,

THE WHOLE, *AE*, IS, ALSO, EVEN.

Q. E. D.

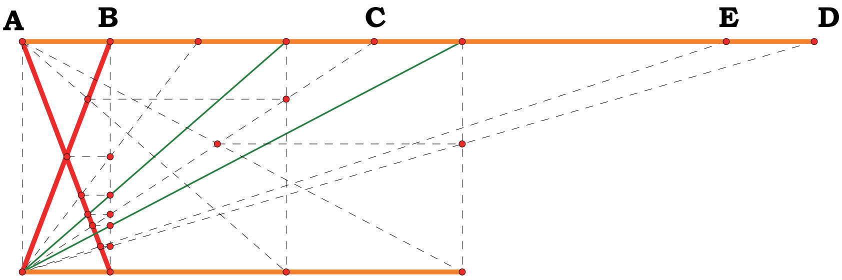
Proposition 23.

If as many odd numbers as we please be added together, and their multitude be odd, the whole will, also, *be odd.*

AB = 1.16417 cm
AD = 10.47750 cm
BC = 3.49250 cm
CD = 5.82083 cm
AC = 4.65667 cm
AE = 9.31333 cm

$\frac{AD}{AB} = 9.00000$ $\frac{AC}{AB} = 4.00000$

$\frac{CD}{AB} = 5.00000$ $\frac{AE}{AB} = 8.00000$



For let,

as many odd numbers as we please,

AB, *BC*, *CD*, the multitude of which is odd,

be added together;

I say that;

the whole, *AD*, is, also, odd.

Let,

the unit, *DE*, be subtracted from *CD*;

[VII. Def. 7]

therefore,

the remainder, *CE*, is even.

[IX. 22]

But,

CA is, also, even;

[IX. 21]

therefore,

the whole, *AE*, is, also, even.

And,

DE is an unit.

[VII. Def. 7]

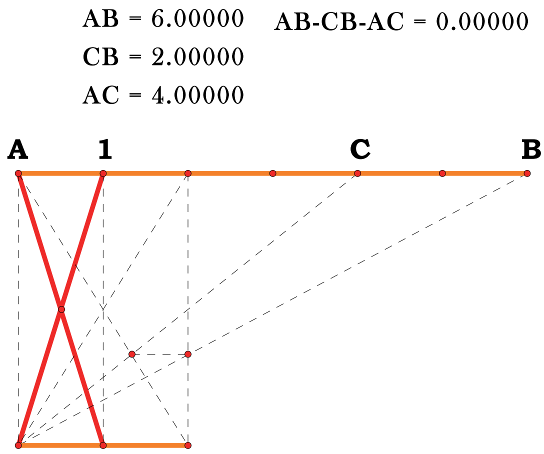
Therefore,

AD is odd.

Q. E. D.

Proposition 24.

If from an even number an even number be subtracted, the remainder will be even.



For let,

from the even number, *AB*,

the even number, *BC*, be subtracted:

I say that;

the remainder, *CA*, is even.

[vii. Def. 6]

For, since,

AB is even, it has a half part.

For,

the same reason, *BC*, also, has a half part;

so that,

the remainder, [*CA*, also, has a half part,

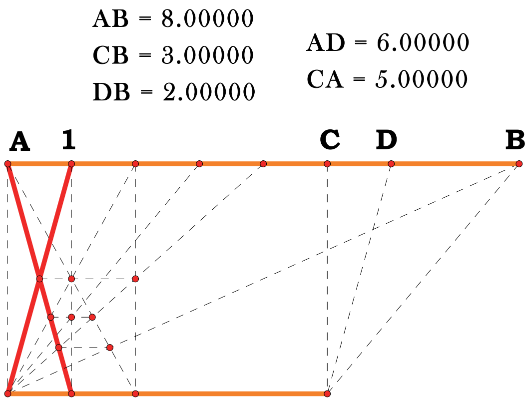
and],

AC is therefore even.

Q. E. D.

Proposition 25.

If from an even number an odd number be subtracted, the remainder will be odd.



For let,

from the even number, *AB*,

the odd number, *BC*, be subtracted;

I say that;

the remainder, *CA*, is odd.

For let,

the unit, *CD*, be subtracted from *BC*;

[vii. Def. 7]

therefore,

DB is even.

But,

AB is, also, even;

[ix. 24]

therefore,

the remainder, *AD*, is, also, even.

And,

CD is an unit;

[vii. Def. 7]

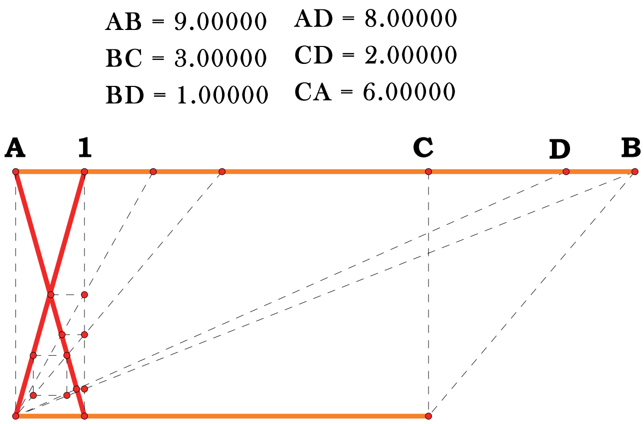
therefore,

CA is odd.

Q. E. D.

PROPOSITION 26.

IF FROM AN ODD NUMBER AN ODD NUMBER BE SUBTRACTED, THE REMAINDER WILL BE EVEN.



FOR LET,

FROM THE ODD NUMBER, *AB*,

THE ODD NUMBER, *BC*, BE SUBTRACTED;

I SAY THAT;

THE REMAINDER, *CA*, IS EVEN.

FOR, SINCE,

AB IS ODD,

LET,

THE UNIT, *BD*, BE SUBTRACTED;

[VII. DEF. 7]

THEREFORE,

THE REMAINDER, *AD*, IS EVEN.

[VII. DEF. 7]

FOR THE SAME REASON,

CD IS, ALSO, EVEN;

[IX. 24]

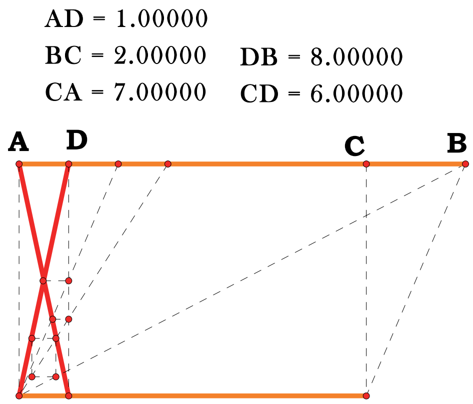
SO THAT,

THE REMAINDER, *CA*, IS, ALSO, EVEN.

Q. E. D.

PROPOSITION 27.

IF FROM AN ODD NUMBER AN EVEN NUMBER BE SUBTRACTED, THE REMAINDER WILL BE ODD.



FOR LET,

FROM THE ODD NUMBER, *AB*,

THE EVEN NUMBER, *BC*, BE SUBTRACTED;

I SAY THAT;

THE REMAINDER, *CA*, IS ODD.

LET,

THE UNIT, *AD*, BE SUBTRACTED;

[VII. DEF. 7]

THEREFORE,

DB IS EVEN.

BUT,

BC IS, ALSO, EVEN;

[IX. 24]

THEREFORE,

THE REMAINDER, *CD*, IS EVEN.

[VII. DEF. 7]

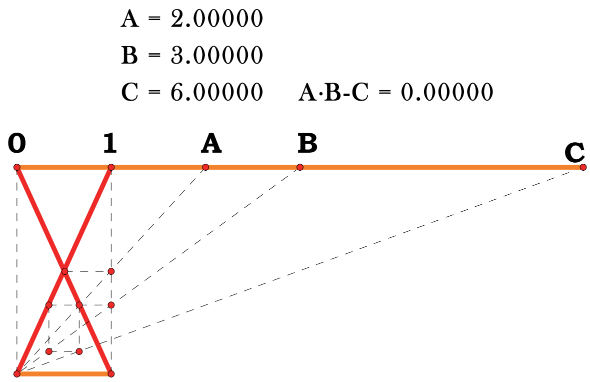
THEREFORE,

CA IS ODD.

Q. E. D.

PROPOSITION 28.

IF AN ODD NUMBER, BY MULTIPLYING AN EVEN NUMBER, MAKE SOME NUMBER, THE PRODUCT WILL BE EVEN.



FOR LET,

THE ODD NUMBER, *A*, BY MULTIPLYING
THE EVEN NUMBER, *B*, MAKE *C*;

I SAY THAT;

C IS EVEN.

FOR, SINCE,

A, BY MULTIPLYING *B*, HAS MADE *C*,

[VII. DEF. 15]
THEREFORE,

C IS MADE UP OF AS MANY NUMBERS EQUAL TO *B*,
AS THERE ARE UNITS IN *A*.

AND,

B IS EVEN;

THEREFORE,

C IS MADE UP OF EVEN NUMBERS.

[IX. 21]
BUT,

IF AS MANY EVEN NUMBERS AS WE PLEASE
BE ADDED TOGETHER, THE WHOLE IS EVEN.

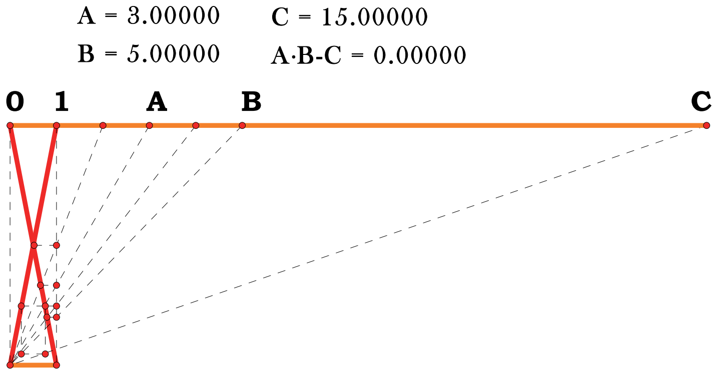
THEREFORE,

C IS EVEN.

Q. E. D.

PROPOSITION 29.

IF AN ODD NUMBER, BY MULTIPLYING AN ODD NUMBER, MAKE SOME NUMBER, THE PRODUCT WILL BE ODD.



FOR LET,

THE ODD NUMBER, A ,BY MULTIPLYING

THE ODD NUMBER, B, MAKE C;

I SAY THAT;

C IS ODD.

FOR, SINCE,

A, BY MULTIPLYING B, HAS MADE C,

[VII. DEF. 15]

THEREFORE,

C IS MADE UP OF AS MANY NUMBERS EQUAL TO B,

AS THERE ARE UNITS IN A.

AND,

EACH, OF THE NUMBERS, A, B, IS ODD;

THEREFORE,

C IS MADE UP OF ODD NUMBERS

THE MULTITUDE OF WHICH IS ODD.

[IX. 23]

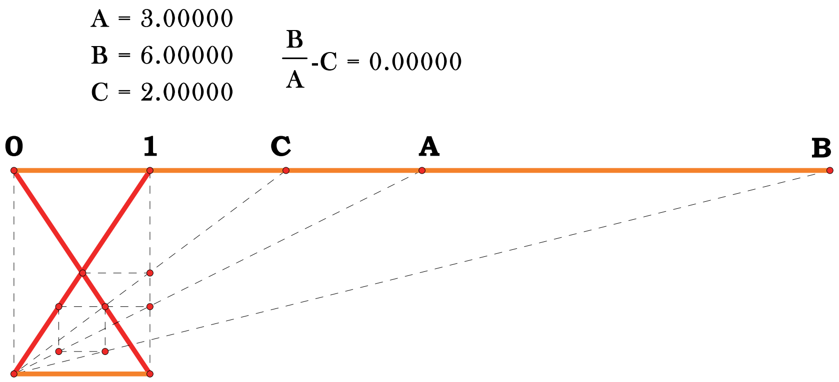
THUS,

C IS ODD.

Q. E. D.

PROPOSITION 30.

IF AN ODD NUMBER MEASURE AN EVEN NUMBER, IT WILL, ALSO, *MEASURE THE HALF OF IT.*



FOR LET,

THE ODD NUMBER, *A*, MEASURE THE EVEN NUMBER, *B*;

I SAY THAT;

IT WILL, ALSO, MEASURE THE HALF OF IT.

FOR, SINCE,

A MEASURES *B*,

LET,

IT MEASURE IT ACCORDING TO *C*;

I SAY THAT;

C IS NOT ODD.

FOR, IF POSSIBLE, LET,

IT BE SO.

THEN, SINCE,

A MEASURES *B*, ACCORDING TO *C*,

THEREFORE,

A, BY MULTIPLYING *C*, HAS MADE *B*.

THEREFORE,

B IS MADE UP OF ODD NUMBERS

THE MULTITUDE OF WHICH IS ODD.

[IX. 23]

THEREFORE,

B IS ODD:

WHICH,

IS ABSURD,

FOR,

BY HYPOTHESIS IT IS EVEN.

THEREFORE,

C IS NOT ODD;

THEREFORE,

C IS EVEN.

THUS,

A MEASURES *B*, AN EVEN NUMBER OF TIMES.

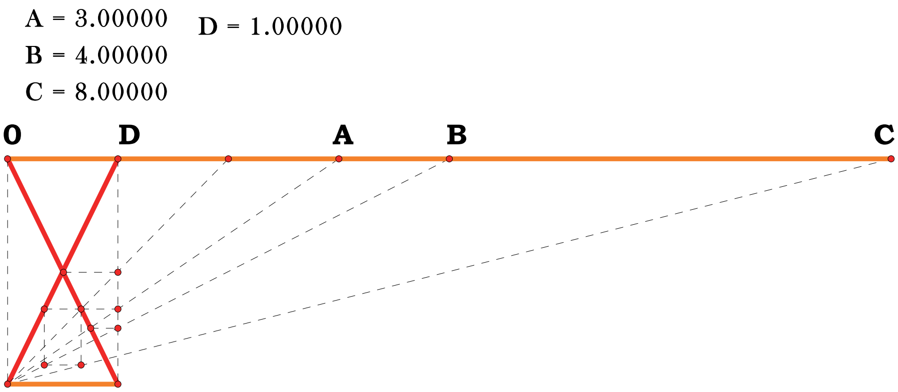
FOR,

THIS REASON THEN IT, ALSO, MEASURES THE HALF OF IT.

Q. E. D.

Proposition 31.

If an odd number be prime to any number, it will, also, *be prime to the double of it*.



For let,

the odd number, *A*, be prime to any number, *B*,

and let,

C be double of *B*;

I say that;

A is prime to *C*.

For,

if they are not prime to one another,

some number will measure them.

Let,

a number measure them,

and let,

it be *D*.

Now,

A is odd;

therefore,

D is, also, odd.

And since,

D, which is odd, measures *C*,

and,

C is even,

[ix. 30]

therefore,

[*D*] will measure the half of *C*, also.

But,

B is half of *C*;

therefore,

D measures *B*.

But,

it, also, measures *A*;

therefore,

D MEASURES *A*, *B*,

WHICH ARE PRIME TO ONE ANOTHER:

WHICH,

IS IMPOSSIBLE.

THEREFORE,

A CANNOT BUT BE PRIME TO *C*.

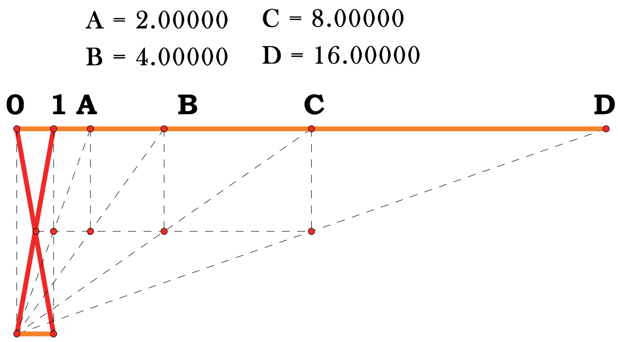
THEREFORE,

A, *C* ARE PRIME TO ONE ANOTHER.

Q. E. D.

Proposition 32.

Each, of the numbers which are continually doubled beginning from a dyad is even-times even only.



For let,

as many numbers as we please,

B, *C*, *D*, have been continually doubled beginning from the dyad, *A*;

I say that;

B, *C*, *D* are even-times even only.

Now,

that each, of

the numbers, *B*, *C*, *D*, is even-times even is manifest;

for it is doubled from a dyad.

I say that;

it is, also, even-times even only.

For let,

an unit be set out.

Since,

then as many numbers as we please,

beginning from an unit, are in continued proportion,

and,

the number, *A*, after the unit is prime,

[ix. 13]

therefore,

D, the greatest of the numbers, *A*, *B*, *C*, *D*,

will not be measured by any other number,

except, *A*, *B*, *C*.

And,

each, of the numbers, *A*, *B*, *C*, is even;

[vii. Def. 8]

therefore,

D is even-times even only.

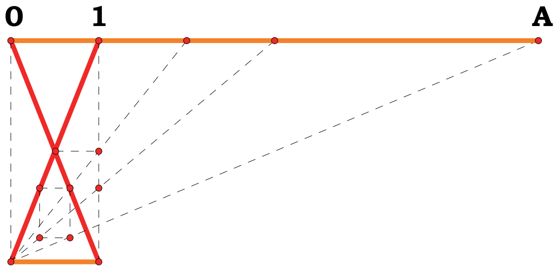
Similarly we can prove that;

each, of the numbers, *B*, *C*, is even-times even only.

Q. E, D.

Proposition 33.

If a number have its half odd, it is even-times odd only.



For let,

the number, *A*, have its half odd;

I say that;

A is even-times odd only.

Now,

that it is even-times odd is manifest;

[vii. Def. 9]

for,

the half of it, being odd,

measures it an even number of times.

I say next that;

it is, also, even-times odd only.

[vii. Def. 8]

For,

if *A* is even-times even, also,

it will be measured by an even number

according to an even number;

so that,

the half of it will, also, be measured by an even number

though it is odd:

which,

is absurd.

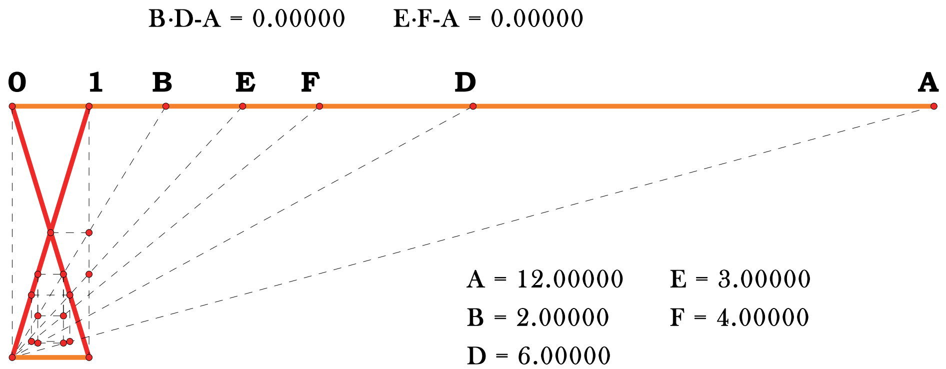
Therefore,

A is even-times odd only.

Q. E. D.

PROPOSITION 34.

IF A NUMBER NEITHER BE ONE OF THOSE WHICH ARE CONTINUALLY DOUBLED FROM A DYAD, NOR HAVE ITS HALF ODD, IT IS BOTH EVEN-TIMES EVEN AND EVEN-TIMES ODD.



FOR LET,

THE NUMBER, *A*, NEITHER BE ONE OF THOSE;

DOUBLED FROM A DYAD, NOR HAVE ITS HALF ODD;

I SAY THAT;

A IS BOTH EVEN-TIMES EVEN AND EVEN-TIMES ODD.

NOW,

THAT *A* IS EVEN-TIMES EVEN IS MANIFEST;

[VII. DEF. 8]

FOR,

IT HAS NOT ITS HALF ODD.

I SAY NEXT THAT;

IT IS, ALSO, EVEN-TIMES ODD.

FOR,

IF WE BISECT *A*, THEN BISECT ITS HALF, AND

DO THIS CONTINUALLY,

WE SHALL COME UPON SOME ODD NUMBER

WHICH WILL MEASURE *A* ACCORDING TO AN EVEN NUMBER.

FOR,

IF NOT, WE SHALL COME UPON A DYAD, AND

A WILL BE AMONG THOSE WHICH ARE DOUBLED FROM A DYAD:

WHICH,

IS CONTRARY TO THE HYPOTHESIS.

THUS,

A IS EVEN-TIMES ODD.

BUT,

IT WAS, ALSO, PROVED EVEN-TIMES EVEN.

THEREFORE,

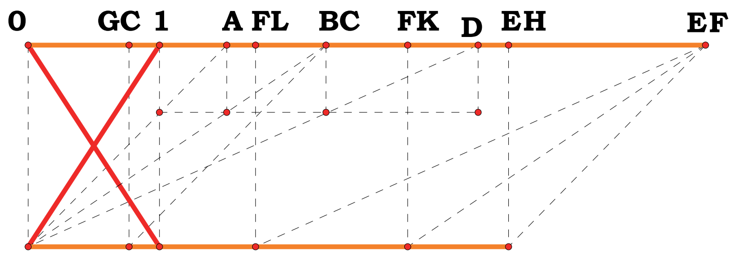
A IS BOTH EVEN-TIMES EVEN AND EVEN-TIMES ODD.

Q. E. D.

PROPOSITION 35.

IF AS MANY NUMBERS AS WE PLEASE BE IN CONTINUED PROPORTION, AND THERE BE SUBTRACTED FROM THE SECOND AND THE LAST NUMBERS EQUAL TO THE FIRST, THEN, AS THE EXCESS OF THE SECOND IS TO THE FIRST, SO WILL THE EXCESS OF THE LAST BE TO ALL THOSE BEFORE IT.

A = 1.50716	CG = 0.76437		
BC = 2.27152	FL = 3.42355	GC = 0.76437	FK-BC = 0.00000
D = 3.42355	HK = 0.76437	FH = 1.50716	FL-D = 0.00000
EF = 5.15982	BG = 1.50716	EL = 1.73628	BG-FH = 0.00000
EH = 3.65267	FK = 2.27152	LK = 1.15202	HK-GC = 0.00000



$$\frac{EF}{D} = 1.50716 \quad \frac{EF}{FL} = 1.50716 \quad \frac{CG}{A} = 0.50716 \quad \frac{LK}{FK} = 0.50716$$

$$\frac{D}{BC} = 1.50716 \quad \frac{FL}{FK} = 1.50716 \quad \frac{EH}{A+BC+D} = 0.50716 \quad \frac{HK}{FH} = 0.50716$$

$$\frac{BC}{A} = 1.50716 \quad \frac{FK}{FH} = 1.50716 \quad \frac{EL}{FL} = 0.50716 \quad \frac{EL+LK+HK}{FL+FK+FH} = 0.50716$$

LET,

THERE BE AS MANY NUMBERS AS WE PLEASE
IN CONTINUED PROPORTION, *A*, *BC*, *D*, *EF*,
BEGINNING FROM *A*, AS LEAST,

AND LET,

THERE BE SUBTRACTED, FROM *BC* AND *EF*,
THE NUMBERS, *BG*, *FH*,
EACH EQUAL TO *A*;

I SAY THAT;

AS *GC* IS TO *A*,
SO IS *EH* TO *A*, *BC*, *D*.

FOR LET,

FK BE MADE EQUAL TO *BC*, AND
FL EQUAL TO *D*.

THEN, SINCE,

FK = *BC*, AND
OF THESE THE PART, *FH*, = THE PART, *BG*,

THEREFORE,

THE REMAINDER, *HK*, = THE REMAINDER, *GC*.

AND SINCE,

AS *EF* IS TO *D*,

so is D to BC, and
BC to A,
while,
$D = FL$,
$BC = FK$, and
$A = FH$,
therefore,
as EF is to FL,
so is LF to FK, and
FK to FH.

[vii. 11, 13]
Separando,
as EL is to LF,
so is LK to FK, and
KH to FH.

[vii. 12]
Therefore also,
as one of the antecedents is to one of
the consequents,
so are all the antecedents to all the consequents;
therefore,
as KH is to FH,
so are EL, LK, KH to LF, FK, HF.
But,
$KH = CG$,
$FH = A$, and
LF, FK, $HF = D$, BC, A;
therefore,
as CG is to A,
so is EH to D, BC, A.
Therefore,
as the excess of the second is to the first,
so is the excess of the last to all those before it

Q. E. D.

PROPOSITION 36.

IF AS MANY NUMBERS AS WE PLEASE BEGINNING FROM AN UNIT BE SET OUT CONTINUOUSLY IN DOUBLE PROPORTION, UNTIL THE SUM OF ALL BECOMES PRIME, AND IF THE SUM MULTIPLIED INTO THE LAST MAKE SOME NUMBER, THE PRODUCT WILL BE PERFECT.

$1 = 1.00000$

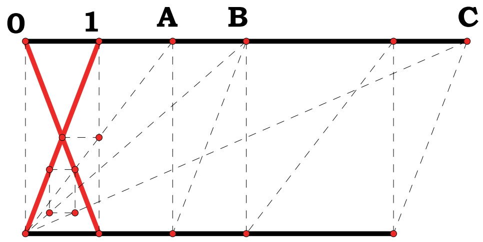
$A = 2.00000$

$B = 3.00000$

$C = 6.00000$

$$\frac{(1+A)\cdot A}{(B-2)+(B-1)+B} = 1.00000$$

$$\frac{6}{1+2+3} = 1.00000$$



$A = 4.00000$

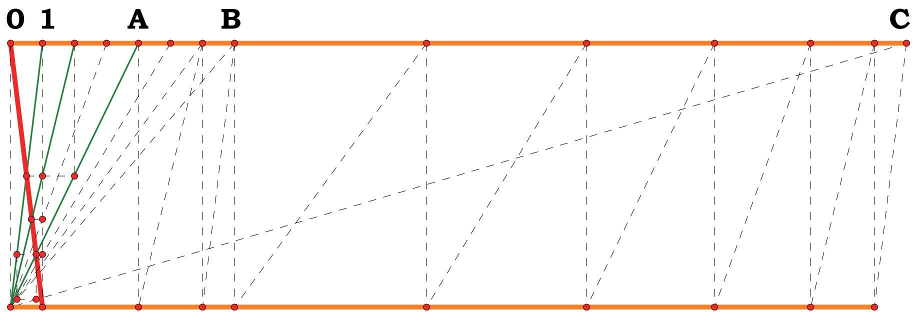
$B = 7.00000$

$C = 28.00000$

$$\frac{C}{B+(B-1)+(B-2)+(B-3)+(B-4)+(B-5)+(B-6)} = 1.00000$$

$$\frac{28}{1+2+3+4+5+6+7} = 1.00000 \qquad 1+2+4+7+14 = 28.00000$$

$$2^0+2^1+2^2+2^0\cdot 7+2^1\cdot 7 = 28.00000$$



FOR LET,

AS MANY NUMBERS AS WE PLEASE, *A*, *B*, *C*, *D*,
BEGINNING FROM AN UNIT BE SET OUT IN
DOUBLE PROPORTION, UNTIL THE SUM OF ALL BECOMES PRIME,

$A = 2.00000$

$D = 16.00000$

$C = 8.00000$

$E = 31.00000$

$M = 248.00000$

$FG = 496.00000$

$HK = 62.00000$

$NK = 31.00000$

$L = 124.00000$

$OG = 465.00000$

$FO = 31.00000$

$B = 4.00000$

$$\frac{E}{1+A+B+C+D} = 1.00000$$

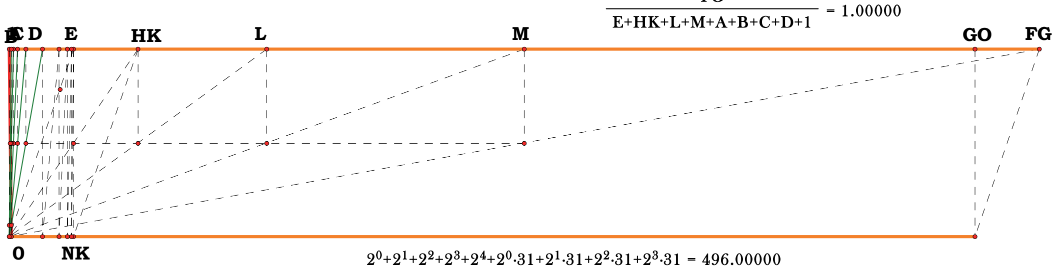
$$(A^5-1)\cdot A^4 = 496.00000$$

$$\frac{FG}{A} = 248.00000 \qquad \frac{FG}{B} = 124.00000 \qquad \frac{FG}{C} = 62.00000$$

$$\frac{FG}{D} = 31.00000 \qquad \frac{FG}{E} = 16.00000 \qquad \frac{FG}{HK} = 8.00000$$

$$\frac{FG}{L} = 4.00000 \qquad \frac{FG}{M} = 2.00000 \qquad A^4\cdot E = 496.00000$$

$$\frac{FG}{E+HK+L+M+A+B+C+D+1} = 1.00000$$



$$2^0+2^1+2^2+2^3+2^4+2^0\cdot 31+2^1\cdot 31+2^2\cdot 31+2^3\cdot 31 = 496.00000$$

$$\frac{A}{D} = 0.12500 \qquad \frac{E}{M} = 0.12500$$

$$E\cdot D-A\cdot M = 0.00000 \qquad E\cdot D-FG = 0.00000 \qquad A\cdot M-FG = 0.00000$$

$$\frac{FG}{M}-A = 0.00000 \qquad \frac{NK}{E} = 1.00000$$

$$\frac{OG}{M+L+HK+E} = 1.00000 \qquad \frac{FG}{E\cdot 2^4} = 1.00000$$

LET,

E BE EQUAL TO THE SUM,

AND LET,

E, BY MULTIPLYING *D*, MAKE *FG*;

I SAY THAT;

FG IS PERFECT.

FOR LET,

HOWEVER MANY, *A*, *B*, *C*, *D*, ARE IN MULTITUDE,
SO MANY, *E*, *HK*, *L*, *M*, BE TAKEN IN DOUBLE PROPORTION,
BEGINNING FROM *E*;

[VII. 14]

THEREFORE, *EX AEQUALI*,

AS *A* IS TO *D*,
SO IS *E* TO *M*.

[VII. 19]

THEREFORE,

THE PRODUCT, OF *E*, *D*, =
THE PRODUCT, OF *A*, *M*.

AND,

THE PRODUCT, OF *E*, *D*, IS *FG*;

THEREFORE,

THE PRODUCT, OF *A*, *M*, IS, ALSO, *FG*.

THEREFORE,

A, BY MULTIPLYING *M*, HAS MADE *FG*;

THEREFORE,

M MEASURES *FG*, ACCORDING TO THE UNITS IN *A*.

AND,

A IS A DYAD;

THEREFORE,

FG IS DOUBLE OF *M*.

BUT,

M, *L*, *HK*, *E* ARE CONTINUOUSLY DOUBLE OF EACH OTHER;

THEREFORE,

E, *HK*, *L*, *M*, *FG* ARE, CONTINUOUSLY PROPORTIONAL, IN DOUBLE PROPORTION.

NOW LET,

THERE BE SUBTRACTED FROM THE SECOND, *HK*, AND
THE LAST, *FG*, THE NUMBERS, *HN*, *FO*,
EACH EQUAL TO THE FIRST, *E*;

[IX. 35]

THEREFORE,

AS THE EXCESS OF THE SECOND IS TO THE FIRST,
SO IS THE EXCESS OF THE LAST TO ALL THOSE BEFORE IT.

THEREFORE,

as NK is to E,

so is OG to M, L, KH, E.

AND,

$NK = E$;

therefore,

$OG = M, L, HK, E$.

BUT,

$FO = E$, and

$E = A, B, C, D$ and the unit.

THEREFORE,

the whole, FG, =

E, HK, L, M and A, B, C, D and the unit; and

it is measured by them.

I say, also, that;

FG will not be measured by any other number,

except, A, B, C, D, E, HK, L, M, and the unit.

FOR, if possible, let,

some number, P, measure FG,

and let,

P not be the same with any of the numbers,

A, B, C, D, E, HK, L, M.

AND,

as many times as P measures FG,

so many units let there be in Q;

therefore,

Q, by multiplying P, has made FG.

BUT, further,

E, has also, by multiplying D, made FG;

[VII. 19]

therefore,

as E is to Q,

so is P to D.

AND, since,

A, B, C, D are continuously proportional

beginning from an unit,

[IX. 13]

therefore,

D will not be measured by any other number,

except, A, B, C.

AND, by hypothesis,

P is not the same with any of the numbers, A, B, C;

therefore,

P will not measure D.

BUT,

AS P IS TO D,
SO IS E TO Q;

[VII. DEF. 20]
THEREFORE,

NEITHER DOES E MEASURE Q.

AND,

E IS PRIME;

[VII. 29]
AND,

ANY PRIME NUMBER IS PRIME TO ANY NUMBER
WHICH IT DOES NOT MEASURE.

THEREFORE,

E, Q ARE PRIME TO ONE ANOTHER.

[VII. 21]
BUT,

PRIMES ARE, ALSO, LEAST,

[VII. 20]
AND,

THE LEAST NUMBERS MEASURE THOSE WHICH HAVE
THE SAME RATIO THE SAME NUMBER OF TIMES,
THE ANTECEDENT THE ANTECEDENT AND,
THE CONSEQUENT THE CONSEQUENT; AND
AS E IS TO Q,
SO IS P TO D;

THEREFORE,

E MEASURES P,
THE SAME NUMBER OF TIMES THAT Q MEASURES D.

BUT,

D IS NOT MEASURED BY ANY OTHER NUMBER,
EXCEPT, A, B, C;

THEREFORE,

Q IS THE SAME WITH ONE OF THE NUMBERS, A, B, C.

LET,

IT BE THE SAME WITH B.

AND LET,

HOWEVER MANY, B, C, D, ARE IN MULTITUDE,
SO MANY, E, HK, L, BE TAKEN BEGINNING FROM E.

NOW,

E, HK, L ARE IN THE SAME RATIO WITH B, C, D,

[VII. 14]
THEREFORE, *EX AEQUALI*,

AS B IS TO D,

so is E to L.

[vii. 19]
Therefore,
the product, of B, L, =
the product, of D, E.
But,
the product, of D, E, =
the product, of Q, P;
therefore,
the product, of Q, P, =
the product, of B, L.

[vii. 19]
Therefore,
as Q is to B,
so is L to P.
And,
Q is the same with B;
therefore,
L is, also, the same with P:
which,
is impossible,
for,
by hypothesis P is not
the same with any of the numbers set out.
Therefore,
no number will measure FG,
except, A, B, C, D, E, HK, L, M, and the unit.
And,
FG was proved
equal to, A, B, C, D, E, HK, Z, M, and the unit;

[vii. Def. 22]
and,
a perfect number is that
which = its own parts;
therefore,
FG is perfect.

Q. E. D.

BOOK X.

OF

EUCLID'S ELEMENTS

TRANSLATED FROM THE TEXT OF HEIBERG

BY

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2013 *EDITION*

REVISED WITH SUBTRACTIONS

REFORMATTED AND ABRIDGED FOR STUDY OF *THE ELEMENTS*.

BY JOHN CLARK.

[There are probably some mistakes in this due to the fact that I am not the sharpest tool in the shed. Trying to glean some understanding from Heath's notes is sometimes no use as the following quote from Book 10 Introduction demonstrates.

"(1) β is equal to $\frac{m^2}{n^2}(\alpha^2 + \beta)$, where m, n are integers, i.e. β is of the form

$$\frac{m^2}{n^2 - m^2}\alpha^2."$$

Now, I don't think one has to be smart to recognize gibberish when they see it. Who in their right mind defines a thing in terms of itself? Sometimes we spend too much time trying to appear smart instead of being clear.

And then, the icing on the cake, 'of the form such and such and not of the form such and such.' which means what? Absolutely nothing.]

BOOK X.

DEFINITIONS.

1. THOSE MAGNITUDES ARE SAID TO BE **COMMENSURABLE** WHICH ARE MEASURED BY THE SAME MEASURE, AND THOSE **INCOMMENSURABLE** WHICH CANNOT HAVE ANY COMMON MEASURE.

2. STRAIGHT LINES ARE **COMMENSURABLE IN SQUARE** WHEN THE SQUARES ON THEM ARE MEASURED BY THE SAME AREA, AND **INCOMMENSURABLE IN SQUARE** WHEN THE SQUARES ON THEM CANNOT POSSIBLY HAVE ANY AREA AS A COMMON MEASURE.

3. WITH THESE HYPOTHESES, IT IS PROVED THAT THERE EXIST STRAIGHT LINES INFINITE IN MULTITUDE WHICH ARE COMMENSURABLE AND INCOMMENSURABLE RESPECTIVELY, SOME IN LENGTH ONLY, AND OTHERS IN SQUARE ALSO, WITH AN ASSIGNED STRAIGHT LINE. LET THEN THE ASSIGNED STRAIGHT LINE BE CALLED **RATIONAL**, AND THOSE STRAIGHT LINES WHICH ARE COMMENSURABLE WITH IT, WHETHER IN LENGTH AND IN SQUARE OR IN SQUARE, ONLY, **RATIONAL**, BUT THOSE WHICH ARE INCOMMENSURABLE WITH IT **IRRATIONAL**.

4. AND LET THE SQUARE, ON THE ASSIGNED STRAIGHT LINE BE CALLED **RATIONAL** AND THOSE AREAS WHICH ARE COMMENSURABLE WITH IT **RATIONAL**, BUT THOSE WHICH ARE INCOMMENSURABLE WITH IT **IRRATIONAL**, AND THE STRAIGHT LINES WHICH PRODUCE THEM IRRATIONAL, THAT IS, IN CASE THE AREAS ARE SQUARES, THE SIDES THEMSELVES, BUT IN CASE THEY ARE ANY OTHER RECTILINEAL FIGURES, THE STRAIGHT LINES ON WHICH ARE DESCRIBED SQUARES EQUAL TO THEM.

NOTES.

DEFINITION 1. *THOSE MAGNITUDES ARE SAID TO BE* COMMENSURABLE *WHICH ARE MEASURED BY THE SAME MEASURE, AND THOSE* INCOMMENSURABLE *WHICH CANNOT HAVE ANY COMMON MEASURE.*

NOTES.

DEFINITION 2. *STRAIGHT LINES ARE* COMMENSURABLE IN SQUARE *WHEN THE SQUARES ON THEM ARE MEASURED BY THE SAME AREA, AND* INCOMMENSURABLE IN SQUARE *WHEN THE SQUARES ON THEM CANNOT POSSIBLY HAVE ANY AREA AS A COMMON MEASURE.*

NOTES.

DEFINITION 3. *WITH THESE HYPOTHESES, IT IS PROVED THAT THERE EXIST STRAIGHT LINES INFINITE IN MULTITUDE WHICH ARE COMMENSURABLE AND INCOMMENSURABLE RESPECTIVELY, SOME IN LENGTH ONLY, AND OTHERS IN SQUARE ALSO, WITH AN ASSIGNED STRAIGHT LINE. LET THEN THE ASSIGNED STRAIGHT LINE BE CALLED* RATIONAL, *AND THOSE STRAIGHT LINES WHICH ARE COMMENSURABLE WITH IT, WHETHER IN LENGTH AND IN SQUARE OR IN SQUARE, ONLY,* RATIONAL, *BUT THOSE WHICH ARE INCOMMENSURABLE WITH IT* IRRATIONAL.

NOTES.

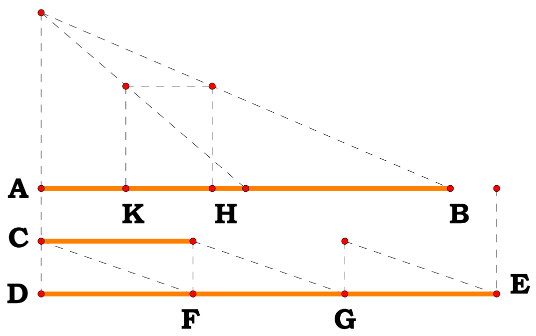
DEFINITION 4. *AND LET THE SQUARE, ON THE ASSIGNED STRAIGHT LINE BE CALLED* RATIONAL *AND THOSE AREAS WHICH ARE COMMENSURABLE WITH IT* RATIONAL, *BUT THOSE WHICH ARE INCOMMENSURABLE WITH IT* IRRATIONAL, *AND THE STRAIGHT LINES WHICH PRODUCE THEM IRRATIONAL*, *THAT IS*, *IN CASE THE AREAS ARE SQUARES*, *THE SIDES THEMSELVES*, *BUT IN CASE THEY ARE ANY OTHER RECTILINEAL FIGURES*, *THE STRAIGHT LINES ON WHICH ARE DESCRIBED SQUARES EQUAL TO THEM*.

BOOK X.

PROPOSITIONS.

PROPOSITION 1.

TWO UNEQUAL MAGNITUDES BEING SET OUT, IF FROM THE GREATER THERE BE SUBTRACTED A MAGNITUDE GREATER THAN ITS HALF AND FROM THAT WHICH IS LEFT A MAGNITUDE GREATER THAN ITS HALF AND IF THIS PROCESS BE REPEATED CONTINUALLY THERE WILL BE LEFT SOME MAGNITUDE WHICH WILL BE LESS THAN THE LESSER MAGNITUDE SET OUT.



LET,

AB, *C*, BE TWO UNEQUAL MAGNITUDES, OF WHICH,
AB IS THE GREATER:

I SAY THAT;

IF FROM *AB*, THERE BE SUBTRACTED
A MAGNITUDE GREATER THAN ITS HALF, AND
FROM THAT WHICH IS LEFT
A MAGNITUDE GREATER THAN ITS HALF, AND
IF THIS PROCESS BE REPEATED CONTINUALLY,
THERE WILL BE LEFT SOME MAGNITUDE
WHICH WILL BE LESS THAN THE MAGNITUDE, *C*.

[CF. V. DEF. 4]

FOR,

C, IF MULTIPLIED WILL SOMETIME BE GREATER THAN *AB*.

LET,

IT BE MULTIPLIED,

AND LET,

DE BE A MULTIPLE OF *C*, AND GREATER THAN *AB*;

LET,

DE BE DIVIDED INTO THE PARTS, *DF*, *FG*, *GE*, EQUAL TO *C*,

LET FROM,

AB THERE BE SUBTRACTED *BH*, GREATER THAN ITS HALF,

AND, FROM,

AH, HK GREATER THAN ITS HALF,

AND LET,

THIS PROCESS BE REPEATED CONTINUALLY
UNTIL THE DIVISIONS IN AB ARE EQUAL IN MULTITUDE
WITH THE DIVISIONS IN DE.

LET, THEN,

AK, KH, HB BE DIVISIONS, WHICH
ARE EQUAL IN MULTITUDE WITH DF, FG, GE.

NOW, SINCE,

DE IS GREATER THAN AB, AND FROM DE,
THERE HAS BEEN SUBTRACTED EG LESS THAN ITS HALF, AND
FROM AB, BH GREATER THAN ITS HALF,

THEREFORE,

THE REMAINDER, GD, IS GREATER THAN
THE REMAINDER, HA.

AND, SINCE,

GD IS GREATER THAN HA, AND
THERE HAS BEEN SUBTRACTED, FROM GD,
THE HALF GF, AND FROM
HA, HK, GREATER THAN ITS HALF,

THEREFORE,

THE REMAINDER, DF, IS GREATER THAN
THE REMAINDER, AK.

BUT,

$DF = C$;

THEREFORE,

C IS, ALSO, GREATER THAN AK.

THEREFORE,

AK IS LESS THAN C.

THEREFORE,

THERE IS LEFT OF THE MAGNITUDE, AB,
THE MAGNITUDE, AK,
WHICH IS LESS THAN THE LESSER MAGNITUDE SET OUT,

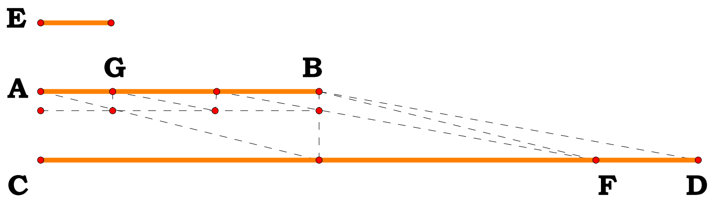
NAMELY,

C.

Q. E. D.

PROPOSITION 2.

IF WHEN THE LESS OF TWO UNEQUAL MAGNITUDES IS CONTINUALLY SUBTRACTED IN TURN FROM THE GREATER, THAT WHICH IS LEFT NEVER MEASURES THE ONE BEFORE IT, THE MAGNITUDES WILL BE INCOMMENSURABLE.



FOR,

THERE BEING TWO UNEQUAL MAGNITUDES, AB, CD, AND
AB BEING THE LESS, WHEN
THE LESS IS CONTINUALLY SUBTRACTED
IN TURN FROM THE GREATER,

LET,

THAT WHICH IS LEFT OVER
NEVER MEASURE THE ONE BEFORE IT;

I SAY THAT;

THE MAGNITUDES, AB, CD, ARE INCOMMENSURABLE.

FOR,

IF THEY ARE COMMENSURABLE,
SOME MAGNITUDE WILL MEASURE THEM.

LET, IF POSSIBLE,

A MAGNITUDE MEASURE THEM,

AND LET,

IT BE E;

LET,

AB, MEASURING FD, LEAVE CF, LESS THAN ITSELF,

LET,

CF MEASURING BG, LEAVE AG, LESS THAN ITSELF,

AND LET,

THIS PROCESS BE REPEATED CONTINUALLY,
UNTIL THERE IS LEFT SOME MAGNITUDE
WHICH IS LESS THAN E.

SUPPOSE,

THIS DONE,

AND LET,

THERE BE LEFT, AG, LESS THAN E.

THEN, SINCE,

E measures *AB*, while
AB measures *DF*,

therefore,
E will, also, measure *FD*.

But,
it measures the whole, *CD*, also;

therefore,
it will, also, measure the remainder, *CF*.

But,
CF measures *BG*;

therefore,
E, also, measures *BG*.

But,
it measures the whole, *AB*, also;

therefore,
it will, also, measure the remainder, *AG*,
the greater the less:

which,
is impossible.

Therefore,
no magnitude will measure the magnitudes, *AB*, *CD*;

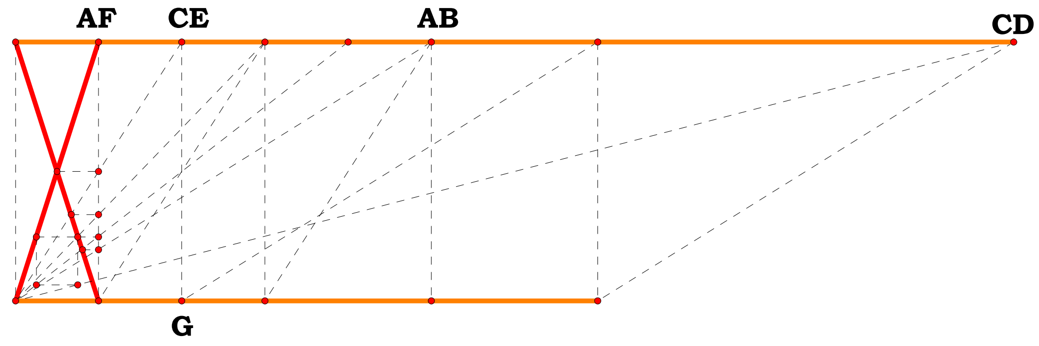
[X. Def. 1]

therefore,
the magnitudes, *AB*, *CD*, are incommensurable.

Therefore etc.

PROPOSITION 3.

GIVEN TWO COMMENSURABLE MAGNITUDES, TO FIND THEIR GREATEST COMMON MEASURE.



LET,

THE TWO GIVEN

COMMENSURABLE MAGNITUDES BE *AB*, *CD*, OF WHICH,

AB IS THE LESS;

THUS IT IS REQUIRED,

TO FIND THE GREATEST COMMON MEASURE OF *AB*, *CD*.

NOW,

THE MAGNITUDE, *AB*, EITHER MEASURES *CD* OR IT DOES NOT.

IF,

THEN IT MEASURES IT—AND IT MEASURES ITSELF ALSO—

AB IS A COMMON MEASURE OF *AB*, *CD*.

AND IT IS MANIFEST,

THAT IT IS, ALSO, THE GREATEST;

FOR,

A GREATER MAGNITUDE THAN

THE MAGNITUDE, *AB*, WILL NOT MEASURE *AB*.

NEXT, LET,

AB NOT MEASURE *CD*.

THEN,

IF THE LESS BE CONTINUALLY SUBTRACTED, IN TURN, FROM THE GREATER,

THAT WHICH IS LEFT OVER WILL SOMETIME MEASURE THE ONE BEFORE IT,

[CF. X. 2]

BECAUSE,

AB, *CD* ARE NOT INCOMMENSURABLE;

LET,

AB, MEASURING *ED*, LEAVE *EC*, LESS THAN ITSELF,

LET,

EC, MEASURING *FB*, LEAVE *AF*, LESS THAN ITSELF,

AND LET,

AF MEASURE *CE*.

SINCE,

THEN, *AF* MEASURES *CE*, WHILE
CE MEASURES *FB*,

THEREFORE,

AF WILL, ALSO, MEASURE *FB*.

BUT,

IT MEASURES ITSELF ALSO;

THEREFORE,

AF WILL, ALSO, MEASURE THE WHOLE, *AB*.

BUT,

AB MEASURES *DE*;

THEREFORE,

AF WILL, ALSO, MEASURE *ED*.

BUT,

IT MEASURES *CE*, ALSO;

THEREFORE,

IT, ALSO, MEASURES THE WHOLE, *CD*.

THEREFORE,

AF IS A COMMON MEASURE OF *AB*, *CD*.

I SAY NEXT THAT;

IT IS, ALSO, THE GREATEST.

FOR,

IF NOT, THERE WILL BE SOME MAGNITUDE
GREATER THAN *AF* WHICH WILL MEASURE *AB*, *CD*.

LET,

IT BE *G*.

SINCE THEN,

G MEASURES *AB*, WHILE
AB MEASURES *ED*,

THEREFORE,

G WILL, ALSO, MEASURE *ED*.

BUT,

IT MEASURES THE WHOLE, *CD*, ALSO;

THEREFORE,

G will, also, measure the remainder, *CE*.

But,

CE measures *FB*;

therefore,

G will, also, measure *FB*.

But,

it measures the whole, *AB*, also,

and therefore,

it will measure the remainder, *AF*,

the greater the less:

which,

is impossible.

Therefore,

no magnitude greater than *AF* will measure *AB*, *CD*;

therefore,

AF is the greatest common measure of *AB*, *CD*.

Therefore,

the greatest common measure of

the two given commensurable magnitudes,

AB, *CD*, has been found.

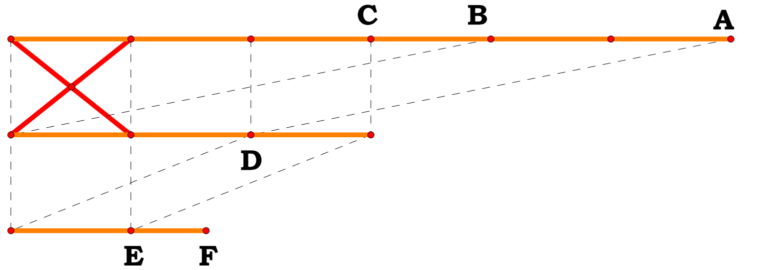
Q. E. D.

Porism.

From this it is manifest that, if a magnitude measure two magnitudes, it will, also, measure their greatest common measure.

PROPOSITION 4.

GIVEN THREE COMMENSURABLE MAGNITUDES, TO FIND THEIR GREATEST COMMON MEASURE.



LET,

A, B, C BE THE THREE GIVEN COMMENSURABLE MAGNITUDES;

THUS IT IS REQUIRED,

TO FIND THE GREATEST COMMON MEASURE, OF A, B, C .

[X. 3]

LET,

THE GREATEST COMMON MEASURE OF

THE TWO MAGNITUDES, A, B, BE TAKEN,

AND LET,

IT BE D;

THEN,

D EITHER MEASURES C, OR DOES NOT MEASURE IT.

FIRST, LET,

IT MEASURE IT.

SINCE THEN,

D MEASURES C, WHILE

IT, ALSO, MEASURES A, B,

THEREFORE,

D IS A COMMON MEASURE, OF A, B, C.

AND IT IS MANIFEST THAT;

IT IS, ALSO, THE GREATEST;

FOR,

A GREATER MAGNITUDE

THAN THE MAGNITUDE, D, DOES NOT MEASURE A, B.

NEXT, LET,

D NOT MEASURE C.

I SAY FIRST THAT;

C, D ARE COMMENSURABLE.

FOR, SINCE,

A, B, C ARE COMMENSURABLE,
SOME MAGNITUDE WILL MEASURE THEM,

AND, OF COURSE,
THIS WILL MEASURE A, B, ALSO;

[X. 3, POR.]

SO THAT,
IT WILL, ALSO, MEASURE
THE GREATEST COMMON MEASURE, OF A, B,

NAMELY,
D.

BUT,
IT, ALSO, MEASURES C;

SO THAT,
THE SAID MAGNITUDE WILL MEASURE C, D;

THEREFORE,
C, D ARE COMMENSURABLE.

[X. 3]

NOW LET,
THEIR GREATEST COMMON MEASURE BE TAKEN,

AND LET,
IT BE E.

SINCE THEN,
E MEASURES D, WHILE
D MEASURES A, B,

THEREFORE,
E WILL, ALSO, MEASURE A, B.

BUT,
IT MEASURES C, ALSO;

THEREFORE,
E MEASURES A, B, C;

THEREFORE,
E IS A COMMON MEASURE, OF A, B, C.

I SAY NEXT THAT;
IT IS, ALSO, THE GREATEST.

FOR, IF POSSIBLE, LET,
THERE BE SOME MAGNITUDE, F, GREATER THAN E,

AND LET,
IT MEASURE A, B, C.

[x. 3, Por.]

Now, since,
    F measures A, B, C,
    it will, also, measure A, B, and
    will measure the greatest common measure, of A, B.

But,
    the greatest common measure, of A, B is D;

therefore,
    F measures D.

But,
    it measures C, also;

therefore,
    F measures C, D;

[x. 3, Por.]

therefore,
    F will, also, measure
    the greatest common measure, of C, D.

But,
    that is E;

therefore,
    F will measure E, the greater the less:

which,
    is impossible.

Therefore,
    no magnitude greater than
    the magnitude, E, will measure A, B, C;

therefore,
    E is the greatest common measure, of A, B, C,
    if D do not measure C, and
    if it measure it,
    D is itself the greatest common measure.

Therefore,
    the greatest common measure of
    the three given commensurable magnitudes
    has been found.

Porism.

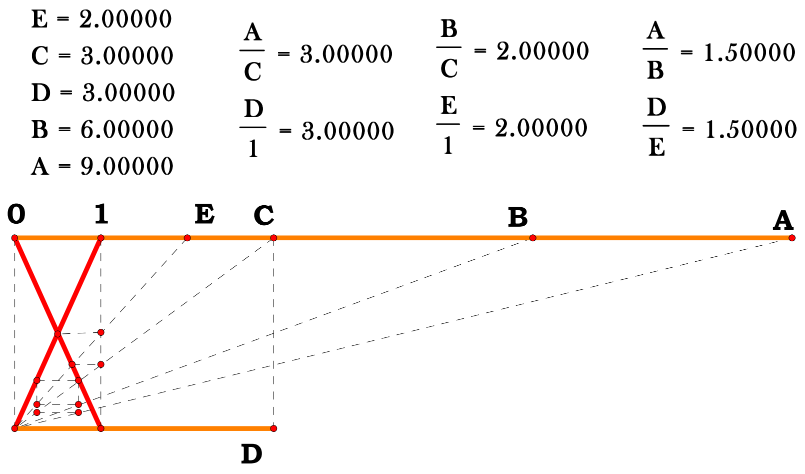
From this it is manifest that, if a magnitude measure three magnitudes, it will, also, measure their greatest common measure.

Similarly too, with more magnitudes, the greatest common measure can be found, and the porism can be extended.

Q. E. D.

Proposition 5.

Commensurable magnitudes have to one another the ratio which a number has to a number.



Let,
  A, *B*, be commensurable magnitudes;

I say that;
  A has to *B*, the ratio which a number has to a number.

For, since,
  A, *B* are commensurable,
  some magnitude will measure them.

Let,
  it measure them,

and let,
  it be *C*.

And let,
  as many times as *C* measures *A*,
  so many units there be in *D*;

and let,
  as many times as *C* measures *B*,
  so many units there be in *E*.

Since,
  then *C* measures *A*, according to the units in *D*, while
  the unit, also, measures *D*, according to the units in it,

therefore,
  the unit measures the number *D*,
  the same number of times as
  the magnitude, *C*, measures *A*;

[vii. Def. 20]

therefore,

AS C IS TO A,
SO IS THE UNIT TO D;

[CF. V. 7, POR.]

THEREFORE, INVERSELY,
AS A IS TO C,
SO IS D TO THE UNIT.

AGAIN, SINCE,
C MEASURES B, ACCORDING TO THE UNITS IN E, WHILE
THE UNIT, ALSO, MEASURES E ACCORDING TO THE UNITS IN IT,

THEREFORE,
THE UNIT MEASURES E,
THE SAME NUMBER OF TIMES AS C MEASURES B;

THEREFORE,
AS C IS TO B,
SO IS THE UNIT TO E.

BUT,
IT WAS, ALSO, PROVED THAT,
AS A IS TO C,
SO IS D TO THE UNIT;

[V. 22]

THEREFORE, *EX AEQUALI*,
AS A IS TO B,
SO IS THE NUMBER, D, TO E.

THEREFORE,
THE COMMENSURABLE MAGNITUDES, A, B,
HAVE TO ONE ANOTHER THE RATIO WHICH
THE NUMBER, D, HAS TO THE NUMBER, E.

Q. E. D.

[NOTE: JOHN CLARK]

$\frac{A}{B} = 1.50000$

01 = 1.14300 cm

0C = 3.42900 cm

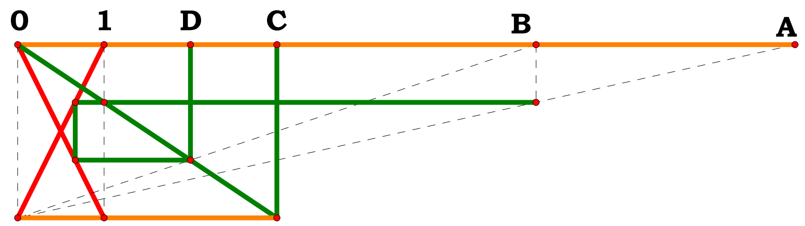
0D = 2.28600 cm

$\frac{0C}{01} = 3.00000$

B = 6.00000

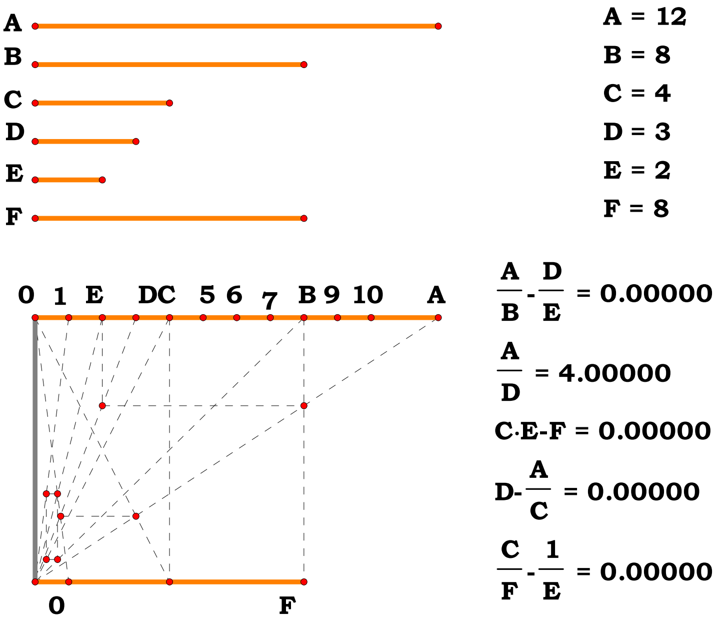
A = 9.00000

$\frac{0D}{01} = 2.00000$



PROPOSITION 6.

IF TWO MAGNITUDES HAVE TO ONE ANOTHER THE RATIO WHICH A NUMBER HAS TO A NUMBER, THE MAGNITUDES WILL BE COMMENSURABLE.



FOR LET,

THE TWO MAGNITUDES, *A*, *B*,
HAVE TO ONE ANOTHER THE RATIO WHICH
THE NUMBER, *D*, HAS TO THE NUMBER, *E*;

I SAY THAT;

THE MAGNITUDES, *A*, *B*, ARE COMMENSURABLE.

FOR LET,

A BE DIVIDED INTO AS MANY EQUAL PARTS
AS THERE ARE UNITS IN *D*,

AND LET,

C BE EQUAL TO ONE OF THEM;

AND LET,

F BE MADE UP OF AS MANY MAGNITUDES EQUAL
TO *C* AS THERE ARE UNITS IN *E*.

SINCE,

THEN THERE ARE IN *A* AS MANY MAGNITUDES EQUAL TO *C*,
AS THERE ARE UNITS IN *D*,
WHATEVER PART THE UNIT IS OF *D*,
THE SAME PART IS *C* OF *A* ALSO;

[VII. DEF. 20]

THEREFORE,

AS *C* IS TO *A*,
SO IS THE UNIT TO *D*.

BUT,

THE UNIT MEASURES THE NUMBER D;

THEREFORE,

C, ALSO, MEASURES A.

AND SINCE,

AS C IS TO A,
SO IS THE UNIT TO D,

[CF. V. 7, POR.]

THEREFORE, INVERSELY,

AS A IS TO C,
SO IS THE NUMBER, D, TO THE UNIT.

AGAIN, SINCE,

THERE ARE IN F AS MANY MAGNITUDES EQUAL TO C,
AS THERE ARE UNITS IN E,

[VII. DEF. 20]

THEREFORE,

AS C IS TO F,
SO IS THE UNIT TO E.

BUT, IT WAS, ALSO, PROVED THAT;

AS A IS TO C,
SO IS D TO THE UNIT;

[V. 22]

THEREFORE, *EX AEQUALI*,

AS A IS TO F,
SO IS D TO E.

BUT,

AS D IS TO E,
SO IS A TO B;

[V. 11]

THEREFORE ALSO,

AS A IS TO B,
SO IS IT TO F, ALSO.

THEREFORE,

A HAS THE SAME RATIO TO EACH, OF THE MAGNITUDES, B, F;

[V. 9]

THEREFORE,

$B = F$.

BUT,

C MEASURES F;

THEREFORE,

IT MEASURES B, ALSO.

FURTHER,

IT MEASURES A, ALSO;

THEREFORE,

C MEASURES A, B.

THEREFORE,

A IS COMMENSURABLE WITH B.

THEREFORE ETC.

PORISM.

FROM THIS IT IS MANIFEST THAT,

IF THERE BE TWO NUMBERS, AS D, E

AND,

A STRAIGHT LINE, AS A,
IT IS POSSIBLE TO MAKE A STRAIGHT LINE [F] SUCH THAT
THE GIVEN STRAIGHT LINE IS TO IT AS
THE NUMBER, D, IS TO THE NUMBER, E.

AND,

IF A MEAN PROPORTIONAL BE, ALSO, TAKEN
BETWEEN A, F, AS B,
AS A IS TO F,
SO WILL THE SQUARE, ON A, BE TO THE SQUARE, ON B,

[VI. 19, POR.]

THAT IS,

AS THE FIRST IS TO THE THIRD,
SO IS THE FIGURE ON THE FIRST
TO THAT WHICH IS SIMILAR AND
SIMILARLY DESCRIBED ON THE SECOND.

BUT,

AS A IS TO F,
SO IS THE NUMBER, D, TO THE NUMBER, E;

THEREFORE,

IT HAS BEEN CONTRIVED THAT,
AS THE NUMBER, D, IS TO THE NUMBER, E,
SO ALSO, IS THE FIGURE, ON
THE STRAIGHT LINE, A, TO
THE FIGURE, ON THE STRAIGHT LINE, B.

Q. E. D.

[NOTE: JOHN CLARK]

01 = 1.43933 cm

0D = 2.87867 cm

0C = 4.31800 cm

0B = 6.28650 cm

0A = 9.42975 cm

$\frac{0D}{01} = 2.00000$

$\frac{0C}{01} = 3.00000$

D = 2.00000

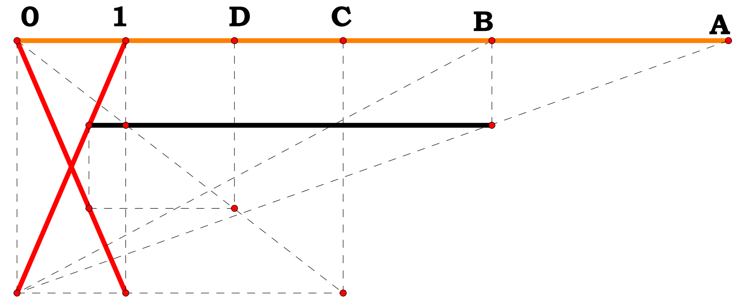
C = 3.00000

A = 6.55147

B = 4.36765

$\frac{A}{B} = 1.50000$

$\frac{C}{D} = 1.50000$



PROPOSITION 7.

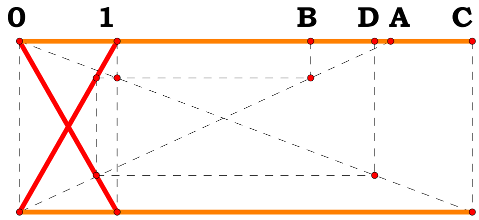
INCOMMENSURABLE MAGNITUDES HAVE NOT TO ONE ANOTHER THE RATIO WHICH A NUMBER HAS TO A NUMBER.

01 = 1.29117 cm
0A = 4.91067 cm
0B = 3.85233 cm
0C = 5.99101 cm
0D = 4.69985 cm

A = 3.80328
B = 2.98361
C = 4.64000
D = 3.64000

$\frac{0A}{0B} = 1.27473$ $\frac{A}{B} = 1.27473$

$\frac{C}{D} = 1.27473$



LET,

A, *B* BE INCOMMENSURABLE MAGNITUDES;

I SAY THAT;

A HAS NOT TO *B*,
THE RATIO WHICH A NUMBER HAS TO A NUMBER.

[X. 6]

FOR,

IF *A* HAS TO *B*
THE RATIO WHICH A NUMBER HAS TO A NUMBER,
A WILL BE COMMENSURABLE WITH *B*.

BUT,

IT IS NOT;

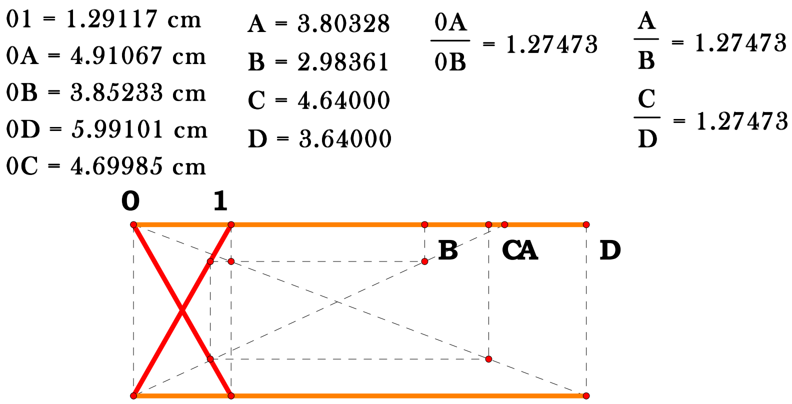
THEREFORE,

A HAS NOT TO *B*
THE RATIO WHICH A NUMBER HAS TO A NUMBER.

THEREFORE ETC.

Proposition 8.

If two magnitudes have not to one another the ratio which a number has to a number, the magnitudes will be incommensurable.



For let,

the two magnitudes, *A*, *B*, not have to one another the ratio which a number has to a number;

I say that;

the magnitudes, *A*, *B*, are incommensurable.

[x. 5]

For,

if they are commensurable, *A* will have to *B* the ratio which a number has to a number.

But,

it has not;

therefore,

the magnitudes, *A*, *B*, are incommensurable.

Therefore etc.

Proposition 9.

The squares on straight lines commensurable in length have to one another the ratio which a square number has to a square number; and squares which *have to one another the ratio which a square number has to a square number will*, also, *have their sides commensurable in length. But the squares on straight lines incommensurable in length have not to one another the ratio which a square number has to a square number; and squares which have not to one another the ratio which a square number has to a square number will not have their sides commensurable in length either.*

A = 3.59434
B = 2.44340
C = 2.12295
D = 1.44316

E = 1.89587
F = 1.56314

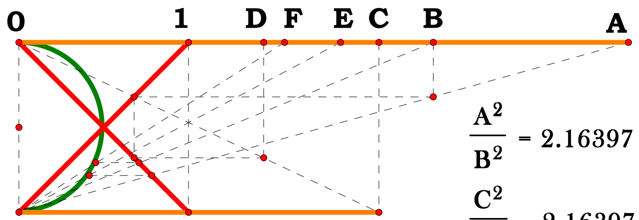
$\frac{A}{B} = 1.47104$

$\frac{C}{D} = 1.47104$

$\frac{E}{F} = 1.21287$

$E^2 - A = 0.00000$

$F^2 - B = 0.00000$



$\frac{A^2}{B^2} = 2.16397$

$\frac{C^2}{D^2} = 2.16397$

$\frac{E}{F}^{2} - \frac{A}{B} = 0.00000$

$\frac{E}{F}^{2} - \frac{C}{D} = 0.00000$

$\frac{E^4}{F^4} = 2.16397$

For let,

A, *B* be commensurable, in length;

I say that;

the square, on *A*, has to the square, on *B*,
the ratio which,
a square number has to a square number.

For, since,

A is commensurable, in length, with *B*,

[x. 5]

therefore,

A has to *B*,
the ratio which a number has to a number.

Let,

it have to it the ratio which *C* has to *D*.

Since then,

as *A* is to *B*,
so is *C* to *D*,

while,

the ratio, of the square, on *A*,

TO THE SQUARE, ON B, IS DUPLICATE OF
THE RATIO, OF A TO B,

[VI. 20, POR.]

FOR,

SIMILAR FIGURES ARE IN
THE DUPLICATE RATIO OF THEIR CORRESPONDING SIDES; AND
THE RATIO, OF THE SQUARE, ON C, TO
THE SQUARE, ON D, IS DUPLICATE OF
THE RATIO, OF C TO D,

FOR,

BETWEEN TWO SQUARE NUMBERS
THERE IS ONE MEAN PROPORTIONAL NUMBER,

[VIII. 11]

AND,

THE SQUARE NUMBER HAS TO THE SQUARE NUMBER
THE RATIO DUPLICATE OF THAT WHICH
THE SIDE HAS TO THE SIDE;

THEREFORE ALSO,

AS THE SQUARE, ON A, IS TO THE SQUARE, ON B,
SO IS THE SQUARE, ON C, TO THE SQUARE, ON D.

NEXT LET,

AS THE SQUARE, ON A, IS TO THE SQUARE, ON B,
SO THE SQUARE, ON C ,BE TO THE SQUARE, ON D;

I SAY THAT;

A IS COMMENSURABLE, IN LENGTH, WITH B.

FOR SINCE,

AS THE SQUARE, ON A, IS TO THE SQUARE, ON B,
SO IS THE SQUARE, ON C, TO THE SQUARE, ON D,

WHILE,

THE RATIO, OF THE SQUARE, ON A,
TO THE SQUARE, ON B, IS DUPLICATE OF
THE RATIO, OF A TO B, AND
THE RATIO, OF THE SQUARE, ON C, TO
THE SQUARE, ON D, IS DUPLICATE OF THE RATIO, OF C TO D,

THEREFORE ALSO,

AS A IS TO B,
SO IS C TO D.

THEREFORE,

A HAS TO B,
THE RATIO WHICH THE NUMBER, C, HAS TO THE NUMBER, D;

THEREFORE,

A IS COMMENSURABLE, IN LENGTH, WITH B.

NEXT, LET,

A BE INCOMMENSURABLE, IN LENGTH, WITH B;

I SAY THAT;

THE SQUARE, ON A, HAS NOT TO THE SQUARE, ON B,
THE RATIO WHICH,
A SQUARE NUMBER HAS TO A SQUARE NUMBER.

FOR,

IF THE SQUARE, ON A, HAS TO THE SQUARE, ON B,
THE RATIO WHICH,
A SQUARE NUMBER HAS TO A SQUARE NUMBER,
A WILL BE COMMENSURABLE WITH B.

BUT,

IT IS NOT;

THEREFORE,

THE SQUARE, ON A, HAS NOT TO THE SQUARE, ON B,
THE RATIO WHICH,
A SQUARE NUMBER HAS TO A SQUARE NUMBER.

AGAIN, LET,

THE SQUARE, ON A, NOT HAVE TO THE SQUARE, ON B,
THE RATIO WHICH,
A SQUARE NUMBER HAS TO A SQUARE NUMBER;

I SAY THAT;

A IS INCOMMENSURABLE, IN LENGTH, WITH B.

FOR,

IF A IS COMMENSURABLE WITH B,
THE SQUARE, ON A, WILL HAVE TO THE SQUARE, ON B,
THE RATIO WHICH,
A SQUARE NUMBER HAS TO A SQUARE NUMBER.

BUT,

IT HAS NOT;

THEREFORE,

A IS NOT COMMENSURABLE, IN LENGTH, WITH B.

THEREFORE ETC.

PORISM.

AND IT IS MANIFEST FROM WHAT HAS BEEN PROVED THAT STRAIGHT LINES COMMENSURABLE, IN LENGTH, ARE ALWAYS

COMMENSURABLE IN SQUARE ALSO, BUT THOSE COMMENSURABLE IN SQUARE ARE NOT ALWAYS COMMENSURABLE, IN LENGTH, ALSO.

[LEMMA.

IT HAS BEEN PROVED IN THE ARITHMETICAL BOOKS THAT SIMILAR PLANE NUMBERS HAVE TO ONE ANOTHER THE RATIO WHICH A SQUARE NUMBER HAS TO A SQUARE NUMBER, [VIII. 26] AND THAT, IF TWO NUMBERS HAVE TO ONE ANOTHER THE RATIO WHICH A SQUARE NUMBER HAS TO A SQUARE NUMBER, THEY ARE SIMILAR PLANE NUMBERS. [CONVERSE OF VIII. 26]

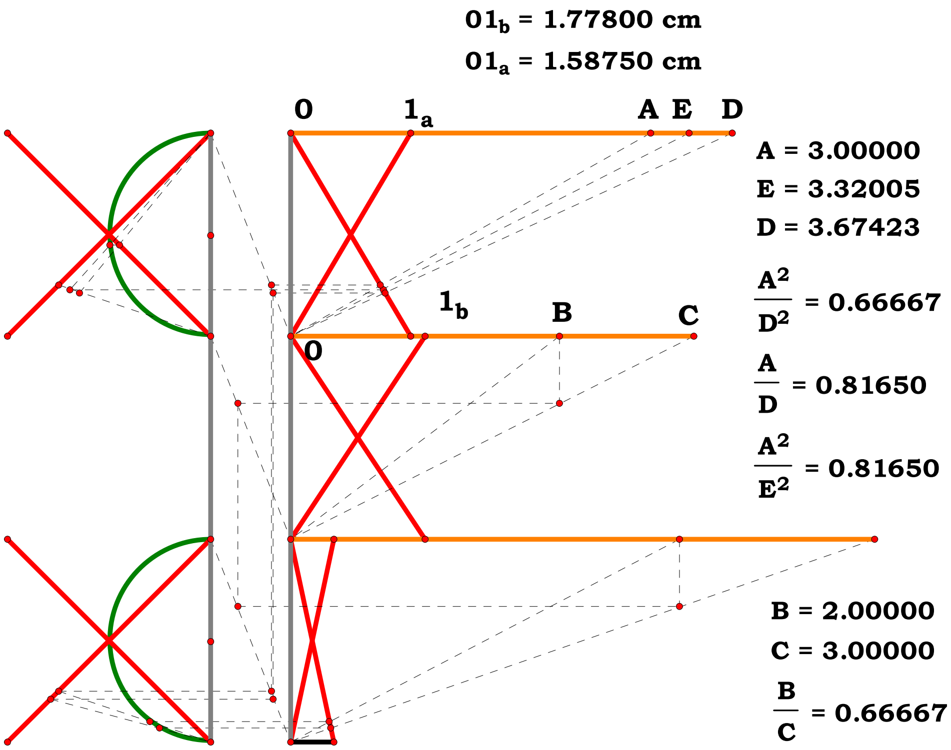
AND IT IS MANIFEST FROM THESE PROPOSITIONS THAT NUMBERS WHICH ARE NOT SIMILAR PLANE NUMBERS, THAT IS, THOSE WHICH HAVE NOT THEIR SIDES PROPORTIONAL, HAVE NOT TO ONE ANOTHER THE RATIO WHICH A SQUARE NUMBER HAS TO A SQUARE NUMBER.

FOR, IF THEY HAVE, THEY WILL BE SIMILAR PLANE NUMBERS: WHICH IS CONTRARY TO THE HYPOTHESIS.

THEREFORE NUMBERS WHICH ARE NOT SIMILAR PLANE NUMBERS HAVE NOT TO ONE ANOTHER THE RATIO WHICH A SQUARE NUMBER HAS TO A SQUARE NUMBER.]

[PROPOSITION 10.

TO FIND TWO STRAIGHT LINES INCOMMENSURABLE, THE ONE IN LENGTH ONLY, AND THE OTHER IN SQUARE ALSO, WITH AN ASSIGNED STRAIGHT LINE.



LET,

A BE THE ASSIGNED STRAIGHT LINE;

THUS IT IS REQUIRED,

TO FIND TWO STRAIGHT LINES INCOMMENSURABLE,
THE ONE IN LENGTH ONLY, AND,
THE OTHER IN SQUARE, ALSO, WITH *A*.

LET,

TWO NUMBERS, *B*, *C*, BE SET OUT, WHICH
HAVE NOT TO ONE ANOTHER, THE RATIO WHICH
A SQUARE NUMBER HAS TO A SQUARE NUMBER,

THAT IS,

WHICH ARE NOT SIMILAR PLANE NUMBERS;

AND LET, IT BE CONTRIVED THAT;

AS *B* IS TO *C*,
SO IS THE SQUARE, ON *A*, TO THE SQUARE, ON *D*.

[X. 6, POR.]

—FOR,

WE HAVE LEARNT HOW TO DO THIS—

[X. 6]

THEREFORE,

THE SQUARE, ON A, IS COMMENSURABLE WITH
THE SQUARE, ON D.

AND, SINCE,

B HAS NOT TO C,
THE RATIO WHICH,
A SQUARE NUMBER HAS TO A SQUARE NUMBER,

THEREFORE,

NEITHER HAS THE SQUARE, ON A, TO THE SQUARE, ON D,
THE RATIO WHICH,
A SQUARE NUMBER HAS TO A SQUARE NUMBER;

[X. 9]

THEREFORE,

A IS INCOMMENSURABLE, IN LENGTH, WITH D.

LET,

E BE TAKEN A MEAN PROPORTIONAL BETWEEN A, D;

[V. DEF. 9]

THEREFORE,

AS A IS TO D,
SO IS THE SQUARE, ON A, TO THE SQUARE, ON E.

BUT,

A IS INCOMMENSURABLE, IN LENGTH, WITH D;

[X. 11]

THEREFORE,

THE SQUARE, ON A, IS, ALSO, INCOMMENSURABLE WITH
THE SQUARE, ON E;

THEREFORE,

A IS INCOMMENSURABLE, IN SQUARE, WITH E.

THEREFORE,

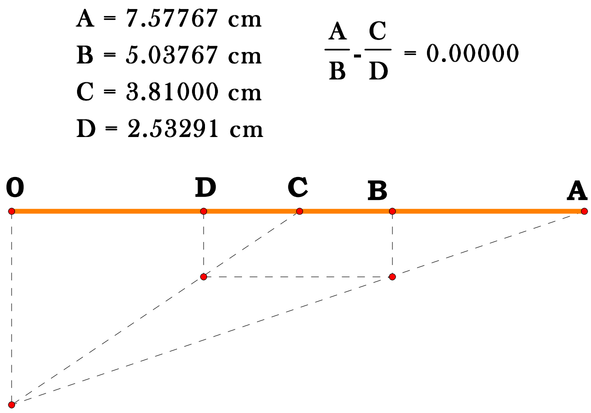
TWO STRAIGHT LINES D, E,
HAVE BEEN FOUND INCOMMENSURABLE,
D IN LENGTH ONLY, AND E, IN SQUARE,

AND, OF COURSE,

IN LENGTH ALSO, WITH THE ASSIGNED STRAIGHT LINE A.]

Proposition 11.

If four magnitudes be proportional, and the first be commensurable with the second, the third will, also, *be commensurable with the fourth; and, if the first be incommensurable with the second, the third will,* also, *be incommensurable with the fourth.*



Let,

A, *B*, *C*, *D* be four magnitudes in proportion,

so that,

as *A* is to *B*,

so is *C* to *D*,

and let,

A be commensurable with *B*;

I say that;

C will, also, be commensurable with *D*.

For, since,

A is commensurable with *B*,

[x. 5]

therefore,

A has to *B*, the ratio which,

a number has to a number.

And,

as *A* is to *B*,

so is *C* to *D*;

therefore also,

C has to *D*, the ratio which,

a number has to a number;

[x. 6]

therefore,

C is commensurable with *D*.

NEXT, LET,

A BE INCOMMENSURABLE WITH B;

I SAY THAT;

C WILL, ALSO, BE INCOMMENSURABLE WITH D.

FOR, SINCE,

A IS INCOMMENSURABLE WITH B,

[X. 7]

THEREFORE,

A HAS NOT TO B, THE RATIO WHICH,
A NUMBER HAS TO A NUMBER.

AND,

AS A IS TO B,
SO IS C TO D;

THEREFORE,

NEITHER HAS C TO D, THE RATIO WHICH,
A NUMBER HAS TO A NUMBER;

[X. 8]

THEREFORE,

C IS INCOMMENSURABLE WITH D.

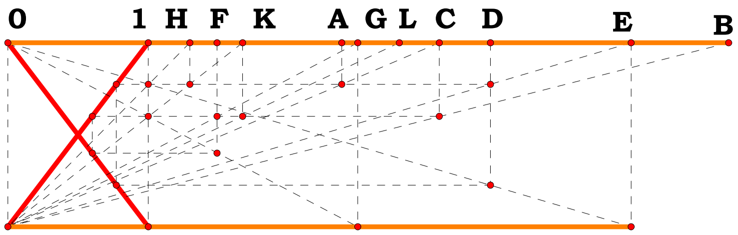
THEREFORE ETC.

Proposition 12.

Magnitudes commensurable with the same magnitude are commensurable with one another also.

$\frac{A}{C}-\frac{D}{E}=0.00000$ $\frac{D}{E}-\frac{H}{K}=0.00000$ $\frac{A}{C}-\frac{H}{K}=0.00000$ $\frac{A}{C}-\frac{H}{K}=0.00000$

$\frac{C}{B}-\frac{F}{G}=0.00000$ $\frac{F}{G}-\frac{K}{L}=0.00000$ $\frac{C}{B}-\frac{K}{L}=0.00000$ $\frac{A}{B}-\frac{H}{L}=0.00000$



For let,

each, of the magnitudes, A, B,
be commensurable with C;

I say that;

A is, also, commensurable with B.

For, since,

A is commensurable with C,

[x. 5]

therefore,

A has to C, the ratio which a number has to a number.

Let,

it have the ratio which D has to E.

Again, since,

C is commensurable with B,

[x. 5]

therefore,

C has to B, the ratio which a number has to a number.

Let,

it have the ratio which F has to G.

And, given,

any number of ratios we please,

namely,

the ratio which D has to E, and
that which F has to G,

[cf. viii. 4]

let,

THE NUMBERS, H, K, L,
BE TAKEN CONTINUOUSLY IN THE GIVEN RATIOS;

SO THAT,
AS D IS TO E,
SO IS H TO K, AND
AS F IS TO G,
SO IS K TO L.

SINCE, THEN,
AS A IS TO C,
SO IS D TO E, WHILE
AS D IS TO E,
SO IS H TO K,

[V. 11]

THEREFORE ALSO,
AS A IS TO C,
SO IS H TO K.

AGAIN, SINCE,
AS C IS TO B,
SO IS F TO G, WHILE
AS F IS TO G,
SO IS K TO L,

[V. 11]

THEREFORE ALSO,
AS C IS TO B,
SO IS K TO L.

BUT ALSO,
AS A IS TO C,
SO IS H TO K;

[V. 22]

THEREFORE, *EX AEQUALI*,
AS A IS TO B,
SO IS H TO L.

THEREFORE,
A HAS TO B, THE RATIO WHICH,
A NUMBER HAS TO A NUMBER;

[X. 6]

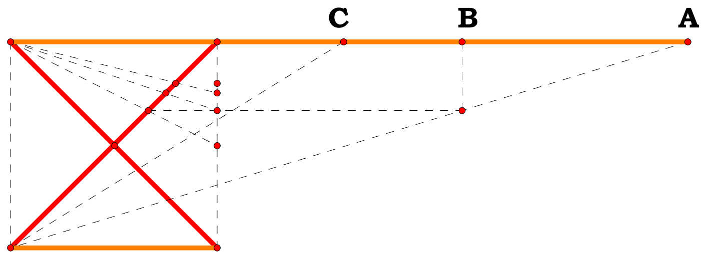
THEREFORE,
A IS COMMENSURABLE WITH B.

THEREFORE ETC.

Q. E. D.

PROPOSITION 13.

IF TWO MAGNITUDES BE COMMENSURABLE, AND THE ONE OF THEM BE INCOMMENSURABLE WITH ANY MAGNITUDE, THE REMAINING ONE WILL, ALSO, *BE INCOMMENSURABLE WITH THE SAME.*



LET,

A, *B* BE TWO COMMENSURABLE MAGNITUDES,

AND LET,

ONE OF THEM, *A*,
BE INCOMMENSURABLE WITH ANY OTHER MAGNITUDE, *C*;

I SAY THAT;

THE REMAINING ONE, *B*,
WILL, ALSO, BE INCOMMENSURABLE WITH *C*.

FOR,

IF *B* IS COMMENSURABLE WITH *C*,

[X. 12]

WHILE,

A IS, ALSO, COMMENSURABLE WITH *B*,
A IS, ALSO, COMMENSURABLE WITH *C*.

BUT,

IT IS, ALSO, INCOMMENSURABLE WITH IT:

WHICH,

IS IMPOSSIBLE.

THEREFORE,

B IS NOT COMMENSURABLE WITH *C*;

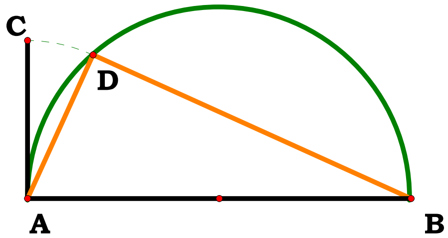
THEREFORE,

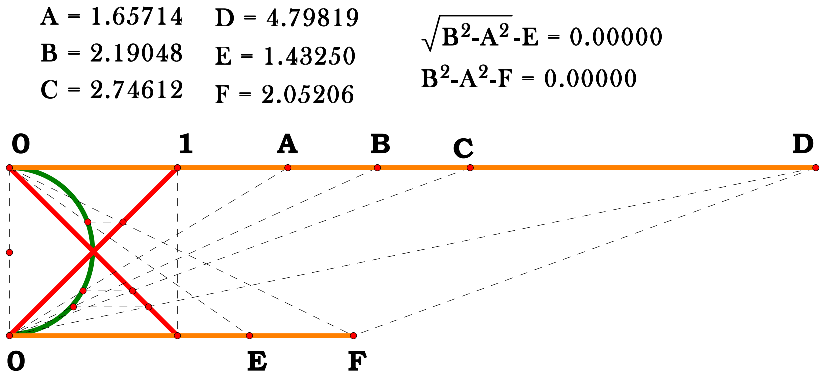
IT IS INCOMMENSURABLE WITH IT.

THEREFORE ETC.

LEMMA.

GIVEN TWO UNEQUAL STRAIGHT LINES, TO FIND BY WHAT SQUARE THE SQUARE, ON THE GREATER IS GREATER THAN THE SQUARE, ON THE LESS.





Let,

AB, *C* be the given two unequal straight lines,

and let,

AB be the greater of them;

thus it is required,

to find by what square
the square, on *AB*, is greater than
the square, on *C*.

[iv. 1]

Let,

the semicircle, *ADB*, be described, on *AB*,

and let,

AD be fitted into it equal to *C*;

let,

DB be joined.

[iii. 31]

It is then manifest that;

∠*ADB*, is right,

and that,

the square, on *AB*, is greater than
the square, on *AD*,

[i. 47]

that is,

C, by the square, on *DB*.

SIMILARLY ALSO,

IF TWO STRAIGHT LINES BE GIVEN,
THE STRAIGHT LINE THE SQUARE, ON WHICH =
THE SUM OF
THE SQUARES ON THEM IS FOUND IN THIS MANNER.

LET,

AD, *DB* BE THE GIVEN TWO STRAIGHT LINES,

AND LET IT BE REQUIRED,

TO FIND THE STRAIGHT LINE,
THE SQUARE, ON WHICH =
THE SUM OF THE SQUARES ON THEM.

LET,

THEM BE PLACED SO AS TO CONTAIN A RIGHT ANGLE,
THAT FORMED BY *AD*, *DB*;

AND LET,

AB BE JOINED.

[I. 47]

IT IS AGAIN MANIFEST THAT;

THE STRAIGHT LINE, THE SQUARE, ON WHICH, =
THE SUM OF THE SQUARES ON *AD*, *DB* IS *AB*.

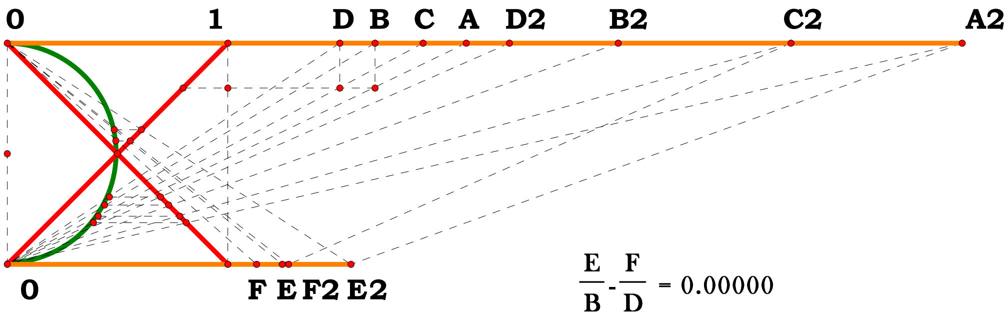
Q. E. D.

PROPOSITION 14.

IF FOUR STRAIGHT LINES BE PROPORTIONAL, AND THE SQUARE, ON THE FIRST BE GREATER THAN THE SQUARE, ON THE SECOND BY THE SQUARE, ON A STRAIGHT LINE COMMENSURABLE WITH THE FIRST, THE SQUARE, ON THE THIRD WILL, ALSO, *BE GREATER THAN THE SQUARE, ON THE FOURTH BY THE SQUARE, ON A STRAIGHT LINE COMMENSURABLE WITH THE THIRD.*

AND, IF THE SQUARE, ON THE FIRST BE GREATER THAN THE SQUARE, ON THE SECOND BY THE SQUARE, ON A STRAIGHT LINE INCOMMENSURABLE WITH THE FIRST, THE SQUARE, ON THE THIRD WILL, ALSO, *BE GREATER THAN THE SQUARE, ON THE FOURTH BY THE SQUARE, ON A STRAIGHT LINE INCOMMENSURABLE WITH THE THIRD.*

A = 2.07971 A2 = 4.32519 $\frac{A}{B}$ = 1.25000 $A2-A^2$ = 0.00000
B = 1.66377 B2 = 2.76812 $B2-B^2$ = 0.00000
C = 1.88406 C2 = 3.54967 $\frac{C}{D}$ = 1.25000 $C2-C^2$ = 0.00000
D = 1.50725 D2 = 2.27179 $D2-D^2$ = 0.00000
F = 1.13043 E2 = 1.55707 A^2-B^2-E2 = 0.00000 $\frac{A2}{B2}-\frac{C2}{D2}$ = 0.00000
E = 1.24783 F2 = 1.27788 C^2-D^2-F2 = 0.00000



$\frac{E}{B}-\frac{F}{D}$ = 0.00000

(B2+E2)-A2 = 0.00000
(D2+F2)-C2 = 0.00000
$\frac{E2}{B2}-\frac{F2}{D2}$ = 0.00000
$\frac{A}{E}-\frac{C}{F}$ = 0.00000

LET,

A, *B*, *C*, *D* BE FOUR STRAIGHT LINES IN PROPORTION,

SO THAT,

AS *A* IS TO *B*,
SO IS *C* TO *D*;

AND LET,

THE SQUARE, ON *A*, BE GREATER THAN
THE SQUARE, ON *B*, BY
THE SQUARE, ON *E*,

AND LET,

THE SQUARE, ON *C*, BE GREATER THAN
THE SQUARE, ON *D*,
BY THE SQUARE, ON *F*;

I SAY THAT;

IF *A* IS COMMENSURABLE WITH *E*,

C IS, ALSO, COMMENSURABLE WITH *F*, AND
IF *A* IS INCOMMENSURABLE WITH *E*,
C IS, ALSO, INCOMMENSURABLE WITH *F*.

FOR SINCE,
AS *A* IS TO *B*,
SO IS *C* TO *D*,

[VI. 22]

THEREFORE, ALSO,
AS THE SQUARE, ON *A*, IS TO THE SQUARE, ON *B*,
SO IS THE SQUARE, ON *C*, TO THE SQUARE, ON *D*.

BUT,
THE SQUARES, ON *E*, *B*, ARE EQUAL TO
THE SQUARE ,ON *A*, AND
THE SQUARES, ON *D*, *F*, ARE EQUAL TO
THE SQUARE, ON *C*.

THEREFORE,
AS THE SQUARES, ON *E*, *B*, ARE TO THE SQUARE, ON *B*
SO ARE THE SQUARES, ON *D*, ,*F* TO THE SQUARE, ON *D*;

[V. 17]

THEREFORE, *SEPARANDO*,
AS THE SQUARE, ON *E*, IS TO THE SQUARE, ON *B*,
SO IS THE SQUARE, ON *F*, TO THE SQUARE, ON *D*;

[VI. 22]

THEREFORE ALSO,
AS *E* IS TO *B*,
SO IS *F* TO *D*;

THEREFORE, INVERSELY,
AS *B* IS TO *E*,
SO IS *D* TO *F*.

BUT,
AS *A* IS TO *B*,
SO, ALSO, IS *C* TO *D*;

[V. 22]

THEREFORE, *EX AEQUALI*,
AS *A* IS TO *E*,
SO IS *C* TO *F*.

[X. 11]

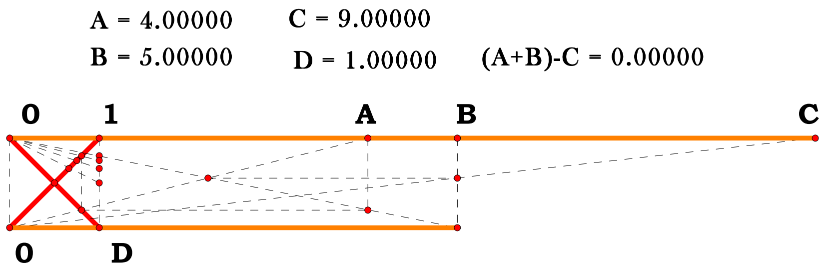
THEREFORE,
IF *A* IS COMMENSURABLE WITH *E*,

C IS, ALSO, COMMENSURABLE WITH F, AND
IF A IS INCOMMENSURABLE WITH E,
C IS, ALSO, INCOMMENSURABLE WITH F.

THEREFORE ETC.

Proposition 15.

If two commensurable magnitudes be added together, the whole will, also, *be commensurable with each, of them; and, if the whole be commensurable with one of them, the original magnitudes will,* also, *be commensurable.*



For let,

 the two commensurable magnitudes,

 AB, *BC*, be added together;

I say that;

 the whole, *AC*, is, also, commensurable with each, of the magnitudes, *AB*, *BC*.

For, since,

 AB, *BC* are commensurable,

 some magnitude will measure them.

Let,

 it measure them,

and let,

 it be *D*.

Since then,

 D measures *AB*, *BC*,

 it will, also, measure the whole, *AC*.

But,

 it measures *AB*, *BC* also;

therefore,

 D measures *AB*, *BC*, *AC*;

[x. Def. 1]

therefore,

 AC is commensurable with each, of the magnitudes, *AB*, *BC*.

Next, let,

 AC be commensurable with *AB*;

I say that;

 AB, *BC* are, also, commensurable.

FOR, SINCE,

AC, *AB* ARE COMMENSURABLE,
SOME MAGNITUDE WILL MEASURE THEM.

LET,

IT MEASURE THEM,

AND LET,

IT BE *D*.

SINCE THEN,

D MEASURES *CA*, *AB*, IT WILL, ALSO, MEASURE THE REMAINDER, *BC*.

BUT,

IT MEASURES *AB* ALSO;

THEREFORE,

D WILL MEASURE *AB*, *BC*;

[X. DEF. 1]

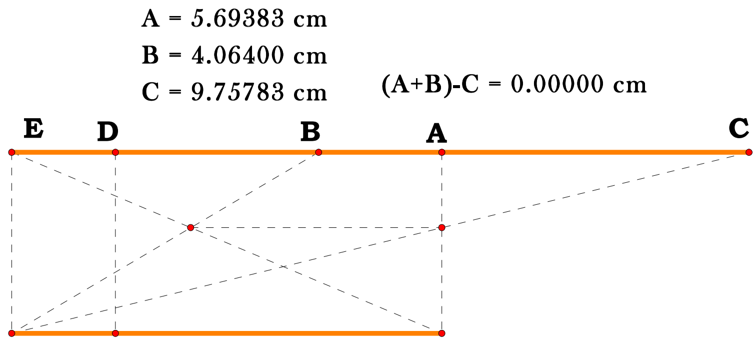
THEREFORE,

AB, *BC* ARE COMMENSURABLE.

THEREFORE ETC.

PROPOSITION 16.

IF TWO INCOMMENSURABLE MAGNITUDES BE ADDED TOGETHER, THE WHOLE WILL, ALSO, BE INCOMMENSURABLE WITH EACH, OF THEM; AND, IF THE WHOLE BE INCOMMENSURABLE WITH ONE OF THEM, THE ORIGINAL MAGNITUDES WILL, ALSO, BE INCOMMENSURABLE.



FOR LET,

THE TWO INCOMMENSURABLE MAGNITUDES,
AB, *BC*, BE ADDED TOGETHER;

I SAY THAT;

THE WHOLE, *AC*, IS, ALSO, INCOMMENSURABLE
WITH EACH, OF THE MAGNITUDES, *AB*, *BC*.

FOR,

IF *CA*, *AB* ARE NOT INCOMMENSURABLE,
SOME MAGNITUDE WILL MEASURE THEM.

LET, IF POSSIBLE,

IT MEASURE THEM,

AND LET,

IT BE *D*.

SINCE THEN,

D MEASURES *CA*, *AB*,

THEREFORE,

IT WILL, ALSO, MEASURE THE REMAINDER, *BC*.

BUT,

IT MEASURES *AB*, ALSO;

THEREFORE,

D MEASURES *AB*, *BC*.

THEREFORE,

AB, *BC* ARE COMMENSURABLE;

BUT BY HYPOTHESIS,

THEY WERE ALSO, INCOMMENSURABLE:

WHICH,

IS IMPOSSIBLE.

THEREFORE,

NO MAGNITUDE WILL MEASURE CA, AB;

[X. DEF. 1]

THEREFORE,

CA, AB ARE INCOMMENSURABLE.

SIMILARLY WE CAN PROVE THAT;

AC, CB ARE, ALSO, INCOMMENSURABLE.

THEREFORE,

AC IS INCOMMENSURABLE
WITH EACH, OF THE MAGNITUDES, AB, BC.

NEXT, LET,

AC BE INCOMMENSURABLE
WITH ONE OF THE MAGNITUDES, AB, BC.

FIRST, LET,

IT BE INCOMMENSURABLE WITH AB.

I SAY THAT;

AB, BC ARE, ALSO, INCOMMENSURABLE.

FOR,

IF THEY ARE COMMENSURABLE,
SOME MAGNITUDE WILL MEASURE THEM.

LET,

IT MEASURE THEM,

AND LET,

IT BE D.

SINCE,

THEN D MEASURES AB, BC,

THEREFORE,

IT WILL, ALSO, MEASURE THE WHOLE, AC.

BUT,

IT MEASURES AB ALSO;

THEREFORE,

D MEASURES CA, AB,

THEREFORE,

CA, AB ARE COMMENSURABLE;

BUT, BY HYPOTHESIS,

THEY WERE, ALSO, INCOMMENSURABLE:

WHICH,

IS IMPOSSIBLE.

THEREFORE,

NO MAGNITUDE WILL MEASURE AB, BC;

[X. DEF. 1]

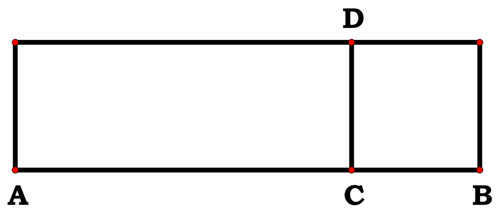
THEREFORE,

AB, BC ARE INCOMMENSURABLE.

THEREFORE ETC.

LEMMA.

IF TO ANY STRAIGHT LINE THERE BE APPLIED A PARALLELOGRAM DEFICIENT BY A SQUARE FIGURE, THE APPLIED PARALLELOGRAM EQUALS THE RECTANGLE CONTAINED BY THE SEGMENTS OF THE STRAIGHT LINE RESULTING FROM THE APPLICATION.



FOR LET,

THERE BE APPLIED TO

THE STRAIGHT LINE, AB, THE PARALLELOGRAM, AD,

DEFICIENT BY THE SQUARE FIGURE, DB;

I SAY THAT;

AD EQUALS THE RECTANGLE, CONTAINED BY AC, CB.

THIS IS INDEED AT ONCE MANIFEST; FOR, SINCE,

DB IS A SQUARE,

$DC = CB$; AND

AD IS THE RECTANGLE, AC, CD,

THAT IS,

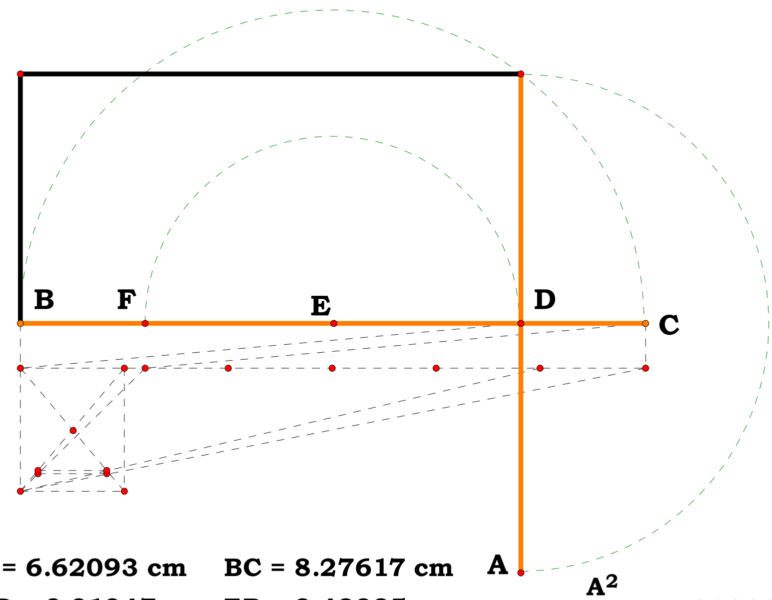
THE RECTANGLE, AC, CB.

THEREFORE ETC.

PROPOSITION 17.

IF THERE BE TWO UNEQUAL STRAIGHT LINES, AND TO THE GREATER THERE BE APPLIED A PARALLELOGRAM EQUAL TO THE FOURTH PART OF THE SQUARE, ON THE LESS AND DEFICIENT BY A SQUARE FIGURE, AND IF IT DIVIDE IT INTO PARTS WHICH ARE COMMENSURABLE IN LENGTH, THEN THE SQUARE, ON THE GREATER WILL BE GREATER THAN THE SQUARE, ON THE LESS BY THE SQUARE, ON A STRAIGHT LINE COMMENSURABLE WITH THE GREATER.

AND, IF THE SQUARE, ON THE GREATER BE GREATER THAN THE SQUARE, ON THE LESS BY THE SQUARE, ON A STRAIGHT LINE COMMENSURABLE WITH THE GREATER, AND IF THERE BE APPLIED TO THE GREATER A PARALLELOGRAM EQUAL TO THE FOURTH PART OF THE SQUARE, ON THE LESS AND DEFICIENT BY A SQUARE FIGURE, IT WILL DIVIDE IT INTO PARTS WHICH ARE COMMENSURABLE IN LENGTH.



A = 6.62093 cm **BC = 8.27617 cm**

AD = 3.31047 cm **ED = 2.48285 cm**

BD = 6.62093 cm **EC = 4.13808 cm**

DC = 1.65523 cm **DF = 4.96570 cm**

$\frac{BD}{DC} = 4.00000$ $\frac{BC}{DF} = 1.66667$

$\frac{A^2}{AD^2} = 4.00000$

$\frac{BD \cdot DC}{AD^2} = 1.00000$

$\frac{BC}{DC} = 5.00000$

$(BD \cdot DC + ED^2) - EC^2 = 0.00000 \text{ cm}^2$

$(4 \cdot BD \cdot DC + 4 \cdot ED^2) - 4 \cdot EC^2 = 0.00000 \text{ cm}^2$

$A^2 - 4 \cdot BD \cdot DC = 0.00000 \text{ cm}^2$

$DF^2 - 4 \cdot ED^2 = 0.00000 \text{ cm}^2$

$BC^2 - 4 \cdot EC^2 = 0.00000 \text{ cm}^2$

$(A^2 + DF^2) - BC^2 = 0.00000 \text{ cm}^2$

LET,

A, *BC*, BE TWO UNEQUAL STRAIGHT LINES, OF WHICH
BC IS THE GREATER,

AND LET,

THERE BE APPLIED TO *BC*, A PARALLELOGRAM EQUAL TO
THE FOURTH PART OF THE SQUARE, ON THE LESS, *A*,

THAT IS,

EQUAL TO THE SQUARE, ON THE HALF OF A, AND
DEFICIENT BY A SQUARE FIGURE.

[CF. LEMMA]

LET,
THIS BE THE RECTANGLE, BD, DC,

AND LET,
BD BE COMMENSURABLE, IN LENGTH, WITH DC;

I SAY THAT;
THE SQUARE, ON BC, IS GREATER THAN THE SQUARE, ON A,
BY THE SQUARE, ON A STRAIGHT LINE
COMMENSURABLE WITH BC.

FOR LET,
BC BE BISECTED, AT THE POINT, E,

AND LET,
EF BE MADE EQUAL TO DE.

THEREFORE,
THE REMAINDER, $DC = BF$.

AND, SINCE,
THE STRAIGHT LINE, BC,
HAS BEEN CUT INTO EQUAL PARTS, AT E, AND
INTO UNEQUAL PARTS, AT D,

[II. 5]

THEREFORE,
THE RECTANGLE, CONTAINED BY BD, DC,
TOGETHER WITH THE SQUARE, ON ED =
THE SQUARE, ON EC;

AND,
THE SAME IS TRUE OF THEIR QUADRUPLES;

THEREFORE,
FOUR TIMES THE RECTANGLE, BD, DC, TOGETHER WITH
FOUR TIMES THE SQUARE, ON DE, =
FOUR TIMES THE SQUARE, ON EC.

BUT,
THE SQUARE, ON A, =
FOUR TIMES THE RECTANGLE, BD, DC; AND
THE SQUARE, ON DF, =
FOUR TIMES THE SQUARE, ON DE,

FOR,
DF IS DOUBLE OF DE.

AND,

THE SQUARE, ON BC, =
FOUR TIMES THE SQUARE, ON EC,

FOR, AGAIN

BC IS DOUBLE OF CE.

THEREFORE,

THE SQUARES, ON A, DF, ARE EQUAL TO
THE SQUARE, ON BC,

SO THAT,

THE SQUARE, ON BC, IS GREATER THAN
THE SQUARE, ON A, BY
THE SQUARE, ON DF.

IT IS TO BE PROVED THAT;

BC IS, ALSO, COMMENSURABLE WITH DF.

SINCE,

BD IS COMMENSURABLE, IN LENGTH, WITH DC,

[X. 15]

THEREFORE,

BC IS, ALSO, COMMENSURABLE, IN LENGTH, WITH CD.

BUT,

CD IS COMMENSURABLE, IN LENGTH, WITH CD, BF,

[X. 6]

FOR,

$CD = BF$.

[X. 12]

THEREFORE,

BC IS, ALSO, COMMENSURABLE, IN LENGTH, WITH BF, CD,

[X. 15]

SO THAT,

BC IS, ALSO, COMMENSURABLE, IN LENGTH, WITH
THE REMAINDER, FD;

THEREFORE,

THE SQUARE, ON BC, IS GREATER THAN THE SQUARE, ON A, BY
THE SQUARE, ON A STRAIGHT LINE COMMENSURABLE WITH BC.

NEXT, LET,

THE SQUARE, ON BC, BE GREATER THAN THE SQUARE, ON, A
BY

THE SQUARE, ON A STRAIGHT LINE COMMENSURABLE WITH BC,

LET,

A PARALLELOGRAM BE APPLIED, TO BC, EQUAL TO
THE FOURTH PART OF THE SQUARE, ON A, AND
DEFICIENT BY A SQUARE FIGURE,

AND LET,

IT BE THE RECTANGLE, BD, DC.

IT IS TO BE PROVED THAT,

BD IS COMMENSURABLE, IN LENGTH, WITH DC.

WITH THE SAME CONSTRUCTION,

WE CAN PROVE SIMILARLY THAT;

THE SQUARE, ON BC, IS GREATER THAN THE SQUARE, ON A, BY
THE SQUARE, ON FD.

BUT,

THE SQUARE, ON BC, IS GREATER THAN THE SQUARE, ON A, BY
THE SQUARE, ON A STRAIGHT LINE COMMENSURABLE WITH BC.

THEREFORE,

BC IS COMMENSURABLE, IN LENGTH, WITH FD,

[X. 15]

SO THAT,

BC IS, ALSO, COMMENSURABLE, IN LENGTH, WITH
THE REMAINDER, THE SUM, OF BF, DC.

[X. 6]

BUT,

THE SUM, OF BF, DC, IS COMMENSURABLE WITH DC,

[X. 12]

SO THAT,

BC IS, ALSO, COMMENSURABLE, IN LENGTH, WITH CD;

[X. 15]

AND THEREFORE, *SEPARANDO*,

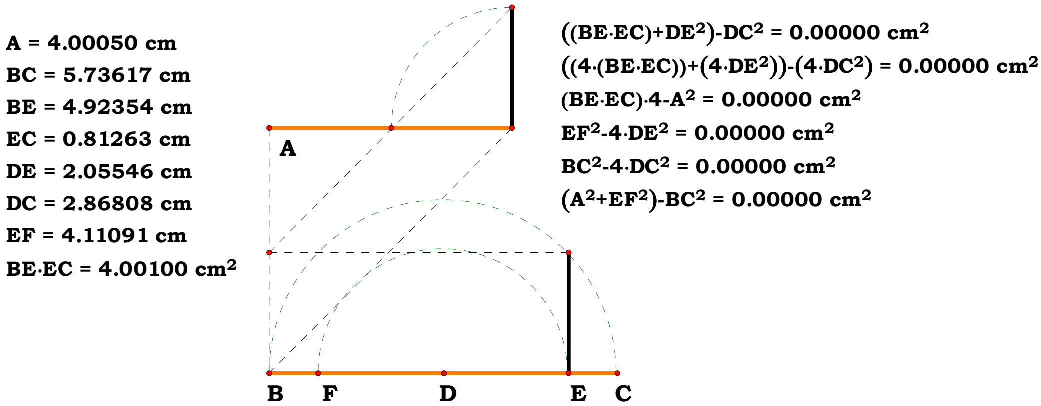
BD IS COMMENSURABLE, IN LENGTH, WITH DC.

THEREFORE ETC.

Proposition 18.

If there be two unequal straight lines, and to the greater there be applied a parallelogram equal to the fourth part of the square, on the less and deficient by a square figure, and if it divide it into parts which are incommensurable, the square, on the greater will be greater than the square, on the less by the square, on a straight line incommensurable with the greater.

And, if the square, on the greater be greater than the square, on the less by the square, on a straight line incommensurable with the greater, and if there be applied to the greater a parallelogram equal to the fourth part of the square, on the less and deficient by a square figure, it divides it into parts which are incommensurable.



LET,

A, *BC*, BE TWO UNEQUAL STRAIGHT LINES, OF WHICH
BC IS THE GREATER,

AND LET,

TO *BC* THERE BE APPLIED
A PARALLELOGRAM EQUAL TO THE FOURTH PART OF
THE SQUARE, ON THE LESS, *A*, AND
DEFICIENT BY A SQUARE FIGURE.

[CF. LEMMA BEFORE X. 17]

LET,

THIS BE THE RECTANGLE, *BD*, *DC*,

AND LET,

BD BE INCOMMENSURABLE, IN LENGTH, WITH *DC*;

I SAY THAT;

THE SQUARE, ON *BC*, IS GREATER THAN
THE SQUARE, ON *A*, BY THE SQUARE, ON A STRAIGHT LINE
INCOMMENSURABLE WITH *BC*.

FOR, WITH THE SAME CONSTRUCTION AS BEFORE,

WE CAN PROVE SIMILARLY THAT;

THE SQUARE, ON *BC*, IS GREATER THAN
THE SQUARE, ON *A*, BY THE SQUARE, ON *FD*.

IT IS TO BE PROVED THAT;

BC IS INCOMMENSURABLE, IN LENGTH, WITH *DF*.

SINCE,

BD IS INCOMMENSURABLE, IN LENGTH, WITH *DC*,

[X. 16]

THEREFORE,

BC IS, ALSO, INCOMMENSURABLE, IN LENGTH, WITH *CD*.

[X. 6]

BUT,

DC IS COMMENSURABLE WITH THE SUM, OF *BF*, *DC*;

[X. 13]

THEREFORE,

BC IS, ALSO, INCOMMENSURABLE WITH THE SUM, OF *BF*, *DC*;

[X. 16]

SO THAT,

BC IS, ALSO, INCOMMENSURABLE, IN LENGTH, WITH
THE REMAINDER, *FD*.

AND,

THE SQUARE, ON *BC*, IS GREATER THAN
THE SQUARE, ON *A*, BY THE SQUARE, ON *FD*;

THEREFORE,

THE SQUARE, ON *BC*, IS GREATER THAN
THE SQUARE, ON *A*, BY THE SQUARE, ON A STRAIGHT LINE
INCOMMENSURABLE WITH *BC*.

AGAIN, LET,

THE SQUARE, ON *BC*, BE GREATER THAN
THE SQUARE, ON *A*, BY THE SQUARE, ON A STRAIGHT LINE
INCOMMENSURABLE WITH *BC*,

AND LET,

THERE BE APPLIED, TO *BC*,
A PARALLELOGRAM EQUAL TO THE FOURTH PART OF
THE SQUARE, ON *A*, AND DEFICIENT BY A SQUARE FIGURE.

LET,

THIS BE THE RECTANGLE *BD*, *DC*.

IT IS TO BE PROVED THAT;

BD IS INCOMMENSURABLE, IN LENGTH, WITH *DC*.

FOR, WITH THE SAME CONSTRUCTION,

WE CAN PROVE SIMILARLY THAT;

THE SQUARE, ON BC, IS GREATER THAN
THE SQUARE, ON A, BY THE SQUARE, ON FD.

BUT,

THE SQUARE, ON BC, IS GREATER THAN
THE SQUARE, ON A, BY THE SQUARE, ON A STRAIGHT LINE
INCOMMENSURABLE WITH BC;

THEREFORE,

BC IS INCOMMENSURABLE, IN LENGTH, WITH FD,

[X. 16]

SO THAT,

BC IS, ALSO, INCOMMENSURABLE WITH THE REMAINDER,
THE SUM, OF BF, DC.

[X. 6]

BUT,

THE SUM OF
BF, DC, IS COMMENSURABLE, IN LENGTH, WITH DC;

[X. 13]

THEREFORE,

BC IS, ALSO, INCOMMENSURABLE, IN LENGTH, WITH DC,

[X. 16]

SO THAT, *SEPARANDO*,

BD IS, ALSO, INCOMMENSURABLE, IN LENGTH, WITH DC.

THEREFORE ETC.

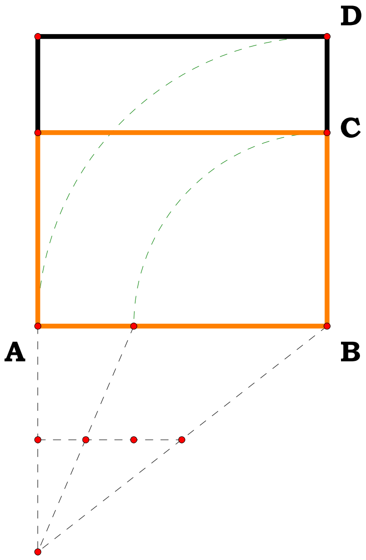
[LEMMA.

SINCE IT HAS BEEN PROVED THAT STRAIGHT LINES COMMENSURABLE IN LENGTH ARE ALWAYS COMMENSURABLE IN SQUARE ALSO, WHILE THOSE COMMENSURABLE IN SQUARE ARE NOT ALWAYS COMMENSURABLE IN LENGTH ALSO, BUT CAN OF COURSE BE EITHER COMMENSURABLE OR INCOMMENSURABLE IN LENGTH, IT IS MANIFEST THAT, IF ANY STRAIGHT LINE BE COMMENSURABLE IN LENGTH WITH A GIVEN RATIONAL STRAIGHT LINE, IT IS CALLED RATIONAL AND COMMENSURABLE WITH THE OTHER NOT ONLY IN LENGTH BUT IN SQUARE ALSO, SINCE STRAIGHT LINES COMMENSURABLE IN LENGTH ARE ALWAYS COMMENSURABLE IN SQUARE ALSO.

BUT, IF ANY STRAIGHT LINE BE COMMENSURABLE IN SQUARE WITH A GIVEN RATIONAL STRAIGHT LINE, THEN, IF IT IS, ALSO,

COMMENSURABLE IN LENGTH WITH IT, IT IS CALLED IN THIS CASE, ALSO, RATIONAL AND COMMENSURABLE WITH IT BOTH IN LENGTH AND IN SQUARE; BUT, IF AGAIN ANY STRAIGHT LINE, BEING COMMENSURABLE IN SQUARE WITH A GIVEN RATIONAL STRAIGHT LINE, BE INCOMMENSURABLE IN LENGTH WITH IT, IT IS CALLED IN THIS CASE, ALSO, RATIONAL BUT COMMENSURABLE IN SQUARE, ONLY.]

PROPOSITION 19.



THE RECTANGLE, CONTAINED BY RATIONAL STRAIGHT LINES COMMENSURABLE IN LENGTH, IS RATIONAL.

FOR LET,

THE RECTANGLE *AC* BE CONTAINED BY

THE RATIONAL STRAIGHT LINES *AB*, *BC*,

COMMENSURABLE, IN LENGTH;

I SAY THAT;

AC IS RATIONAL.

FOR LET,

ON *AB*, THE SQUARE, *AD*, BE DESCRIBED;

[X. DEF. 4]

THEREFORE,

AD IS RATIONAL.

AND, SINCE,

AB IS COMMENSURABLE, IN LENGTH, WITH *BC*, WHILE

AB = *BD*,

THEREFORE,

BD IS COMMENSURABLE, IN LENGTH, WITH *BC*.

[VI. 1]

AND,

AS *BD* IS TO *BC*,

SO IS *DA* TO *AC*.

[X. 11]

THEREFORE,

DA IS COMMENSURABLE WITH *AC*.

BUT,

DA IS RATIONAL;

[X. DEF. 4]

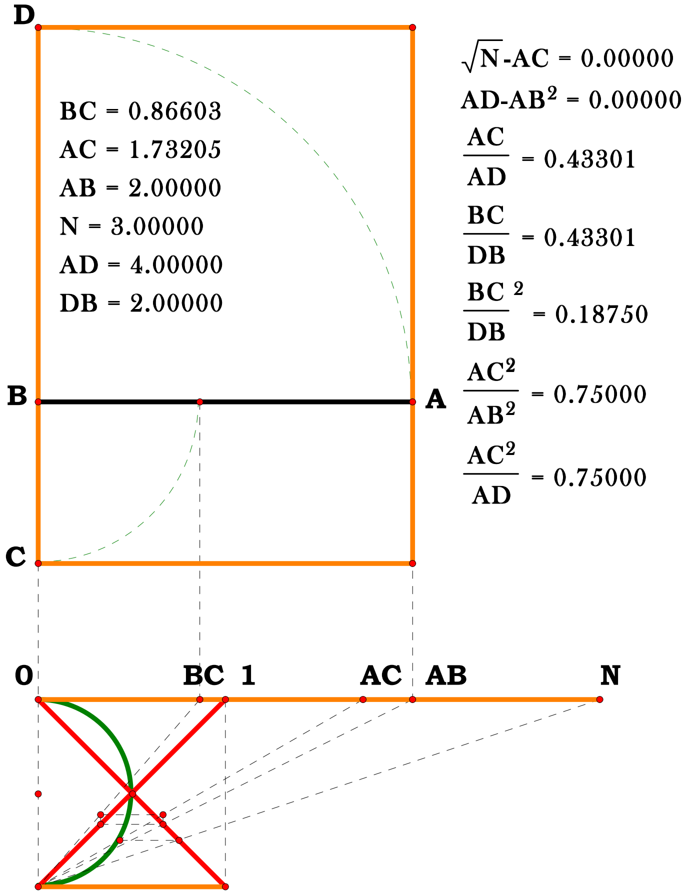
THEREFORE,

AC IS, ALSO, RATIONAL.

THEREFORE ETC.

PROPOSITION 20.

IF A RATIONAL AREA BE APPLIED TO A RATIONAL STRAIGHT LINE, IT PRODUCES AS BREADTH A STRAIGHT LINE RATIONAL AND COMMENSURABLE, IN LENGTH, WITH THE STRAIGHT LINE TO WHICH IT IS APPLIED.



FOR LET,

THE RATIONAL AREA, *AC*, BE APPLIED, TO *AB*,
A STRAIGHT LINE ONCE MORE RATIONAL
IN ANY OF THE AFORESAID WAYS,
PRODUCING *BC*, AS BREADTH;

I SAY THAT;

BC IS RATIONAL, AND COMMENSURABLE, IN LENGTH, WITH *BA*.

FOR LET,

ON *AB*, THE SQUARE, *AD*, BE DESCRIBED;

[X. DEF. 4]

THEREFORE,

AD IS RATIONAL.

BUT,

AC IS, ALSO, RATIONAL;

THEREFORE,

DA IS COMMENSURABLE WITH AC.

[VI. 1]

AND,

AS DA IS TO AC,
SO IS DB TO BC.

[X. 11]

THEREFORE,

DB IS, ALSO, COMMENSURABLE WITH BC; AND
$DB = BA$;

THEREFORE,

AB IS, ALSO, COMMENSURABLE WITH BC.

BUT,

AB IS RATIONAL;

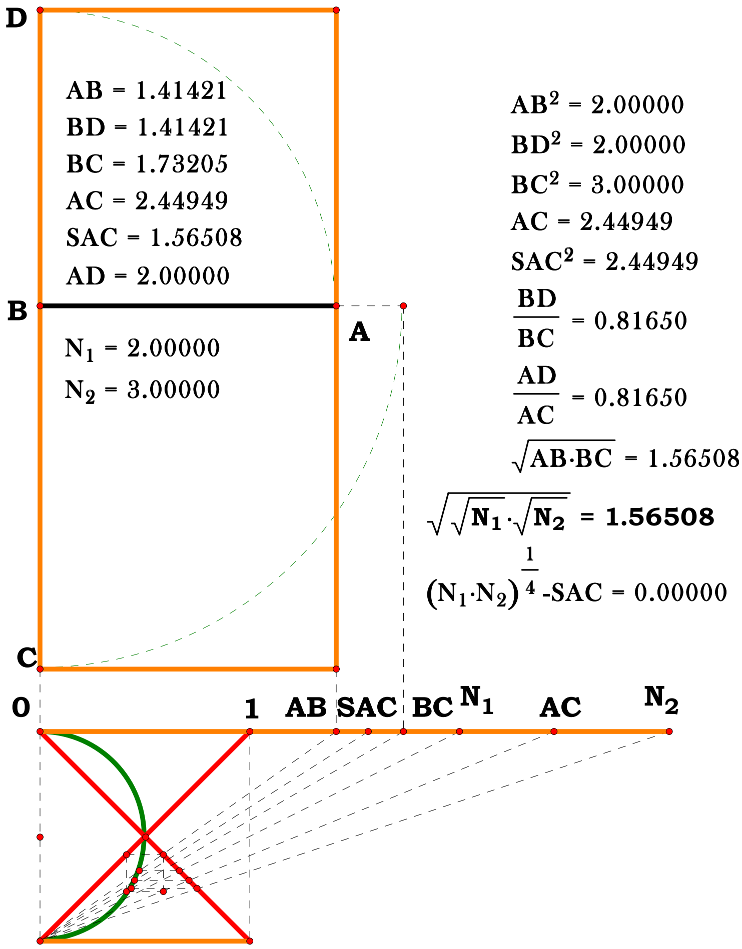
THEREFORE,

BC IS, ALSO, RATIONAL, AND
COMMENSURABLE, IN LENGTH, WITH AB.

THEREFORE ETC.

PROPOSITION 21.

THE RECTANGLE CONTAINED BY RATIONAL STRAIGHT LINES COMMENSURABLE IN SQUARE ONLY IS IRRATIONAL, AND THE SIDE OF THE SQUARE EQUAL TO IT IS IRRATIONAL. LET THE LATTER BE CALLED **MEDIAL.**



FOR LET,

THE RECTANGLE, *AC*, BE CONTAINED BY
THE RATIONAL STRAIGHT LINES, *AB*, *BC*,
COMMENSURABLE, IN SQUARE, ONLY;

I SAY THAT;

AC IS IRRATIONAL, AND
THE SIDE OF THE SQUARE EQUAL TO IT IS IRRATIONAL;

AND LET,

THE LATTER BE CALLED **MEDIAL.**

FOR LET,

ON *AB*, THE SQUARE, *AD*, BE DESCRIBED;

[X. DEF. 4]

THEREFORE,

AD IS RATIONAL.

AND, SINCE,

AB IS INCOMMENSURABLE, IN LENGTH, WITH BC,

FOR,

BY HYPOTHESIS,

THEY ARE COMMENSURABLE, IN SQUARE, ONLY,

WHILE,

$AB = BD$,

THEREFORE,

DB IS, ALSO, INCOMMENSURABLE, IN LENGTH, WITH BC.

[VI. 1]

AND,

AS DB IS TO BC,

SO IS AD TO AC;

[X. 11]

THEREFORE,

DA IS INCOMMENSURABLE WITH AC.

BUT,

DA IS RATIONAL;

THEREFORE,

AC IS IRRATIONAL,

[X. DEF. 4]

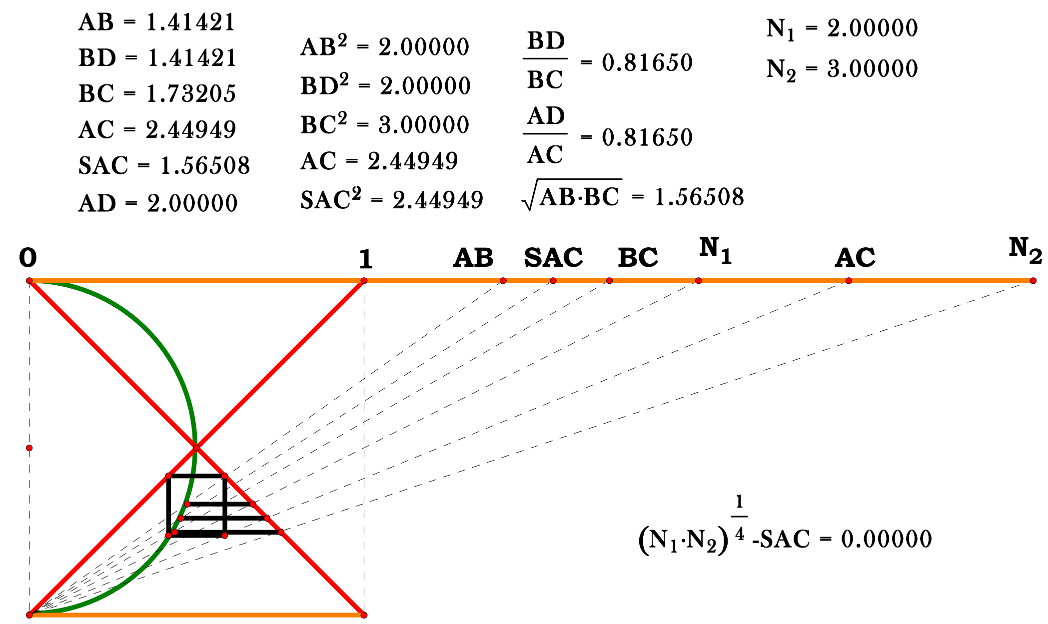
SO THAT,

THE SIDE OF THE SQUARE EQUAL TO AC IS, ALSO, IRRATIONAL.

AND LET,

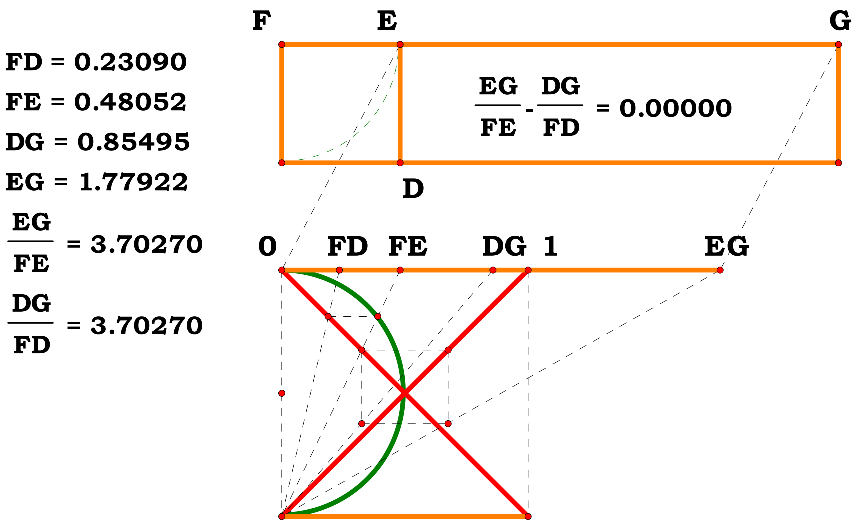
THE LATTER BE CALLED **MEDIAL**.

Q. E. D.



LEMMA.

IF THERE BE TWO STRAIGHT LINES, THEN, AS THE FIRST IS TO THE SECOND, SO IS THE SQUARE, ON THE FIRST TO THE RECTANGLE CONTAINED BY THE TWO STRAIGHT LINES.



LET,

FE, *EG* BE TWO STRAIGHT LINES.

I SAY THAT;

AS *FE* IS TO *EG*,

SO IS THE SQUARE, ON *FE*, TO THE RECTANGLE, *FE*, *EG*.

FOR LET,

ON *FE*, THE SQUARE, *DF*, BE DESCRIBED,

AND LET,

GD BE COMPLETED.

[VI. 1]

SINCE THEN,

AS *FE* IS TO *EG*,
SO IS *FD* TO *DG*, AND
FD IS THE SQUARE, ON *FE*, AND THE RECTANGLE, *DE*, *EG*,

THAT IS,

THE RECTANGLE *FE*, *EG*,

THEREFORE,

AS *FE* IS TO *EG*,
SO IS THE SQUARE, ON *FE*, TO THE RECTANGLE, *FE*, *EG*.

SIMILARLY ALSO,

AS THE RECTANGLE, *GE*, *EF*, IS TO THE SQUARE, ON *EF*,

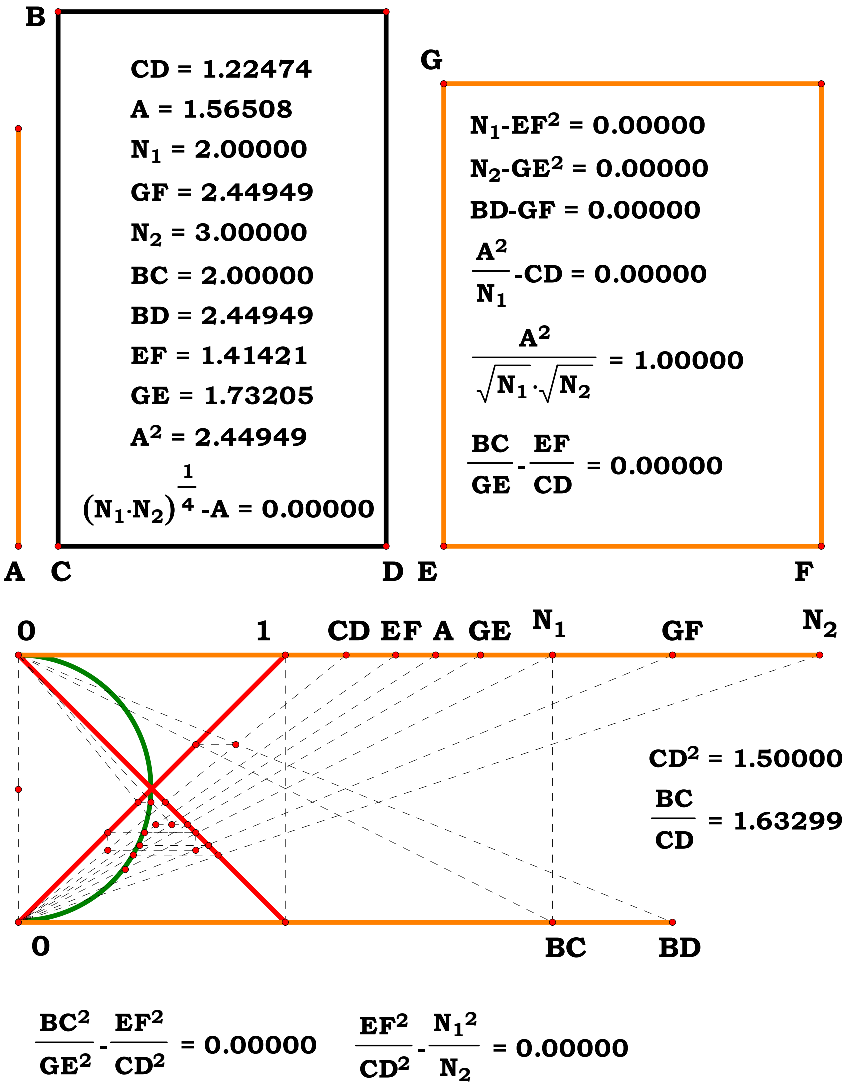
THAT IS,

AS *GD* IS TO *FD*,
SO IS *GE* TO *EF*.

Q. E. D.

Proposition 22.

The square, on a medial straight line, if applied to a rational straight line, produces as breadth a straight line rational and incommensurable, in length, with that to which it is applied.



Let,

A be medial and,

CB rational,

and let,

a rectangular area, *BD*, equal to the square, on *A*, be applied, to *BC*, producing *CD*, as breadth;

I say that;

CD is rational and incommensurable, in length, with *CB*.

[x. 21]

For, since,

A is medial,

THE SQUARE, ON IT EQUAL TO A RECTANGULAR AREA
CONTAINED BY RATIONAL STRAIGHT LINES
COMMENSURABLE, IN SQUARE, ONLY.

LET,
THE SQUARE, ON IT BE EQUAL TO GF.

BUT,
THE SQUARE, ON IT IS ALSO EQUAL TO BD;

THEREFORE,
$BD = GF$.

BUT,
IT IS, ALSO, EQUIANGULAR WITH IT;

[VI. 14]

AND,
IN EQUAL AND EQUIANGULAR PARALLELOGRAMS
THE SIDES ABOUT
THE EQUAL ANGLES ARE RECIPROCALLY PROPORTIONAL;

THEREFORE, PROPORTIONALLY,
AS BC IS TO EG,
SO IS EF TO CD.

[VI. 22]

THEREFORE, ALSO,
AS THE SQUARE, ON BC, IS TO THE SQUARE, ON EG,
SO IS THE SQUARE, ON EF, TO THE SQUARE, ON CD.

BUT,
THE SQUARE, ON CB, IS COMMENSURABLE WITH
THE SQUARE, ON EG,

FOR,
EACH, OF THESE STRAIGHT LINES IS RATIONAL;

[X. 11]

THEREFORE,
THE SQUARE, ON EF, IS, ALSO, COMMENSURABLE WITH
THE SQUARE, ON CD.

BUT,
THE SQUARE, ON EF, IS RATIONAL;

[X. DEF. 4]

THEREFORE,
THE SQUARE, ON CD, IS, ALSO, RATIONAL;

THEREFORE,

CD IS RATIONAL.

AND, SINCE,

EF IS INCOMMENSURABLE, IN LENGTH, WITH *EG*,

[LEMMA]

FOR,

THEY ARE COMMENSURABLE, IN SQUARE, ONLY, AND
AS *EF* IS TO *EG*,
SO IS THE SQUARE, ON *EF*, TO THE RECTANGLE, *FE*, *EG*,

[X. 11]

THEREFORE,

THE SQUARE, ON *EF*, IS INCOMMENSURABLE WITH
THE RECTANGLE, *FE*, *EG*.

BUT,

THE SQUARE, ON *CD*, IS COMMENSURABLE WITH
THE SQUARE, ON *EF*,

FOR,

THE STRAIGHT LINES ARE RATIONAL, IN SQUARE; AND
THE RECTANGLE, *DC*, *CB*, IS COMMENSURABLE WITH
THE RECTANGLE, *FE*, *EG*,

FOR,

THEY ARE EQUAL TO THE SQUARE, ON *A*;

[X. 13]

THEREFORE,

THE SQUARE, ON *CD*, IS, ALSO, INCOMMENSURABLE WITH
THE RECTANGLE, *DC*, *CB*.

[LEMMA]

BUT,

AS THE SQUARE, ON *CD*, IS TO THE RECTANGLE, *DC*, *CB*,
SO IS *DC* TO *CB*;

[X. 11]

THEREFORE,

DC IS INCOMMENSURABLE, IN LENGTH, WITH *CB*.

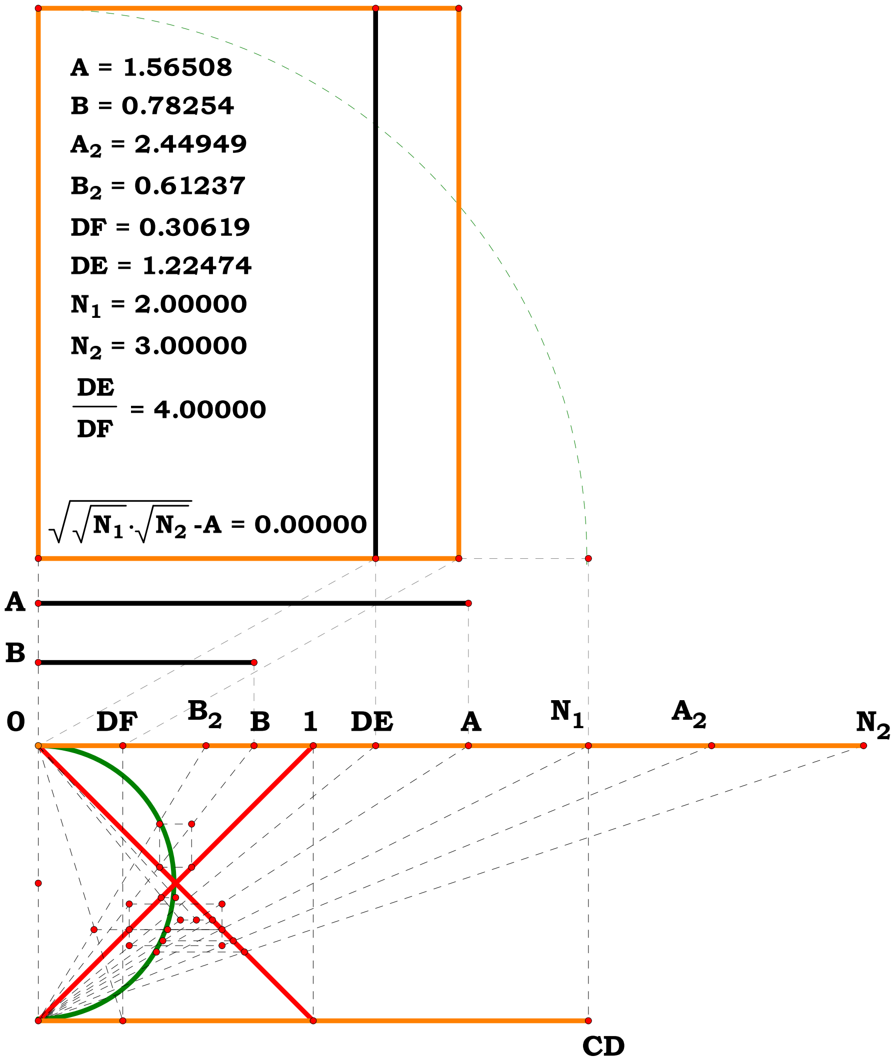
THEREFORE,

CD IS RATIONAL AND
INCOMMENSURABLE, IN LENGTH, WITH *CB*.

Q. E. D.

Proposition 23.

A straight line commensurable with a medial straight line is medial.



Let,

A be medial,

and let,

B be commensurable with *A*;

I say that;

B is, also, medial.

For let,

a rational straight line, *CD*, be set out,

and let,

to *CD*,

the rectangular area, *CE*, equal to

the square, on *A*, be applied, producing *ED*, as breadth;

[x. 22]

therefore,

ED IS RATIONAL, AND
INCOMMENSURABLE, IN LENGTH, WITH *CD*.

AND LET,
THE RECTANGULAR AREA, *CF*, EQUAL TO
THE SQUARE, ON *B*, BE APPLIED, TO *CD*,
PRODUCING *DF*, AS BREADTH.

SINCE THEN,
A IS COMMENSURABLE WITH *B*,
THE SQUARE, ON *A*, IS, ALSO, COMMENSURABLE WITH
THE SQUARE, ON *B*.

BUT,
EC = THE SQUARE, ON *A*, AND
CE = THE SQUARE, ON *B*;

THEREFORE,
EC IS COMMENSURABLE WITH *CF*.

[VI. 1]

AND,
AS *EC* IS TO *CF*,
SO IS *ED* TO *DF*;

[X. 11]

THEREFORE,
ED IS COMMENSURABLE IN, LENGTH, WITH *DF*.

BUT,
ED IS RATIONAL AND
INCOMMENSURABLE, IN LENGTH, WITH *DC*;

[X. DEF. 3]

THEREFORE,
DF IS, ALSO, RATIONAL

[X. 13]

AND,
INCOMMENSURABLE, IN LENGTH, WITH *DC*.

THEREFORE,
CD, *DF* ARE RATIONAL AND
COMMENSURABLE, IN SQUARE, ONLY.

[X. 21]

BUT,
THE STRAIGHT LINE, THE SQUARE, ON WHICH =
THE RECTANGLE CONTAINED BY RATIONAL STRAIGHT LINES
COMMENSURABLE, IN SQUARE, ONLY, IS MEDIAL;

THEREFORE,

THE SIDE OF THE SQUARE EQUAL TO
THE RECTANGLE, CD, DF, IS MEDIAL.

AND,

B IS THE SIDE OF THE SQUARE EQUAL TO
THE RECTANGLE, CD, DF;

THEREFORE,

B IS MEDIAL.

PORISM.

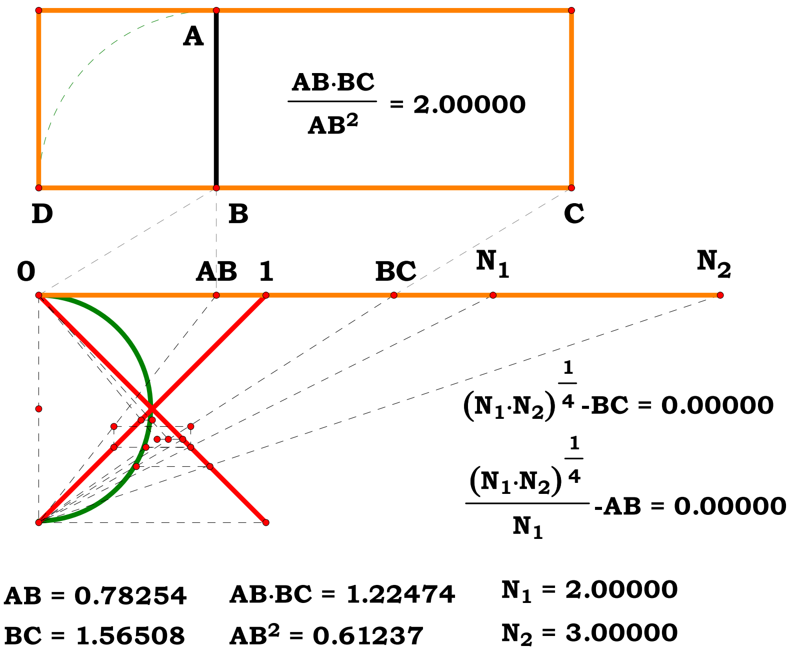
FROM THIS IT IS MANIFEST THAT AN AREA COMMENSURABLE WITH A MEDIAL AREA IS MEDIAL.

[AND IN THE SAME WAY AS WAS EXPLAINED IN THE CASE OF RATIONALS [LEMMA FOLLOWING X. 18] IT FOLLOWS, AS REGARDS MEDIALS, THAT A STRAIGHT LINE COMMENSURABLE IN LENGTH WITH A MEDIAL STRAIGHT LINE IS CALLED MEDIAL AND COMMENSURABLE WITH IT NOT ONLY IN LENGTH BUT IN SQUARE ALSO, SINCE, IN GENERAL, STRAIGHT LINES COMMENSURABLE IN LENGTH ARE ALWAYS COMMENSURABLE IN SQUARE ALSO.

BUT, IF ANY STRAIGHT LINE BE COMMENSURABLE IN SQUARE WITH A MEDIAL STRAIGHT LINE, THEN, IF IT IS, ALSO, COMMENSURABLE IN LENGTH WITH IT, THE STRAIGHT LINES ARE CALLED, IN THIS CASE TOO, MEDIAL AND COMMENSURABLE IN LENGTH AND IN SQUARE, BUT, IF IN SQUARE, ONLY, THEY ARE CALLED MEDIAL STRAIGHT LINES COMMENSURABLE IN SQUARE, ONLY.]

PROPOSITION 24.

THE RECTANGLE CONTAINED BY MEDIAL STRAIGHT LINES COMMENSURABLE IN LENGTH IS MEDIAL.



FOR LET,

THE RECTANGLE, *AC*, BE CONTAINED BY
THE MEDIAL STRAIGHT LINES, *AB*, *BC*,
WHICH ARE COMMENSURABLE, IN LENGTH;

I SAY THAT;

AC IS MEDIAL.

FOR LET,

ON *AB*,
THE SQUARE, *AD*, BE DESCRIBED;

THEREFORE,

AD IS MEDIAL.

AND, SINCE,

AB IS COMMENSURABLE, IN LENGTH, WITH *BC*, WHILE
AB = *BD*,

THEREFORE,

DB IS, ALSO, COMMENSURABLE, IN LENGTH, WITH *BC*;

[VI. 1, X. 11]

SO THAT,

DA IS, ALSO, COMMENSURABLE WITH *AC*.

BUT,

DA IS MEDIAL;

[X. 23, POR.]

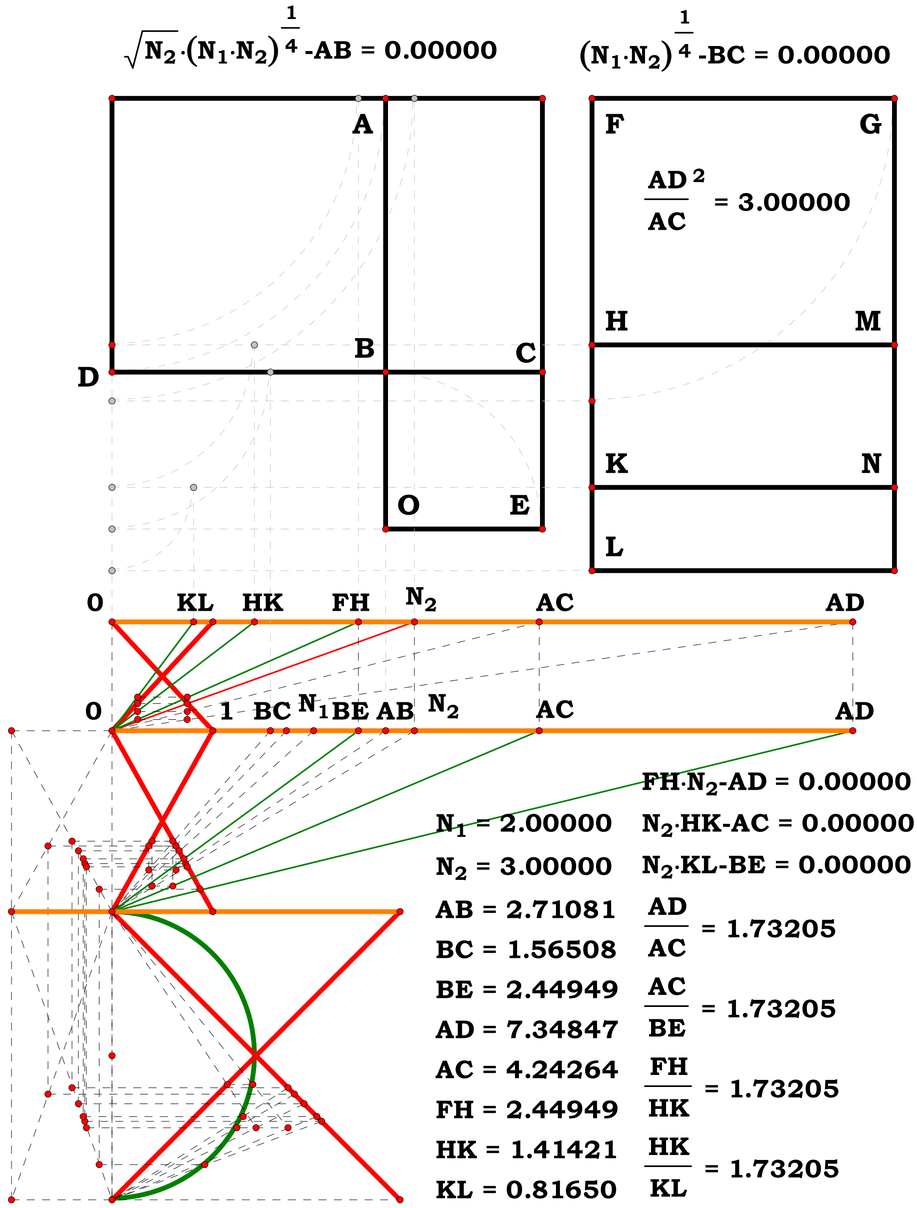
THEREFORE,

AC IS, ALSO, MEDIAL.

Q. E. D.

PROPOSITION 25.

THE RECTANGLE CONTAINED BY MEDIAL STRAIGHT LINES COMMENSURABLE IN SQUARE, ONLY IS EITHER RATIONAL OR MEDIAL.



FOR LET,

THE RECTANGLE, *AC*, BE CONTAINED BY
THE MEDIAL STRAIGHT LINES, *AB*, *BC*, WHICH,
ARE COMMENSURABLE, IN SQUARE, ONLY;

I SAY THAT;

AC IS EITHER RATIONAL OR MEDIAL.

FOR LET,

ON *AB*, *BC*,
THE SQUARES, *AD*, *BE*, BE DESCRIBED;

THEREFORE,

EACH, OF THE SQUARES, *AD*, *BE*, IS MEDIAL.

LET,

A RATIONAL STRAIGHT LINE, *FG*, BE SET OUT,

LET,

TO *FG*, THERE BE APPLIED
THE RECTANGULAR PARALLELOGRAM, *GH*,
EQUAL TO *AD*, PRODUCING *FH*, AS BREADTH,

LET,

TO *HM*, THERE BE APPLIED
THE RECTANGULAR PARALLELOGRAM, *MK*,
EQUAL TO *AC*, PRODUCING *HK*, AS BREADTH,

AND FURTHER LET,

TO *KN*, THERE BE, SIMILARLY, APPLIED
NL, EQUAL TO *BE*, PRODUCING *KL*, AS BREADTH;

THEREFORE,

FH, *HK*, *KL* ARE IN A STRAIGHT LINE.

SINCE,

THEN EACH, OF THE SQUARES, *AD*, *BE*, IS MEDIAL, AND
AD = *GH*, AND
BE TO *NL*,

THEREFORE,

EACH, OF THE RECTANGLES, *GH*, *NL*, IS, ALSO, MEDIAL.

AND,

THEY ARE APPLIED TO THE RATIONAL STRAIGHT LINE *EG*;

[X. 22]

THEREFORE,

EACH, OF THE STRAIGHT LINES, *FH*, *KL*, IS RATIONAL AND,
INCOMMENSURABLE, IN LENGTH, WITH *FG*.

AND, SINCE,

AD IS COMMENSURABLE WITH *BE*,

THEREFORE,

GH IS, ALSO, COMMENSURABLE WITH *NL*.

[VI. 1]

AND,

AS *GH* IS TO *NL*,
SO IS *FH* TO *KL*;

[X. 11]

THEREFORE,

FH IS COMMENSURABLE, IN LENGTH, WITH *KL*.

THEREFORE,

FH, *KL* ARE RATIONAL STRAIGHT LINES,

COMMENSURABLE, IN LENGTH;

[X. 19]

THEREFORE,

THE RECTANGLE, *FH*, *KL*, IS RATIONAL.

AND, SINCE,

DB = *BA*, AND

OB = *BC*,

THEREFORE,

AS *DB* IS TO *BC*,

SO IS *AB* TO *BO*.

[VI. 1]

BUT,

AS *DB* IS TO *BC*,

SO IS *DA* TO *AC*,

[*ID*.]

AND,

AS *AB* IS TO *BO*,

SO IS *AC* TO *CO*;

THEREFORE,

AS *DA* IS TO *AC*,

SO IS *AC* TO *CO*.

BUT,

AD = *GH*,

AC TO *MK*, AND

CO TO *NL*;

THEREFORE,

AS *GH* IS TO *MK*,

SO IS *MK* TO *NL*;

[VI. 1, V. 11]

THEREFORE ALSO,

AS *FH* IS TO *HK*,

SO IS *HK* TO *KL*;

[VI. 17]

THEREFORE,

THE RECTANGLE, *FH*, *KL* = THE SQUARE, ON *HK*.

BUT,

THE RECTANGLE, *FH*, *KL*, IS RATIONAL;

THEREFORE,

THE SQUARE, ON HK, IS, ALSO, RATIONAL.

THEREFORE,

HK IS RATIONAL.

[X. 19]

AND,

IF IT IS COMMENSURABLE, IN LENGTH, WITH FG,

HN IS RATIONAL;

BUT,

IF IT IS INCOMMENSURABLE, IN LENGTH, WITH FG,

KH, HM ARE RATIONAL STRAIGHT LINES

COMMENSURABLE, IN SQUARE, ONLY,

[X. 21]

AND THEREFORE,

HN IS MEDIAL.

THEREFORE,

HN IS EITHER RATIONAL OR MEDIAL.

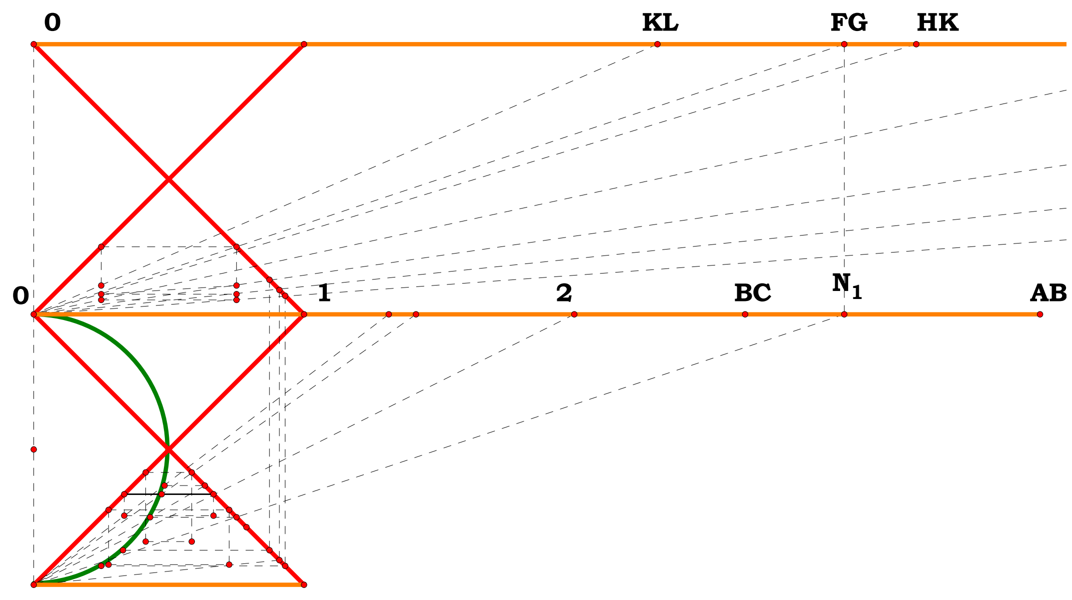
BUT,

$HN = AC$;

THEREFORE,

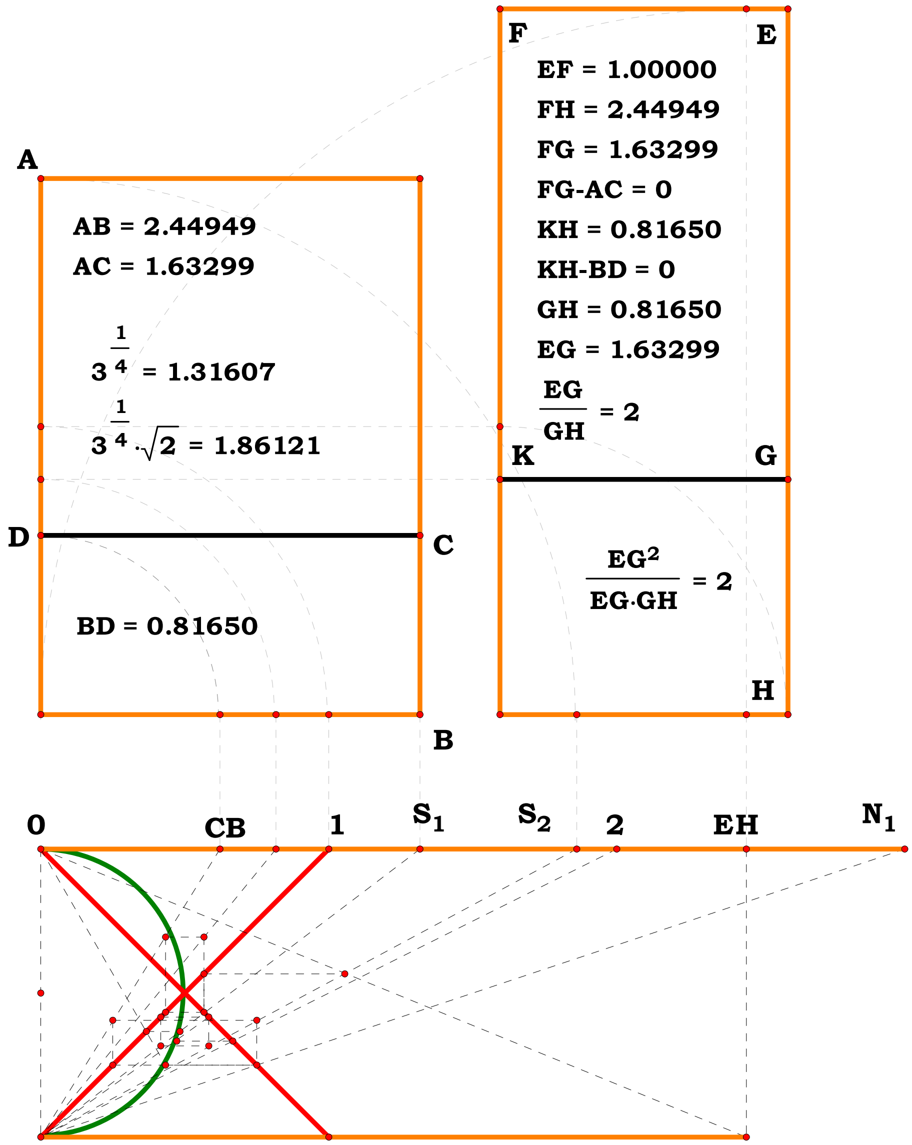
AC IS EITHER RATIONAL OR MEDIAL.

THEREFORE ETC.



Proposition 26.

A medial area does not exceed a medial area by a rational area.



FOR, IF POSSIBLE, LET,

THE MEDIAL AREA, *AB*, EXCEED THE MEDIAL AREA, *AC*, BY THE RATIONAL AREA, *DB*,

AND LET,

A RATIONAL STRAIGHT LINE, *EF*, BE SET OUT;

LET,

TO *EF*, THERE BE APPLIED,
THE RECTANGULAR PARALLELOGRAM, *FH*,
EQUAL TO *AB*, PRODUCING *EH*, AS BREADTH,

AND LET,

THE RECTANGLE, *FG*, EQUAL TO *AC*, BE SUBTRACTED;

THEREFORE,

THE REMAINDERS, *BD* = *KH*.

But,

DB is rational;

therefore,

KH is, also, rational.

Since,

then, each, of the rectangles, AB, AC, is medial, and
$AB = FH$, and
AC to FG,

therefore,

each, of the rectangles, FH, FG, is, also, medial.

And,

they are applied to the rational straight line, EF;

[x. 22]

therefore,

each, of the straight lines, HE, EG, is rational and incommensurable, in length, with EF.

And, since,

[DB is rational, and
$= KH$,

therefore],

KH is [also] rational; and
it is applied to the rational straight line, EF;

[x. 20]

therefore,

GH is rational and commensurable, in length, with EF.

But,

EG is, also, rational, and
is incommensurable, in length, with EF;

[x. 13]

therefore,

EG is incommensurable, in length, with GH.

And,

as EG is to GH,
so is the square, on EG, to the rectangle, EG, GH;

[x. 11]

therefore,

the square, on EG, is incommensurable with
the rectangle, EG, GH.

But,

THE SQUARES, ON *EG*, *GH*, ARE COMMENSURABLE WITH
THE SQUARE, ON *EG*, FOR BOTH ARE RATIONAL;

[X. 6]

AND,

TWICE THE RECTANGLE, *EG*, *GH*, IS COMMENSURABLE WITH
THE RECTANGLE, *EG*, *GH*, FOR IT IS DOUBLE OF IT;

[X. 13]

THEREFORE,

THE SQUARES, ON *EG*, *GH*, ARE INCOMMENSURABLE WITH
TWICE THE RECTANGLE, *EG*, *GH*;

[II. 4]

THEREFORE ALSO,

THE SUM OF THE SQUARES, ON *EG*, *GH*, AND
TWICE THE RECTANGLE, *EG*, *GH*,

[X. 16]

THAT IS,

THE SQUARE, ON *EH*, IS INCOMMENSURABLE WITH
THE SQUARES, ON *EG*, *GH*.

But,

THE SQUARES, ON *EG*, *GH*, ARE RATIONAL;

[X. Def. 4]

THEREFORE,

THE SQUARE, ON *EH*, IS IRRATIONAL.

Therefore,

EH IS IRRATIONAL.

But,

IT IS, ALSO, RATIONAL:

WHICH,

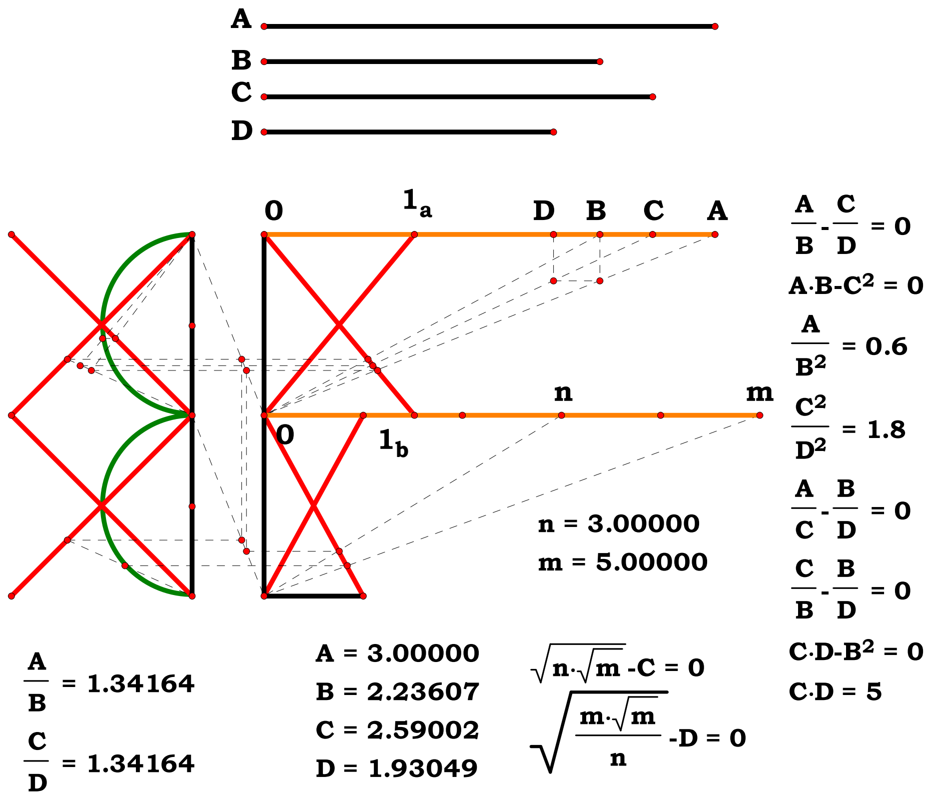
IS IMPOSSIBLE.

Therefore etc.

Q. E. D.

PROPOSITION 27.

TO FIND MEDIAL STRAIGHT LINES COMMENSURABLE IN SQUARE ONLY WHICH CONTAIN A RATIONAL RECTANGLE.



LET,

TWO RATIONAL STRAIGHT LINES A, B,
COMMENSURABLE, IN SQUARE, ONLY,
BE SET OUT;

[VI. 13]

LET,

C, BE TAKEN A MEAN PROPORTIONAL BETWEEN A, B,

[VI. 12]

AND LET, IT BE CONTRIVED THAT;

AS A IS TO B,
SO IS C TO D.

[VI 17]

THEN, SINCE,

A, B ARE RATIONAL AND COMMENSURABLE, IN SQUARE, ONLY,
THE RECTANGLE, A, B,

[X. 21]

THAT IS,

THE SQUARE, ON C, IS MEDIAL.

[X. 21]

THEREFORE,
C IS MEDIAL.

AND SINCE,
AS A IS TO B,
SO IS C TO D, AND
A, B ARE COMMENSURABLE, IN SQUARE, ONLY,

[X. 11]

THEREFORE,
C, D ARE, ALSO, COMMENSURABLE, IN SQUARE, ONLY.

AND,
C IS MEDIAL;

[X. 23, ADDITION]

THEREFORE,
D IS, ALSO, MEDIAL.

THEREFORE,
C, D ARE MEDIAL AND COMMENSURABLE, IN SQUARE, ONLY.

I SAY THAT;
THEY, ALSO, CONTAIN A RATIONAL RECTANGLE.

FOR SINCE,
AS A IS TO B,
SO IS C TO D,

[V. 16]

THEREFORE, ALTERNATELY,
AS A IS TO C,
SO IS B TO D.

BUT,
AS A IS TO C,
SO IS C TO B;

THEREFORE ALSO,
AS C IS TO B,
SO IS B TO D;

THEREFORE,
THE RECTANGLE, C, D, = THE SQUARE, ON B.

BUT,
THE SQUARE, ON B, IS RATIONAL;

THEREFORE,
THE RECTANGLE, C, D, IS, ALSO, RATIONAL.

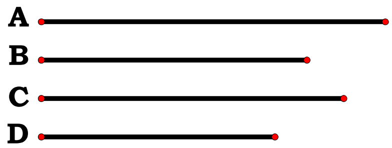
THEREFORE,

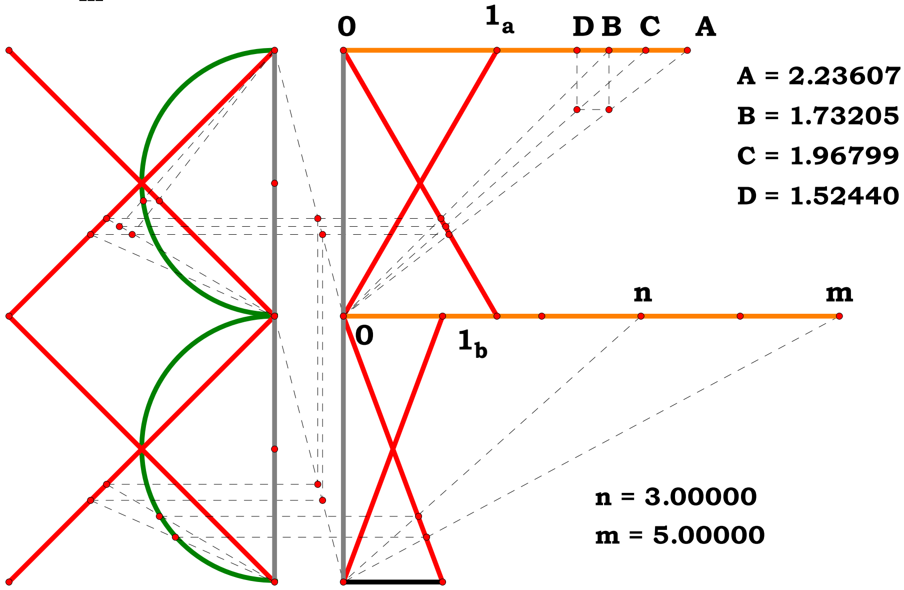
MEDIAL STRAIGHT LINES COMMENSURABLE, IN SQUARE, ONLY HAVE BEEN FOUND WHICH CONTAIN A RATIONAL RECTANGLE.

Q. E. D.

$$(n\cdot m)^{\frac{1}{4}} - C = 0$$

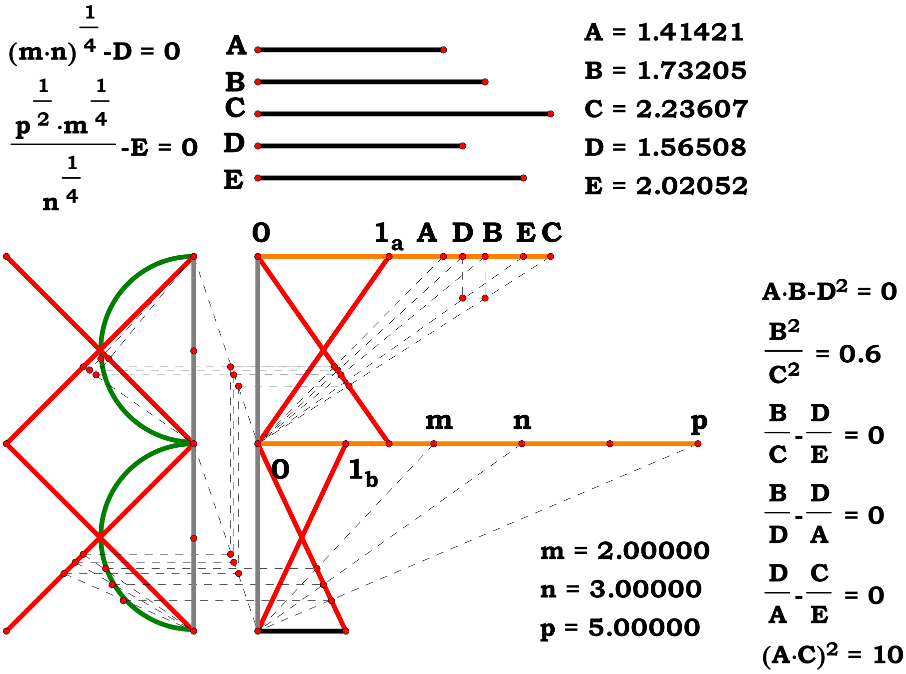
$$\frac{n^{\frac{3}{4}}}{m^{\frac{1}{4}}} - D = 0$$





PROPOSITION 28.

TO FIND MEDIAL STRAIGHT LINES COMMENSURABLE, IN SQUARE, ONLY WHICH CONTAIN A MEDIAL RECTANGLE.



LET,

THE RATIONAL STRAIGHT LINES *A*, *B*, *C*,
COMMENSURABLE, IN SQUARE, ONLY, BE SET OUT;

[VI. 13]

LET,

D BE TAKEN A MEAN PROPORTIONAL BETWEEN *A*, *B*,

[VI. 12]

AND LET IT BE CONTRIVED THAT;

AS *B* IS TO *C*,
SO IS *D* TO *E*.

SINCE,

A, *B* ARE RATIONAL STRAIGHT LINES
COMMENSURABLE, IN SQUARE, ONLY,

[VI. 17],

THEREFORE,

[X. 21]

THE RECTANGLE, *A*, *B*, THAT IS,
THE SQUARE, ON *D*, IS MEDIAL.

[X. 21]

THEREFORE,

D IS MEDIAL.

AND SINCE,

B, *C* ARE COMMENSURABLE, IN SQUARE, ONLY, AND
AS *B* IS TO *C*,
SO IS *D* TO *E*,

[X. 11]

THEREFORE,

D, *E* ARE, ALSO, COMMENSURABLE, IN SQUARE, ONLY.

BUT,

D IS MEDIAL;

[X. 23, ADDITION]

THEREFORE,

E IS, ALSO, MEDIAL.

THEREFORE,

D, *E* ARE MEDIAL STRAIGHT LINES
COMMENSURABLE, IN SQUARE, ONLY.

I SAY NEXT THAT;

THEY, ALSO, CONTAIN A MEDIAL RECTANGLE.

FOR SINCE,

AS *B* IS TO *C*,
SO IS *D* TO *E*,

[V. 16]

THEREFORE, ALTERNATELY,

AS *B* IS TO *D*,
SO IS *C* TO *E*.

BUT,

AS *B* IS TO *D*,
SO IS *D* TO *A*;

THEREFORE ALSO,

AS *D* IS TO *A*,
SO IS *C* TO *E*;

[VI. 16]

THEREFORE,

THE RECTANGLE, *A*, *C*, =
THE RECTANGLE, *D*, *E*.

[X. 21]

BUT,

THE RECTANGLE *A*, *C* IS MEDIAL;

THEREFORE,

THE RECTANGLE, *D*, *E*, IS, ALSO, MEDIAL.

THEREFORE,

MEDIAL STRAIGHT LINES COMMENSURABLE, IN SQUARE, ONLY HAVE BEEN FOUND WHICH CONTAIN A MEDIAL RECTANGLE.

Q. E. D.

LEMMA 1.

TO FIND TWO SQUARE NUMBERS SUCH THAT THEIR SUM IS, ALSO, *SQUARE*.

AB = 24
AC = 18
BC = 6
CD = 9
BD = 15

$(AB \cdot BC + CD^2) - BD^2 = 0$

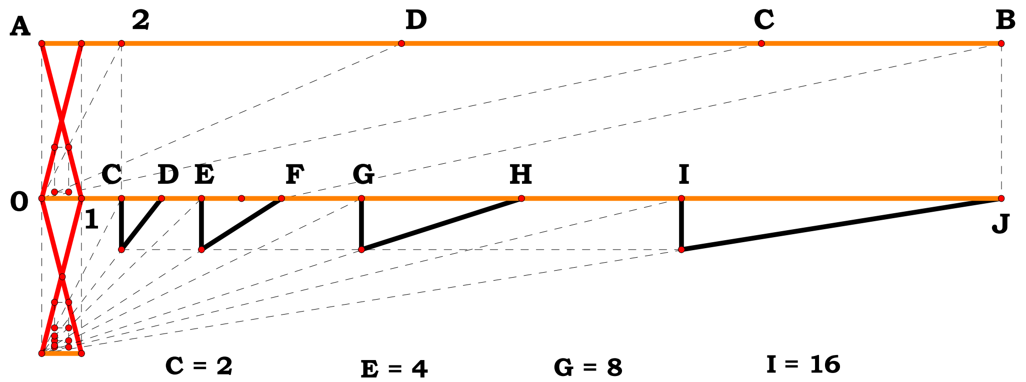
$(AB \cdot BC)^{\frac{1}{2}} = 12$

$\sqrt{AB \cdot BC} = 12$

$\sqrt{AB \cdot BC}^2 = 144$

$CD^2 = 81$

$BD^2 = 225$



C = 2
D = 3
E = 4
F = 6
G = 8
H = 12
I = 16
J = 24

LET,

TWO NUMBERS, *AB*, *BC*, BE SET OUT,

AND LET,

THEM BE EITHER BOTH EVEN OR BOTH ODD.

[IX. 24, 26]

THEN SINCE,

WHETHER AN EVEN NUMBER IS SUBTRACTED FROM AN EVEN NUMBER, OR AN ODD NUMBER FROM AN ODD NUMBER, THE REMAINDER IS EVEN,

THEREFORE,

THE REMAINDER, *AC*, IS EVEN.

LET,

AC BE BISECTED AT *D*.

LET,

AB, *BC*, ALSO, BE EITHER SIMILAR PLANE NUMBERS, OR SQUARE NUMBERS,

WHICH ARE THEMSELVES, ALSO, SIMILAR PLANE NUMBERS.

[II. 6]

NOW,

THE PRODUCT, OF *AB*, *BC*, TOGETHER WITH
THE SQUARE, ON *CD*, EQUALS THE SQUARE, ON *BD*.

[IX. 1]

AND,

THE PRODUCT, OF *AB*, *BC*, IS SQUARE,

INASMUCH AS IT WAS PROVED THAT,
IF TWO SIMILAR PLANE NUMBERS,
BY MULTIPLYING ONE ANOTHER,
MAKE SOME NUMBER THE PRODUCT IS SQUARE,

THEREFORE,

TWO SQUARE NUMBERS,
THE PRODUCT, OF *AB*, *BC*, AND THE SQUARE, ON *CD*,
HAVE BEEN FOUND WHICH,
WHEN ADDED TOGETHER, MAKE THE SQUARE, ON *BD*.

AND IT IS MANIFEST THAT;

TWO SQUARE NUMBERS,
THE SQUARE, ON *BD*, AND THE SQUARE, ON CD,
HAVE AGAIN BEEN FOUND SUCH THAT
THEIR DIFFERENCE, THE PRODUCT, OF *AB*, *BC*, IS A SQUARE,

WHENEVER,

AB, *BC* ARE SIMILAR PLANE NUMBERS.

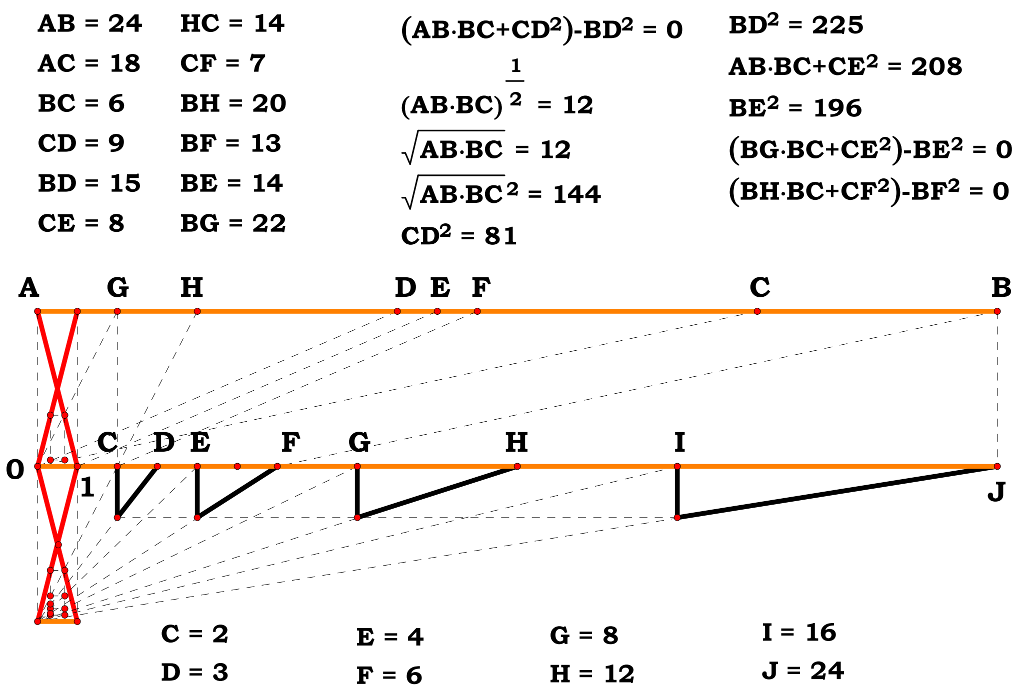
BUT,

WHEN THEY ARE NOT SIMILAR PLANE NUMBERS,
TWO SQUARE NUMBERS,
THE SQUARE, ON *BD*, AND THE SQUARE, ON *DC*,
HAVE BEEN FOUND SUCH THAT THEIR DIFFERENCE,
THE PRODUCT, OF *AB*, *BC*, IS NOT SQUARE.

Q. E. D.

LEMMA 2.

TO FIND TWO SQUARE NUMBERS SUCH THAT THEIR SUM IS NOT SQUARE.



FOR LET,

THE PRODUCT, OF *AB*, *BC*, AS WE SAID, BE SQUARE, AND *CA* EVEN,

AND LET,

CA BE BISECTED BY *D*.

[SEE LEMMA 1]

IT IS THEN MANIFEST THAT;

THE SQUARE, PRODUCT OF *AB*, *BC*, TOGETHER WITH THE SQUARE, ON *CD*, = THE SQUARE, ON *BD*.

LET,

THE UNIT, *DE*, BE SUBTRACTED;

THEREFORE,

THE PRODUCT, OF *AB*, *BC*, TOGETHER WITH THE SQUARE, ON *CE*, IS LESS THAN THE SQUARE, ON *BD*.

I SAY THEN THAT;

THE SQUARE, PRODUCT OF *AB*, *BC*, TOGETHER WITH THE SQUARE, ON *CE*, WILL NOT BE SQUARE.

FOR,

IF IT IS SQUARE,
IT IS EITHER EQUAL TO THE SQUARE, ON *BE*, OR LESS THAN THE SQUARE, ON *BE*,

BUT,

CANNOT ANY MORE BE GREATER,
LEST THE UNIT BE DIVIDED.

FIRST, IF POSSIBLE, LET,

THE PRODUCT, OF AB, BC, TOGETHER WITH
THE SQUARE, ON CE ,BE EQUAL TO THE SQUARE, ON BE,

AND LET,

GA BE DOUBLE OF THE UNIT, DE.

SINCE THEN,

THE WHOLE, AC, IS DOUBLE OF THE WHOLE, CD, AND
IN THEM, AG, IS DOUBLE OF DE,

THEREFORE,

THE REMAINDER, GC, IS, ALSO, DOUBLE OF
THE REMAINDER, EC;

THEREFORE,

GC IS BISECTED BY E.

[II. 6]

THEREFORE,

THE PRODUCT, OF GB, BC, TOGETHER WITH
THE SQUARE, ON CE, = THE SQUARE, ON BE.

BUT,

THE PRODUCT, OF AB, BC, TOGETHER WITH
THE SQUARE, ON CE, IS ALSO, BY HYPOTHESIS,
EQUAL TO THE SQUARE, ON BE;

THEREFORE,

THE PRODUCT, OF GB, BC, TOGETHER WITH
THE SQUARE, ON CE, =
THE PRODUCT, OF AB, BC, TOGETHER WITH
THE SQUARE, ON CE.

AND,

IF THE COMMON SQUARE, ON CE, BE SUBTRACTED,
IT FOLLOWS THAT $AB = GB$:

WHICH,

IS ABSURD.

THEREFORE,

THE PRODUCT, OF AB, BC, TOGETHER WITH
THE SQUARE, ON CE, IS NOT EQUAL TO THE SQUARE, ON BE.

I SAY NEXT THAT;

NEITHER IS IT LESS THAN THE SQUARE, ON BE.

FOR, IF POSSIBLE, LET,

IT BE EQUAL TO THE SQUARE, ON BF,

AND LET,

HA be double of *DF*.

Now, it will again follow that;
HC is double of *CF*;

so that,
CH has, also, been bisected at *F*,

[II. 6]

and for this reason,
the product, of *HB*, *BC*, together with
the square, on *FC*, = the square, on *BF*.

But, by hypothesis,
the product, of *AB*, *BC*, together with
the square, on *CE*, = the square, on *BF*.

Thus,
the product, of *HB*, *BC*, together with
the square, on *CE*, will, also, be equal to
the product, of *AB*, *BC*, together with
the square, on *CE*:

which,
is absurd.

Therefore,
the product, of *AB*, *BC*, together with
the square, on *CE*, is not less than the square, on *BE*.

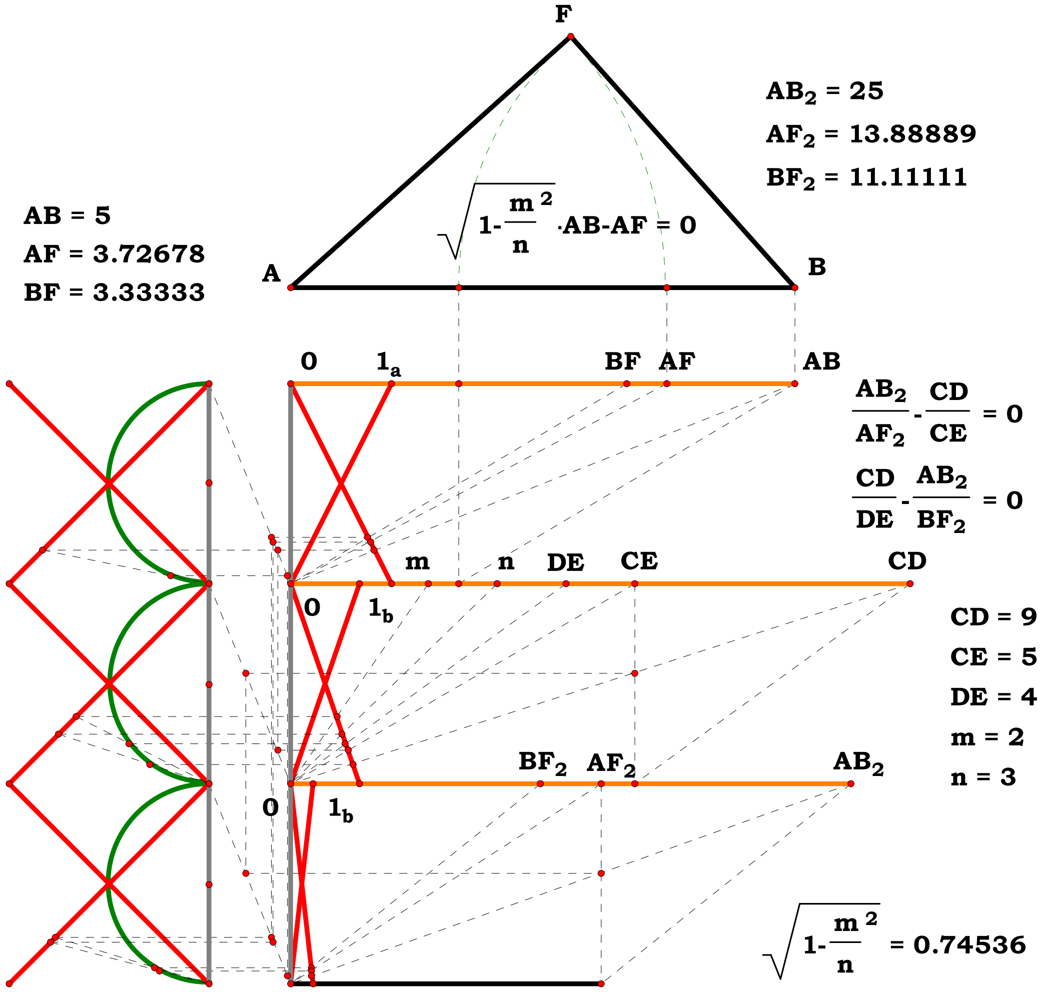
And it was proved that;
neither is it equal to the square, on *BE*.

Therefore,
the product, of *AB*, *BC*, together with
the square, on *CE*, is not square.

Q. E. D.

Proposition 29.

To find two rational straight lines commensurable, in square, only and such that the square, on the greater is greater than the square, on the less by the square, on a straight line commensurable in length with the greater.



[Lemma 1]

For let,

THERE BE SET OUT ANY RATIONAL STRAIGHT LINE, *AB*, AND TWO SQUARE NUMBERS, *CD*, *DE*, SUCH THAT THEIR DIFFERENCE, *CE*, IS NOT SQUARE;

LET,

THERE BE DESCRIBED, ON *AB*, THE SEMICIRCLE, *AFB*,

[X. 6, Por.]

AND LET IT BE CONTRIVED THAT;

AS *DC* IS TO *CE*, SO IS THE SQUARE, ON *BA*, TO THE SQUARE, ON *AF*.

Let,

FB BE JOINED.

Since,

as the square, on BA, is to the square, on AF,
so is DC to CE,

therefore,

the square, on BA, has to the square, on AF,
the ratio which
the number, DC, has to the number, CE;

[x. 6]

therefore,

the square, on BA, is commensurable with
the square, on AF.

[x. Def. 4]

But,

the square, on AB, is rational;

[*id.*]

therefore,

the square, on AF, is, also, rational;

therefore,

AF is, also, rational.

And, since,

DC has not to CE, the ratio which
a square number has to a square number,
neither has the square, on BA, to the square, on AF,
the ratio which,
a square number has to a square number;

[x. 9]

therefore,

AB is incommensurable, in length, with AF.

Therefore,

BA, AF are rational straight lines
commensurable, in square, only.

And since,

as DC is to CE,
so is the square, on BA, to the square, on AF,

[v. 19, Por. iii. 31, i. 47]

therefore, *convertendo*,

as CD is to DE,
so is the square, on AB, to the square, on BF.

But,

CD HAS TO *DE*, THE RATIO WHICH,
A SQUARE NUMBER HAS TO A SQUARE NUMBER:

THEREFORE ALSO,
THE SQUARE, ON *AB*, HAS TO THE SQUARE, ON *BF*,
THE RATIO WHICH,
A SQUARE NUMBER HAS TO A SQUARE NUMBER;

[X. 9]

THEREFORE,
AB IS COMMENSURABLE, IN LENGTH, WITH *BF*.

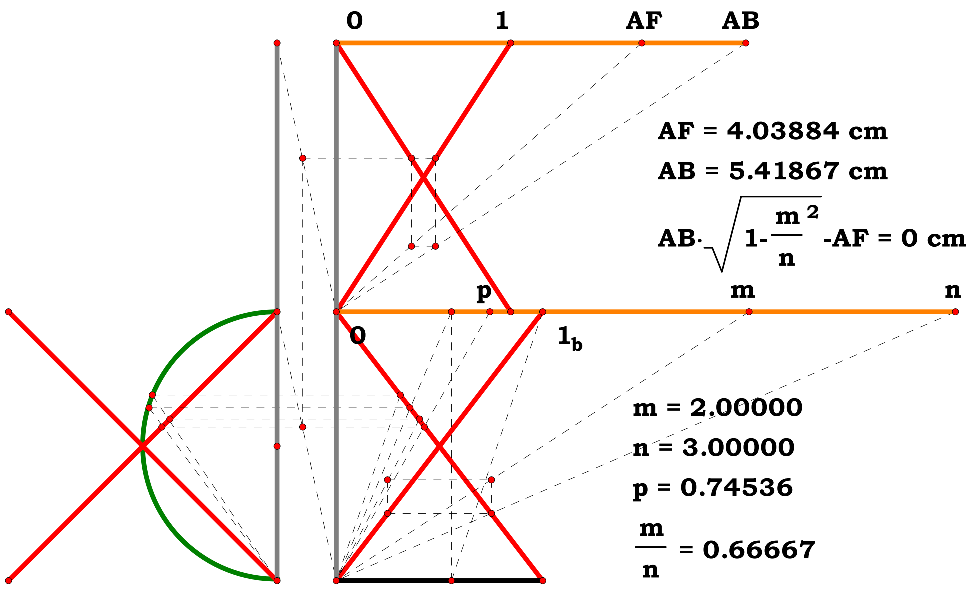
AND,
THE SQUARE, ON *AB*, =
THE SQUARES, ON *AF*, *FB*;

THEREFORE,
THE SQUARE, ON *AB*, IS GREATER THAN
THE SQUARE, ON *AF*, BY
THE SQUARE, ON *BF*, COMMENSURABLE WITH *AB*.

THEREFORE,
THERE HAVE BEEN FOUND TWO RATIONAL STRAIGHT LINES,
BA, *AF*, COMMENSURABLE, IN SQUARE, ONLY, AND

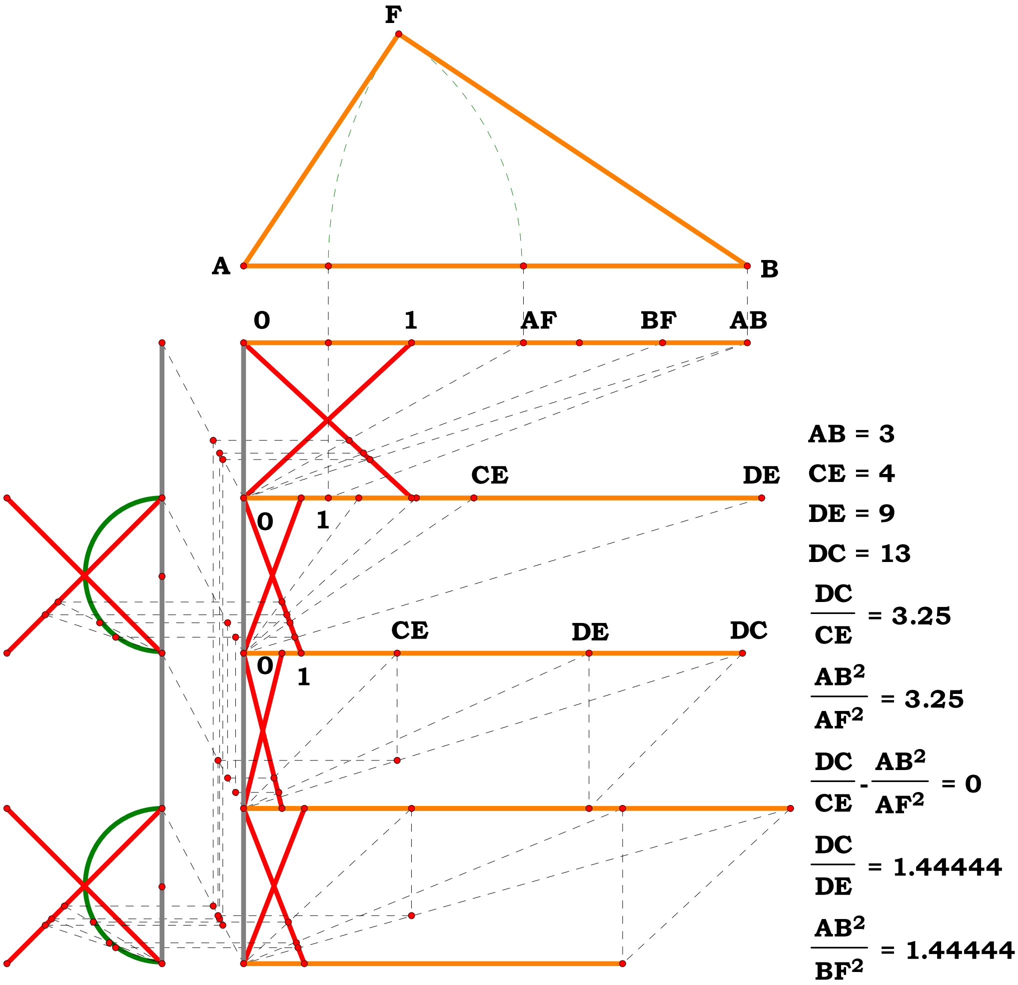
SUCH THAT,
THE SQUARE, ON THE GREATER *AB*, IS GREATER THAN
THE SQUARE, ON THE LESS *AF*, BY THE SQUARE, ON *BF*,
COMMENSURABLE, IN LENGTH, WITH *AB*.

$$\sqrt{1-\frac{m}{n}^2} = 0.74536 \qquad p-\sqrt{1-\frac{m}{n}^2} = 0$$



Proposition 30.

To find two rational straight lines commensurable in square, only and such that the square, on the greater is greater than the square, on the less by the square, on a straight line incommensurable in length with the greater.



[Lemma 2]

Let,

There be set out a rational straight line, *AB*, and two square numbers, *CE*, *ED*, such that their sum, *CD*, is not square;

let,

There be described, on *AB*, the semicircle, *AFB*,

[x. 6, Por.]

let it be contrived that;

as *DC* is to *CE*,

so is the square, on *BA*, to the square, on *AF*,

and let,

FB be joined.

Then, in a similar manner to the preceding,

WE CAN PROVE THAT;

BA, AF ARE RATIONAL STRAIGHT LINES
COMMENSURABLE, IN SQUARE, ONLY.

AND SINCE,

AS DC IS TO CE,
SO IS THE SQUARE, ON BA, TO THE SQUARE, ON AF,

[V. 19, POR. III. 31, I. 47]

THEREFORE, *CONVERTENDO*,

AS CD IS TO DE,
SO IS THE SQUARE, ON AB, TO THE SQUARE, ON BF.

BUT,

CD HAS NOT TO DE, THE RATIO WHICH,
A SQUARE NUMBER HAS TO A SQUARE NUMBER;

THEREFORE, NEITHER HAS

THE SQUARE, ON AB, TO THE SQUARE, ON BF,
THE RATIO WHICH,
A SQUARE NUMBER HAS TO A SQUARE NUMBER;

[X. 9]

THEREFORE,

AB IS INCOMMENSURABLE, IN LENGTH, WITH BF.

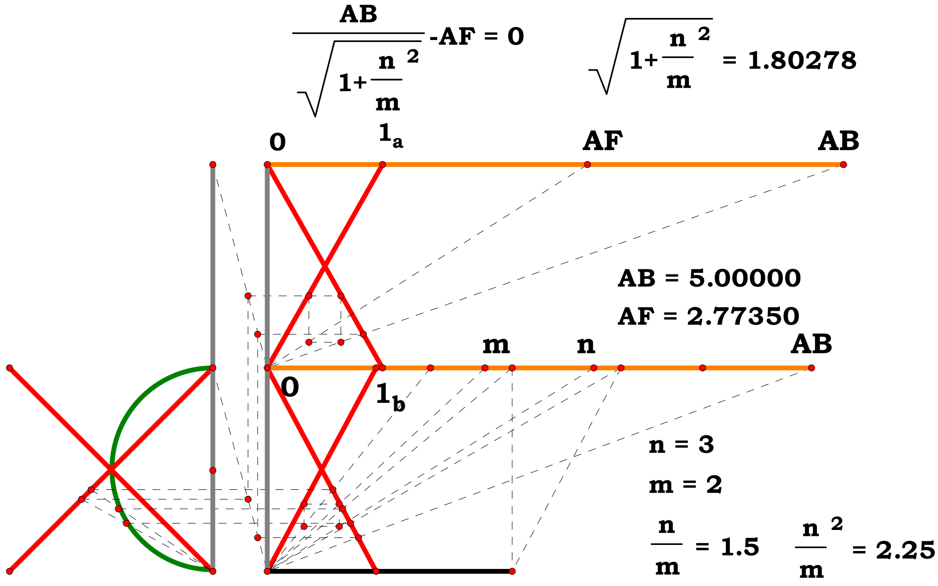
AND,

THE SQUARE, ON AB, IS GREATER THAN
THE SQUARE, ON AF, BY
THE SQUARE, ON FB, INCOMMENSURABLE WITH AB.

THEREFORE,

AB, AF ARE RATIONAL STRAIGHT LINES
COMMENSURABLE, IN SQUARE, ONLY, AND
THE SQUARE, ON AB, IS GREATER THAN
THE SQUARE, ON AF, BY THE SQUARE, ON FB,
INCOMMENSURABLE, IN LENGTH, WITH AB.

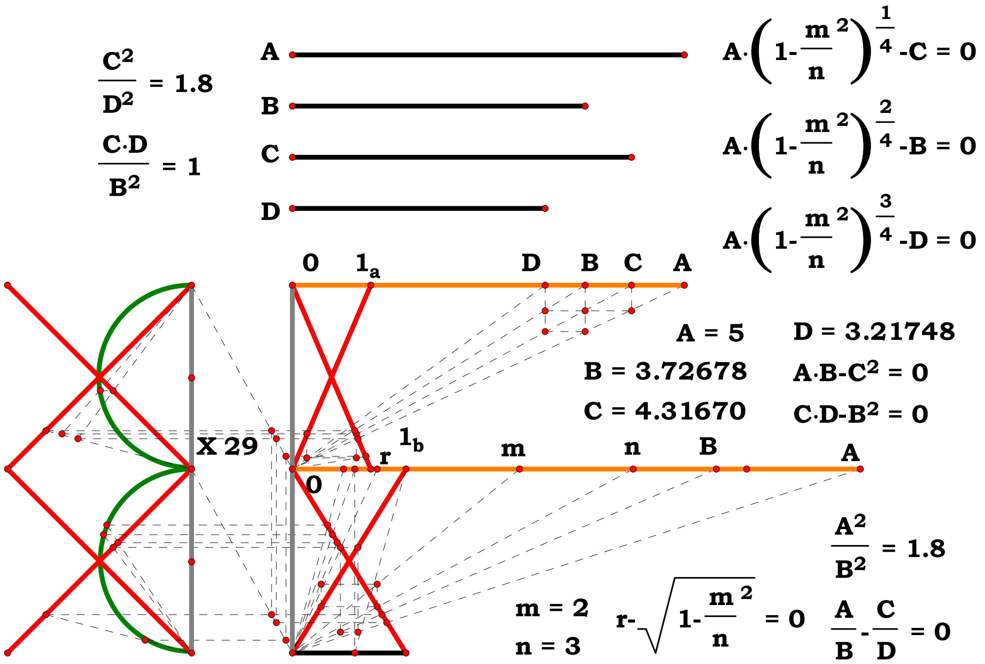
Q. E. D.



$\frac{AB}{\sqrt{1+\frac{n}{m}^2}} - AF = 0$
$\sqrt{1+\frac{n}{m}^2} = 1.80278$
0
1_a
AF
AB
AB = 5.00000
AF = 2.77350
m
n
AB
0
1_b
n = 3
m = 2
$\frac{n}{m} = 1.5$
$\frac{n}{m}^2 = 2.25$

PROPOSITION 31.

TO FIND TWO MEDIAL STRAIGHT LINES COMMENSURABLE IN SQUARE, ONLY, CONTAINING A RATIONAL RECTANGLE, AND SUCH THAT THE SQUARE, ON THE GREATER IS GREATER THAN THE SQUARE, ON THE LESS BY THE SQUARE, ON A STRAIGHT LINE COMMENSURABLE IN LENGTH WITH THE GREATER.



LET,

THERE BE SET OUT TWO RATIONAL STRAIGHT LINES, *A*, *B*, COMMENSURABLE, IN SQUARE, ONLY

[X. 29]

AND, SUCH THAT;

THE SQUARE, ON *A*, BEING THE GREATER,
IS GREATER THAN THE SQUARE, ON *B*,
THE LESS, BY THE SQUARE, ON
A STRAIGHT LINE COMMENSURABLE, IN LENGTH, WITH *A*.

AND LET,

THE SQUARE, ON *C*, BE EQUAL TO THE RECTANGLE, *A*, *B*.

[X. 21]

NOW,

THE RECTANGLE, *A*, *B*, IS MEDIAL;

THEREFORE,

THE SQUARE, ON *C*, IS, ALSO, MEDIAL;

[X. 21]

THEREFORE,

C IS, ALSO, MEDIAL.

LET,
THE RECTANGLE, C, D, BE EQUAL TO THE SQUARE, ON B.

NOW,
THE SQUARE, ON B, IS RATIONAL;

THEREFORE,
THE RECTANGLE, C, D, IS, ALSO, RATIONAL.

AND SINCE,
AS A IS TO B,
SO IS THE RECTANGLE, A, B, TO THE SQUARE, ON B, WHILE
THE SQUARE, ON C, = THE RECTANGLE, A, B, AND
THE RECTANGLE, C, D, = THE SQUARE, ON B,

THEREFORE,
AS A IS TO B,
SO IS THE SQUARE, ON C, TO THE RECTANGLE, C, D.

BUT,
AS THE SQUARE, ON C, IS TO THE RECTANGLE, C, D,
SO IS C TO D;

THEREFORE ALSO,
AS A IS TO B,
SO IS C TO D.

BUT,
A IS COMMENSURABLE WITH B, IN SQUARE, ONLY;

[X. 11]

THEREFORE,
C IS, ALSO, COMMENSURABLE WITH D, IN SQUARE, ONLY.

AND,
C IS MEDIAL;

[X. 23, ADDITION]

THEREFORE,
D IS, ALSO, MEDIAL.

AND SINCE,
AS A IS TO B,
SO IS C TO D, AND
THE SQUARE, ON A, IS GREATER THAN
THE SQUARE, ON B, BY
THE SQUARE, ON A STRAIGHT LINE COMMENSURABLE WITH A,

[X.14]

THEREFORE ALSO,
THE SQUARE, ON C, IS GREATER THAN

THE SQUARE, ON D, BY
THE SQUARE, ON A STRAIGHT LINE COMMENSURABLE WITH C.

THEREFORE,

TWO MEDIAL STRAIGHT LINES, C, D,
COMMENSURABLE, IN SQUARE, ONLY, AND
CONTAINING A RATIONAL RECTANGLE, HAVE BEEN FOUND, AND
THE SQUARE, ON C, IS GREATER THAN
THE SQUARE, ON D, BY THE SQUARE, ON A STRAIGHT LINE
COMMENSURABLE, IN LENGTH, WITH C.

[X. 30]

SIMILARLY ALSO, IT CAN BE PROVED THAT;

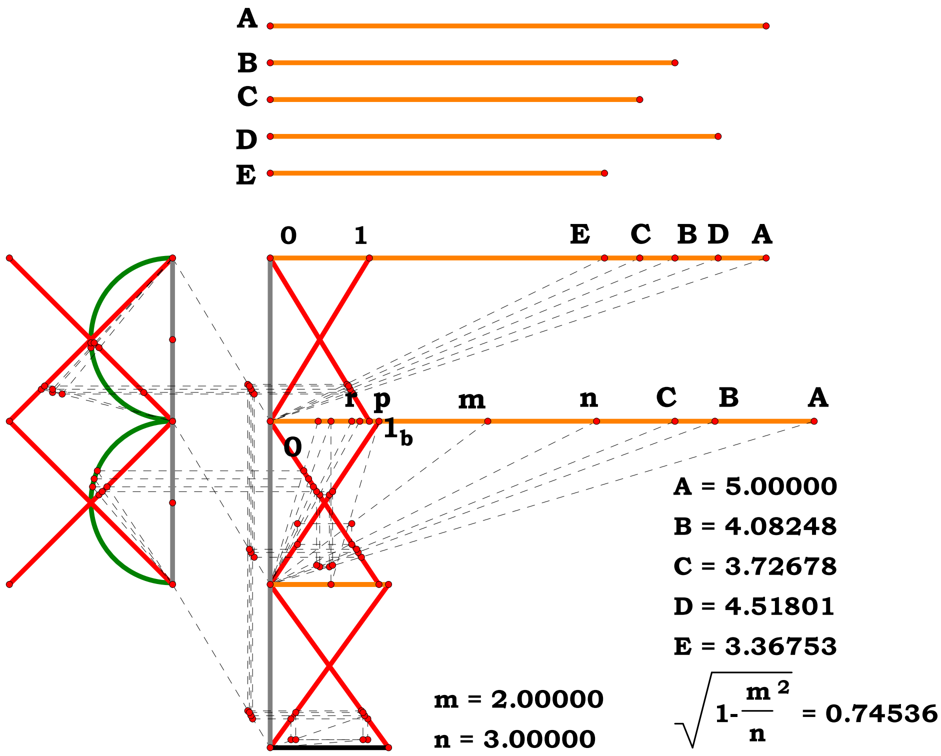
THE SQUARE, ON C, EXCEEDS
THE SQUARE, ON D, BY
THE SQUARE, ON A STRAIGHT LINE INCOMMENSURABLE WITH C,

WHEN,

THE SQUARE, ON A, IS GREATER THAN
THE SQUARE, ON B, BY
THE SQUARE, ON A STRAIGHT LINE INCOMMENSURABLE WITH A.

PROPOSITION 32.

TO FIND TWO MEDIAL STRAIGHT LINES COMMENSURABLE, IN SQUARE, ONLY, CONTAINING A MEDIAL RECTANGLE, AND SUCH THAT THE SQUARE, ON THE GREATER IS GREATER THAN THE SQUARE, ON THE LESS BY THE SQUARE, ON A STRAIGHT LINE COMMENSURABLE WITH THE GREATER.



$\frac{A^2}{C^2} = 1.80000$

$\frac{A^2}{B^2} = 1.50000$

$A \cdot B - D^2 = 0$

$D \cdot E - B \cdot C = 0$

$\frac{A}{C} - \frac{D^2}{D \cdot E} = 0$

$\frac{A}{C} - \frac{D}{E} = 0$

$\frac{\sqrt{D^2 - E^2}}{D} = 0.66667$

[X. 29]

LET,

THERE BE SET OUT THREE RATIONAL STRAIGHT LINES, *A*, *B*, *C*, COMMENSURABLE, IN SQUARE, ONLY, AND SUCH THAT;
THE SQUARE, ON *A*, IS GREATER THAN
THE SQUARE, ON *C*, BY
THE SQUARE, ON A STRAIGHT LINE COMMENSURABLE WITH *A*,

AND LET,

THE SQUARE, ON *D*, BE EQUAL TO
THE RECTANGLE, *A*, *B*.

THEREFORE,

THE SQUARE, ON *D*, IS MEDIAL;

[X. 21]

THEREFORE,

D IS, ALSO, MEDIAL.

LET,

THE RECTANGLE, D, E, BE EQUAL TO
THE RECTANGLE, B, C.

THEN SINCE,

AS THE RECTANGLE, A, B, IS TO
THE RECTANGLE, B, C,
SO IS A TO C,

WHILE,

THE SQUARE, ON D, =
THE RECTANGLE, A, B, AND
THE RECTANGLE D, E =
THE RECTANGLE B, C,

THEREFORE,

AS A IS TO C,
SO IS THE SQUARE, ON D, TO
THE RECTANGLE D, E.

BUT,

AS THE SQUARE, ON D, IS TO
THE RECTANGLE, D, E,
SO IS D TO E;

THEREFORE ALSO,

AS A IS TO C,
SO IS D TO E.

BUT

A IS COMMENSURABLE WITH C, IN SQUARE, ONLY;

[X. 11]

THEREFORE,

D IS, ALSO, COMMENSURABLE WITH E, IN SQUARE, ONLY.

[X. 23, ADDITION]

BUT,

D IS MEDIAL;

THEREFORE,

E IS, ALSO, MEDIAL.

AND, SINCE,

AS A IS TO C,
SO IS D TO E,

WHILE,

THE SQUARE, ON A, IS GREATER THAN
THE SQUARE, ON C, BY
THE SQUARE, ON A STRAIGHT LINE COMMENSURABLE WITH A,

[X. 14]

THEREFORE ALSO,
THE SQUARE, ON D, WILL BE GREATER THAN
THE SQUARE, ON E, BY
THE SQUARE, ON A STRAIGHT LINE COMMENSURABLE WITH D.

I SAY NEXT THAT;
THE RECTANGLE D, E IS, ALSO, MEDIAL.

[X. 21]

FOR, SINCE,
THE RECTANGLE, B, C, =
THE RECTANGLE, D, E,

WHILE,
THE RECTANGLE, B, C, IS MEDIAL,

THEREFORE,
THE RECTANGLE, D, E, IS, ALSO, MEDIAL.

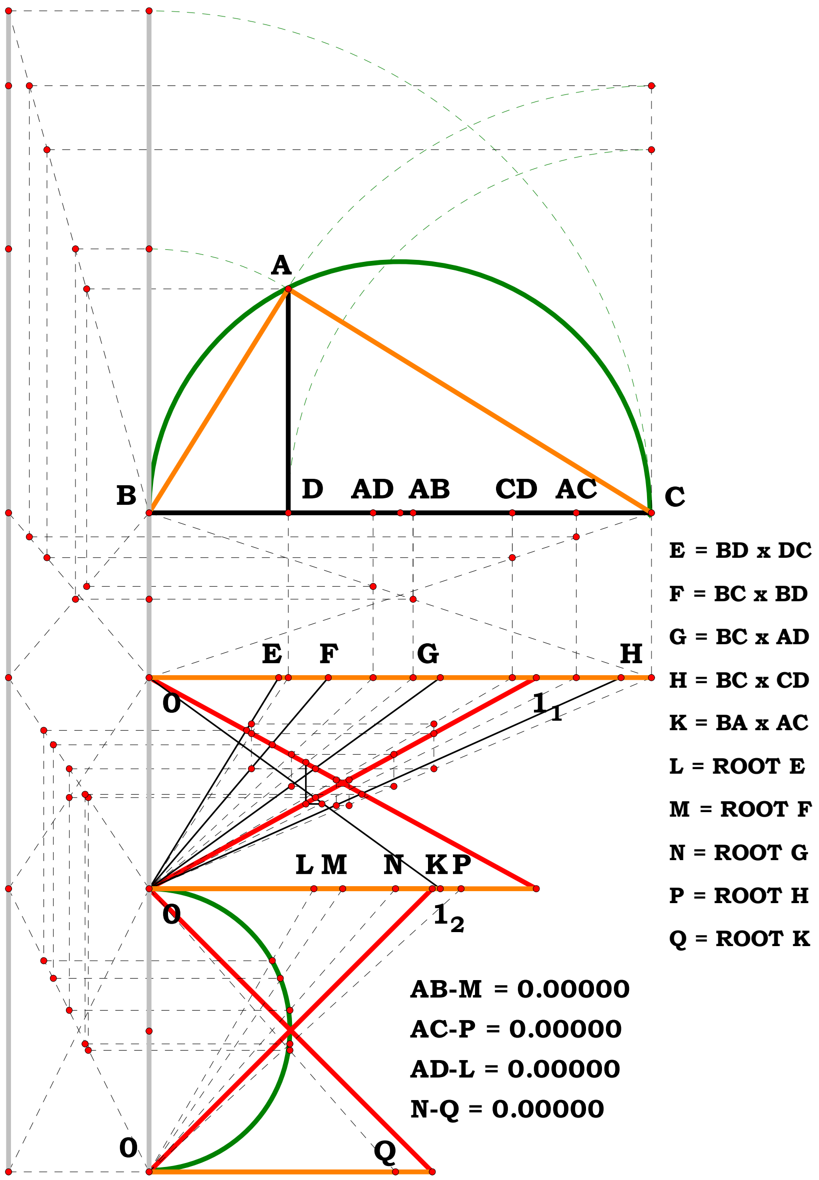
THEREFORE,
TWO MEDIAL STRAIGHT LINES, D, E,
COMMENSURABLE, IN SQUARE, ONLY, AND
CONTAINING A MEDIAL RECTANGLE,
HAVE BEEN FOUND SUCH THAT
THE SQUARE, ON THE GREATER IS GREATER THAN
THE SQUARE, ON THE LESS BY THE SQUARE, ON A STRAIGHT LINE
COMMENSURABLE WITH THE GREATER.

[X. 30]

SIMILARLY AGAIN, IT CAN BE PROVED THAT;
THE SQUARE, ON D, IS GREATER THAN
THE SQUARE, ON E, BY
THE SQUARE, ON A STRAIGHT LINE INCOMMENSURABLE WITH D,
WHEN THE SQUARE, ON A, IS GREATER THAN
THE SQUARE, ON C, BY
THE SQUARE, ON A STRAIGHT LINE INCOMMENSURABLE WITH A.

LEMMA.

LET ABC BE A RIGHT-ANGLED TRIANGLE HAVING $\angle A$, RIGHT, AND LET THE PERPENDICULAR, AD, BE DRAWN;



I SAY THAT;

THE RECTANGLE, *CB*, *BD*, = THE SQUARE, ON *BA*,
THE RECTANGLE, *BC*, *CD*, EQUAL TO THE SQUARE, ON *CA*,
THE RECTANGLE, *BD*, *DC*, EQUAL TO THE SQUARE, ON *AD*,

AND, FURTHER,

THE RECTANGLE, *BC*, *AD*, EQUAL TO
THE RECTANGLE, *BA*, *AC*.

AND FIRST THAT;

THE RECTANGLE, *CB*, *BD*, =
THE SQUARE, ON *BA*.

FOR, SINCE,

IN A RIGHT-ANGLED TRIANGLE, *AD*, HAS BEEN DRAWN
FROM THE RIGHT ANGLE PERPENDICULAR TO THE BASE,

[VI. 8]

THEREFORE,

THE TRIANGLES, ABD, ADC, ARE SIMILAR BOTH TO
THE WHOLE, ABC, AND TO ONE ANOTHER.

AND SINCE,

ΔABC, IS SIMILAR TO ΔABD,

[VI. 4]

THEREFORE,
AS CB IS TO BA,
SO IS BA TO BD;

[VI. 17]

THEREFORE,
THE RECTANGLE, CB, BD, =
THE SQUARE, ON AB.

FOR THE SAME REASON,
THE RECTANGLE, BC, CD, =
THE SQUARE, ON AC.

[VI. 8, POR.]

AND SINCE,
IF IN A RIGHT-ANGLED TRIANGLE A PERPENDICULAR BE DRAWN
FROM THE RIGHT ANGLE TO THE BASE,
THE PERPENDICULAR SO DRAWN IS A MEAN PROPORTIONAL
BETWEEN THE SEGMENTS OF THE BASE,

THEREFORE,
AS BD IS TO DA,
SO IS AD TO DC;

[VI. 17]

THEREFORE,
THE RECTANGLE, BD, DC, = THE SQUARE, ON AD.

I SAY THAT;
THE RECTANGLE, BC, AD, =
THE RECTANGLE, BA, AC.

FOR SINCE, AS WE SAID,
ABC IS SIMILAR TO ABD,

[VI. 4]

THEREFORE,
AS BC IS TO CA,
SO IS BA TO AD.

[VI. 16]

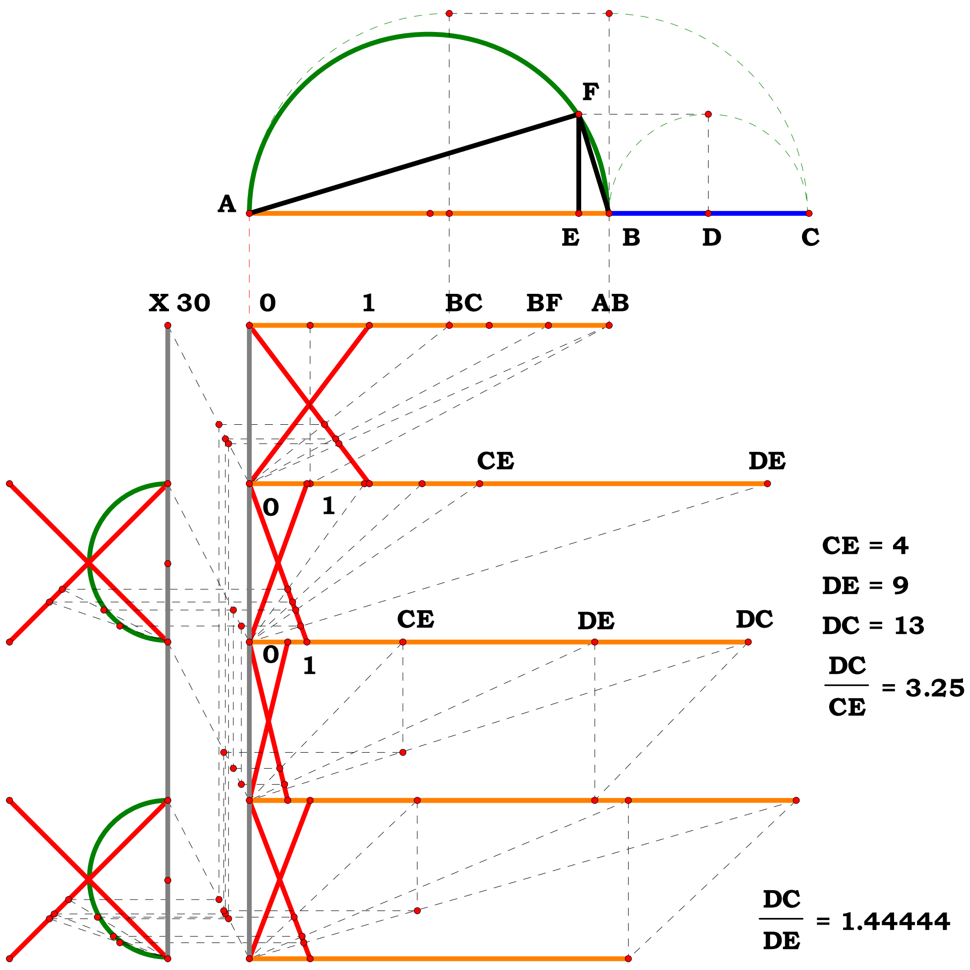
THEREFORE,

THE RECTANGLE, BC, AD, =
THE RECTANGLE, BA, AC.

Q. E. D.

PROPOSITION 33.

TO FIND TWO STRAIGHT LINES INCOMMENSURABLE, IN SQUARE, WHICH MAKE THE SUM OF THE SQUARES ON THEM RATIONAL BUT THE RECTANGLE CONTAINED BY THEM MEDIAL.



[x. 30]

LET,

THERE BE SET OUT TWO RATIONAL STRAIGHT LINES,

AB, *BC*, COMMENSURABLE, IN SQUARE, ONLY

AND SUCH THAT;

THE SQUARE, ON THE GREATER *AB*, IS GREATER THAN

THE SQUARE, ON THE LESS *BC*, BY

THE SQUARE, ON A STRAIGHT LINE,

INCOMMENSURABLE WITH *AB*,

LET,

BC BE BISECTED AT *D*,

LET,

THERE BE APPLIED TO *AB*, A PARALLELOGRAM EQUAL TO

THE SQUARE, ON EITHER OF

THE STRAIGHT LINES, *BD*, *DC*, AND

DEFICIENT BY A SQUARE FIGURE,

[VI. 28]

AND LET,

IT BE THE RECTANGLE, AE, EB;

LET,

THE SEMICIRCLE, AFB, BE DESCRIBED ON AB,

LET,

EF BE DRAWN AT RIGHT ANGLES TO AB,

AND LET,

AF, FB BE JOINED.

THEN, SINCE,

AB, BC ARE UNEQUAL STRAIGHT LINES, AND,
THE SQUARE, ON AB, IS GREATER THAN
THE SQUARE, ON BC, BY
THE SQUARE, ON A STRAIGHT LINE
INCOMMENSURABLE WITH AB, WHILE
THERE HAS BEEN APPLIED TO AB,
A PARALLELOGRAM EQUAL TO THE FOURTH PART OF
THE SQUARE, ON BC,

THAT IS,

TO THE SQUARE, ON HALF OF IT, AND
DEFICIENT BY A SQUARE FIGURE,
MAKING THE RECTANGLE AE, EB,

[X. 18]

THEREFORE,

AE IS INCOMMENSURABLE WITH EB. AND
AS AE IS TO EB,
SO IS THE RECTANGLE, BA, AE, TO
THE RECTANGLE, AB, BE, WHILE
THE RECTANGLE, BA, AE, =
THE SQUARE, ON AF, AND
THE RECTANGLE, AB, BE, TO THE SQUARE, ON BF;

THEREFORE,

THE SQUARE, ON AF, IS INCOMMENSURABLE WITH
THE SQUARE, ON FB;

THEREFORE,

AF, FB ARE INCOMMENSURABLE, IN SQUARE.

AND, SINCE,

AB IS RATIONAL,

[I. 47]

THEREFORE,

THE SQUARE, ON AB, IS, ALSO, RATIONAL; SO THAT
THE SUM OF THE SQUARES, ON AF, FB, IS, ALSO, RATIONAL.

AND SINCE, AGAIN,
THE RECTANGLE, AE, EB, =
THE SQUARE, ON EF, AND, BY HYPOTHESIS,
THE RECTANGLE, AE, EB, =
THE SQUARE, ON BD,

THEREFORE,
$EF = BD$;

THEREFORE,
BC IS DOUBLE OF FE,

SO THAT,
THE RECTANGLE, AB, BC, IS, ALSO, COMMENSURABLE WITH
THE RECTANGLE, AB, EF.

[X. 21]

BUT,
THE RECTANGLE, AB, BC, IS MEDIAL;

[X. 23, POR.]

THEREFORE,
THE RECTANGLE, AB, EF, IS, ALSO, MEDIAL.

[LEMMA]

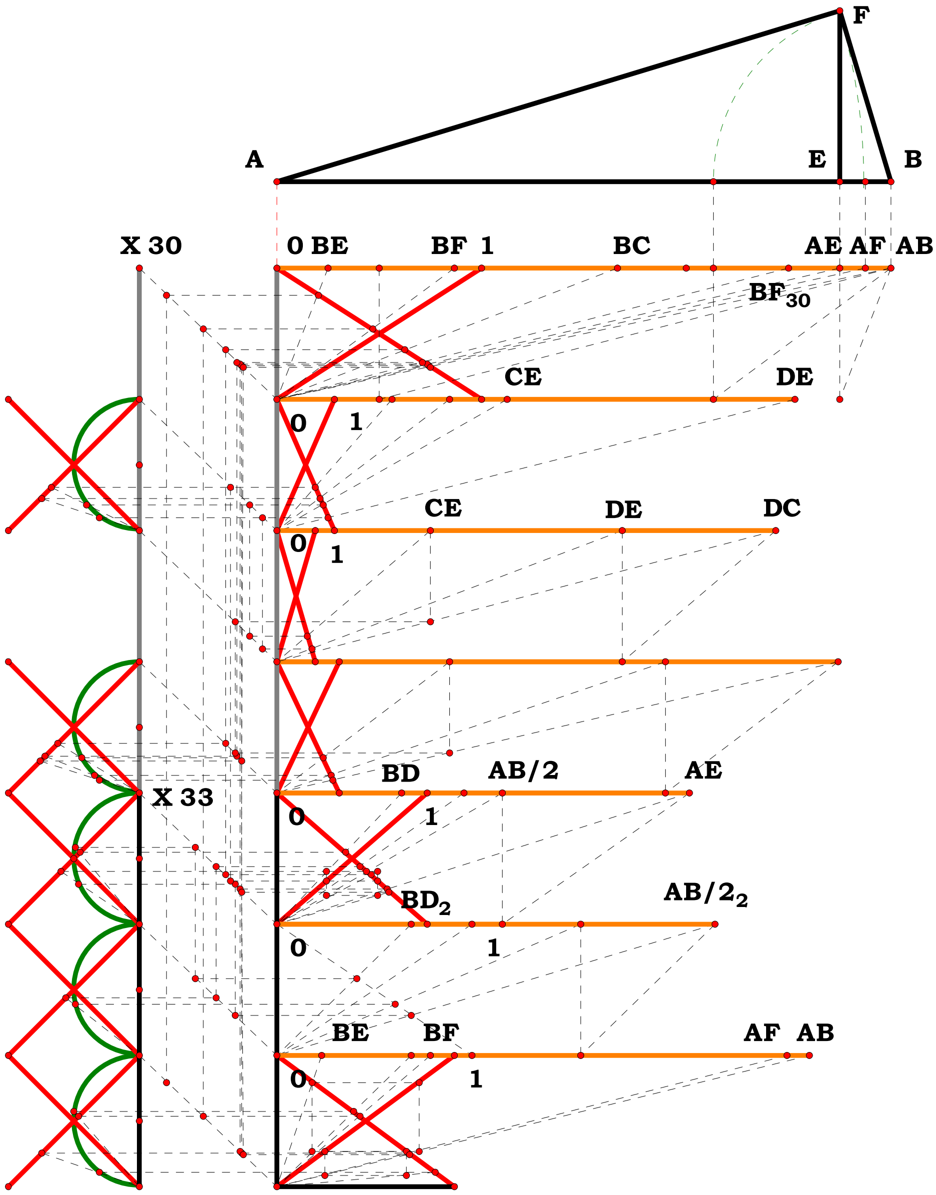
BUT,
THE RECTANGLE, AB, EF, =
THE RECTANGLE, AF, FB;

THEREFORE,
THE RECTANGLE, AF, FB, IS, ALSO, MEDIAL.

BUT,
IT WAS, ALSO, PROVED THAT THE SUM OF
THE SQUARES ON THESE STRAIGHT LINES IS RATIONAL.

THEREFORE,
TWO STRAIGHT LINES, AF, FB,
INCOMMENSURABLE, IN SQUARE,
HAVE BEEN FOUND WHICH MAKE THE SUM OF
THE SQUARES ON THEM RATIONAL, BUT
THE RECTANGLE CONTAINED BY THEM MEDIAL.

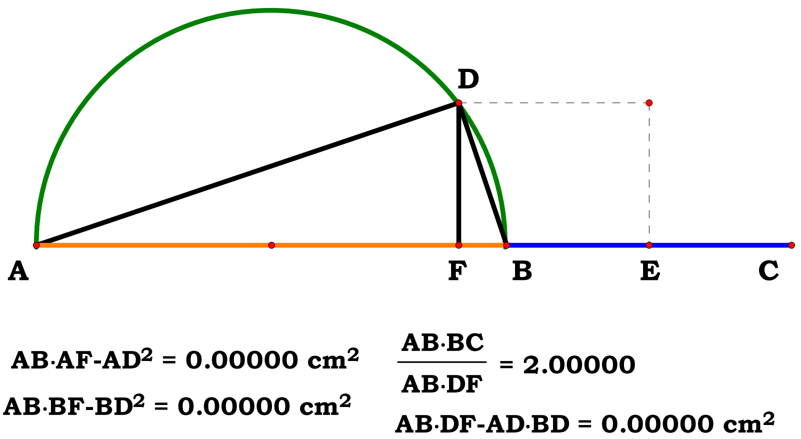
Q. E. D.



F
A
E
B
X 30
0 BE
BF 1
BC
AE AF AB
BF_{30}
CE
DE
0
1
CE
DE
DC
0
1
BD
AB/2
AE
X 33
0
1
BD_2
$AB/2_2$
0
1
BE
BF
AF AB
0
1

Proposition 34.

To find two straight lines incommensurable, in square, which make the sum of the squares on them medial but the rectangle contained by them rational.



[X. 31, *ad fin.*]

Let,

there be set out two medial straight lines, *AB*, *BC*, commensurable, in square, only,

such that,

the rectangle which they contain is rational, and
the square, on *AB*, is greater than
the square, on *BC*,
by the square, on a straight line
incommensurable with *AB*;

let,

the semicircle, *ADB*, be described, on *AB*,

let,

BC be bisected at *E*,

[VI. 28]

let,

there be applied to *AB*, a parallelogram equal to
the square, on *BE*, and deficient by a square figure,

namely,

the rectangle *AF*, *FB*;

[X. 18]

therefore,

AF is incommensurable, in length, with *FB*.

Let,

FD be drawn, from *F*, at right angles, to *AB*,

AND LET,

AD, *DB* BE JOINED.

SINCE,

AF IS INCOMMENSURABLE, IN LENGTH, WITH *FB*,

[X. 11]

THEREFORE,

THE RECTANGLE, *BA*, *AF*, IS, ALSO, INCOMMENSURABLE WITH THE RECTANGLE, *AB*, *BF*.

BUT,

THE RECTANGLE, *BA*, *AF*, =
THE SQUARE, ON *AD*, AND
THE RECTANGLE, *AB*, *BF*, TO THE SQUARE, ON *DB*;

THEREFORE,

THE SQUARE, ON *AD*, IS, ALSO, INCOMMENSURABLE WITH THE SQUARE, ON *DB*.

AND, SINCE,

THE SQUARE, ON *AB* IS MEDIAL,

[III. 31, I. 47]

THEREFORE,

THE SUM OF THE SQUARES, ON *AD*, *DB*, IS, ALSO, MEDIAL.

AND, SINCE,

BC IS DOUBLE OF *DF*,

THEREFORE,

THE RECTANGLE, *AB*, *BC*, IS, ALSO, DOUBLE OF THE RECTANGLE, *AB*, *FD*.

BUT,

THE RECTANGLE, *AB*, *BC*, IS RATIONAL;

[X. 6]

THEREFORE,

THE RECTANGLE, *AB*, *FD*, IS, ALSO, RATIONAL.

[LEMMA]

BUT,

THE RECTANGLE, *AB*, *FD*, =
THE RECTANGLE, *AD*, *DB*;

SO THAT,

THE RECTANGLE, *AD*, *DB*, IS, ALSO, RATIONAL.

THEREFORE,

TWO STRAIGHT LINES, *AD*, *DB*,

INCOMMENSURABLE, IN SQUARE, HAVE BEEN FOUND WHICH
MAKE THE SUM OF THE SQUARES ON THEM MEDIAL,

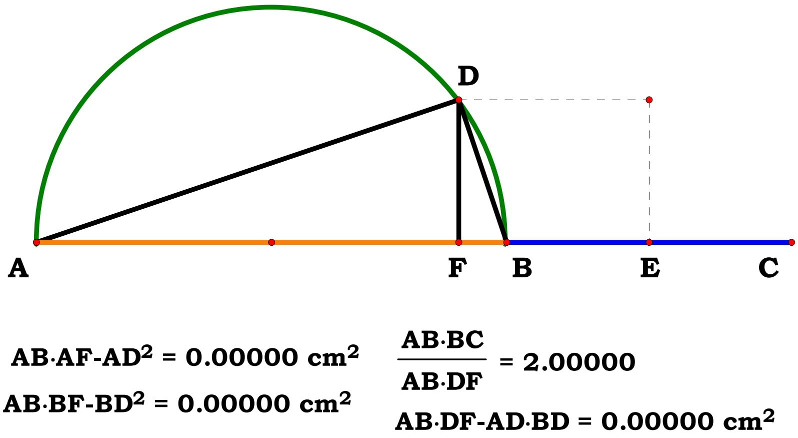
BUT,

THE RECTANGLE CONTAINED BY THEM RATIONAL.

Q. E. D.

Proposition 35.

To find two straight lines incommensurable, in square, which make the sum of the squares on them medial and the rectangle contained by them medial and moreover incommensurable with the sum of the squares on them.



[x. 32, *ad fin.*]

Let,

there be set out two medial straight lines, *AB*, *BC*,
commensurable, in square, only,
containing a medial rectangle,

and such that,

the square, on *AB*, is greater than
the square, on *BC*, by the square, on
a straight line incommensurable with *AB*;

let,

the semicircle, *ADB*, be described, on *AB*,

and let,

the rest of the construction be as above.

[x. 18]

Then, since,

AF is incommensurable, in length, with *FB*,

[x. 11]

AD is, also, incommensurable, in square, with *DB*.

And, since,

the square, on *AB*, is medial,

[iii. 31, i. 47]

therefore,

the sum of
the squares, on *AD*, *DB*, is, also, medial.

AND, SINCE,

THE RECTANGLE, AF, FB, =
THE SQUARE, ON EACH, OF THE STRAIGHT LINES, BE, DF,

THEREFORE,

$BE = DF$;

THEREFORE,

BC IS DOUBLE OF FD,

SO THAT,

THE RECTANGLE, AB, BC, IS, ALSO, DOUBLE OF
THE RECTANGLE, AB, FD.

BUT,

THE RECTANGLE, AB, BC, IS MEDIAL;

[X. 32, POR.]

THEREFORE,

THE RECTANGLE, AB, FD, IS, ALSO, MEDIAL.

[LEMMA AFTER X. 32]

AND,

IT = THE RECTANGLE, AD, DB;

THEREFORE,

THE RECTANGLE, AD, DB, IS, ALSO, MEDIAL.

AND, SINCE,

AB IS INCOMMENSURABLE, IN LENGTH, WITH BC, WHILE
CB IS COMMENSURABLE WITH BE,

[X. 13]

THEREFORE,

AB IS, ALSO, INCOMMENSURABLE, IN LENGTH, WITH BE,

[X. 11]

SO THAT,

THE SQUARE, ON AB, IS, ALSO, INCOMMENSURABLE WITH
THE RECTANGLE, AB, BE.

[I. 47]

BUT,

THE SQUARES, ON AD, DB, ARE EQUAL TO
THE SQUARE, ON AB, AND
THE RECTANGLE, AB, FD, THAT IS
THE RECTANGLE, AD, DB, =
THE RECTANGLE, AB, BE;

THEREFORE,

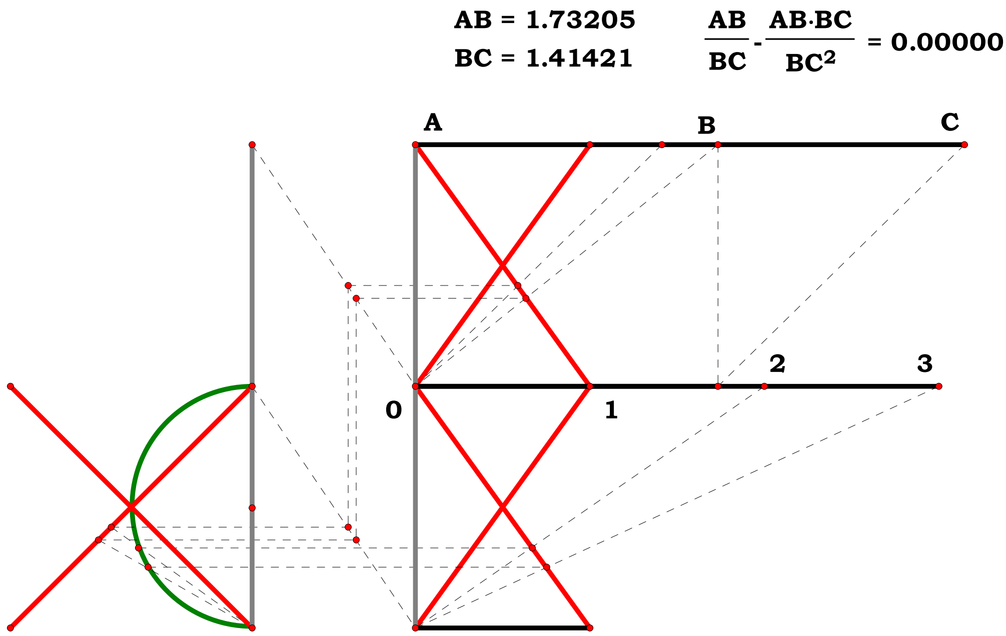
THE SUM OF
THE SQUARES, ON AD, DB, IS INCOMMENSURABLE
WITH THE RECTANGLE, AD, DB.

THEREFORE,
TWO STRAIGHT LINES, AD, DB,
INCOMMENSURABLE, IN SQUARE,
HAVE BEEN FOUND WHICH MAKE THE SUM OF
THE SQUARES ON THEM MEDIAL AND
THE RECTANGLE CONTAINED BY THEM MEDIAL AND
MOREOVER INCOMMENSURABLE WITH
THE SUM OF THE SQUARES ON THEM.

Q. E. D.

PROPOSITION 36.

IF TWO RATIONAL STRAIGHT LINES COMMENSURABLE, IN SQUARE, ONLY BE ADDED TOGETHER, THE WHOLE IS IRRATIONAL; AND LET IT BE CALLED **BINOMIAL**.



FOR LET,

TWO RATIONAL STRAIGHT LINES, *AB*, *BC*,
COMMENSURABLE, IN SQUARE, ONLY, BE ADDED TOGETHER;

I SAY THAT;

THE WHOLE, *AC*, IS IRRATIONAL.

FOR, SINCE,

AB IS INCOMMENSURABLE, IN LENGTH, WITH *BC*—

FOR,

THEY ARE COMMENSURABLE, IN SQUARE, ONLY—AND
AS *AB* IS TO *BC*,
SO IS THE RECTANGLE *AB*, *BC* TO THE SQUARE, ON *BC*,

[X. 11]

THEREFORE,

THE RECTANGLE, *AB*, *BC*, IS INCOMMENSURABLE WITH
THE SQUARE, ON *BC*.

[X. 6]

BUT,

TWICE THE RECTANGLE, *AB*, *BC*, IS COMMENSURABLE WITH
THE RECTANGLE, *AB*, *BC*, AND
THE SQUARES, ON *AB*, *BC*, ARE COMMENSURABLE WITH
THE SQUARE, ON *BC*—

[X. 15]

FOR,

AB, *BC* ARE RATIONAL STRAIGHT LINES
COMMENSURABLE, IN SQUARE, ONLY—

[X. 13]

THEREFORE,

TWICE THE RECTANGLE, *AB*, *BC*, IS INCOMMENSURABLE
WITH THE SQUARES, ON *AB*, *BC*.

AND, *COMPONENDO*,

TWICE THE RECTANGLE, *AB*, *BC*, TOGETHER WITH
THE SQUARES, ON *AB*, *BC*,

[II. 4]

THAT IS,

[X. 16]

THE SQUARE, ON *AC*, IS INCOMMENSURABLE WITH
THE SUM OF THE SQUARES, ON *AB*, *BC*.

BUT,

THE SUM OF THE SQUARES, ON *AB*, *BC*, IS RATIONAL;

THEREFORE,

THE SQUARE, ON *AC*, IS IRRATIONAL,

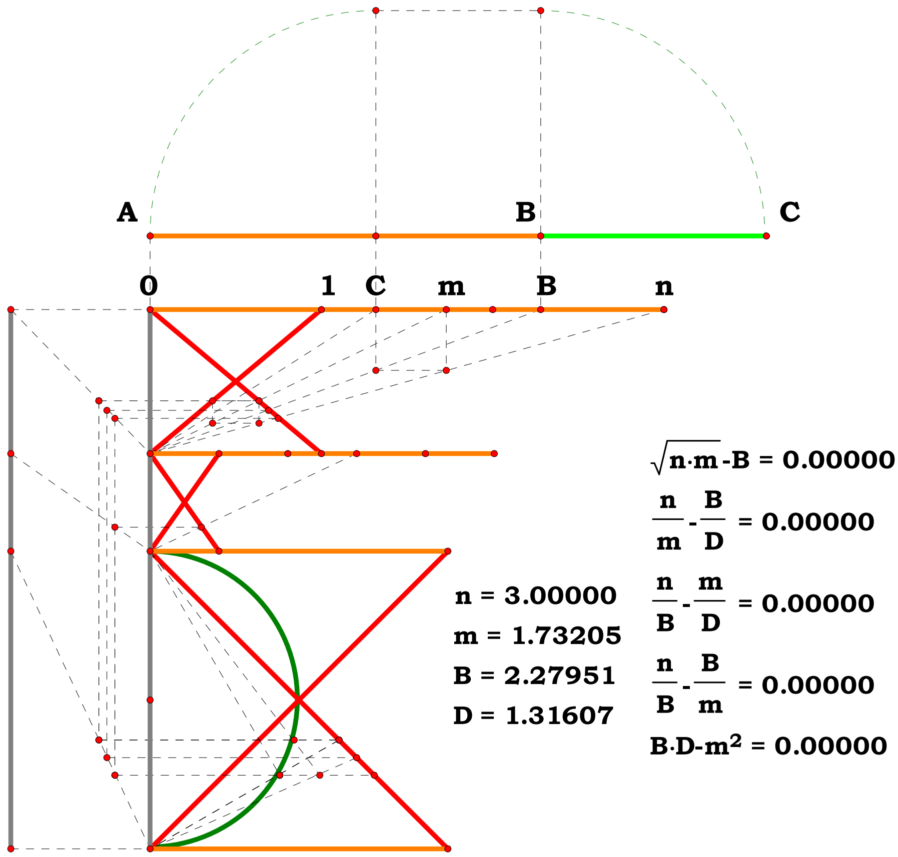
[X. DEF. 4]

SO THAT,

AC IS, ALSO, IRRATIONAL.
AND LET IT BE CALLED **BINOMIAL**.

PROPOSITION 37.

IF TWO MEDIAL STRAIGHT LINES COMMENSURABLE, IN SQUARE, ONLY AND CONTAINING A RATIONAL RECTANGLE BE ADDED TOGETHER, THE WHOLE IS IRRATIONAL; AND LET IT BE CALLED A **FIRST BIMEDIAL** *STRAIGHT LINE.*



FOR LET,

TWO MEDIAL STRAIGHT LINES, *AB*, *BC*,
COMMENSURABLE, IN SQUARE, ONLY AND
CONTAINING A RATIONAL RECTANGLE BE ADDED TOGETHER;

I SAY THAT;

THE WHOLE, *AC*, IS IRRATIONAL.

FOR, SINCE,

AB IS INCOMMENSURABLE, IN LENGTH, WITH *BC*,

[CF. X. 36, II. 9—20]

THEREFORE,

THE SQUARES, ON *AB*, *BC*, ARE, ALSO, INCOMMENSURABLE
WITH TWICE THE RECTANGLE, *AB*, *BC*; AND, *COMPONENDO*,
THE SQUARES, ON *AB*, *BC*,
TOGETHER WITH TWICE THE RECTANGLE, *AB*, *BC*,

[II. 4]

THAT IS,

THE SQUARE, ON *AC*,

[X. 16]

IS INCOMMENSURABLE WITH THE RECTANGLE, *AB*, *BC*.

BUT,

THE RECTANGLE, *AB*, *BC*, IS RATIONAL,

FOR, BY HYPOTHESIS,

AB, *BC* ARE STRAIGHT LINES
CONTAINING A RATIONAL RECTANGLE;

[X. DEF. 4]

THEREFORE,

THE SQUARE, ON *AC* IS IRRATIONAL;

THEREFORE,

AC IS IRRATIONAL.

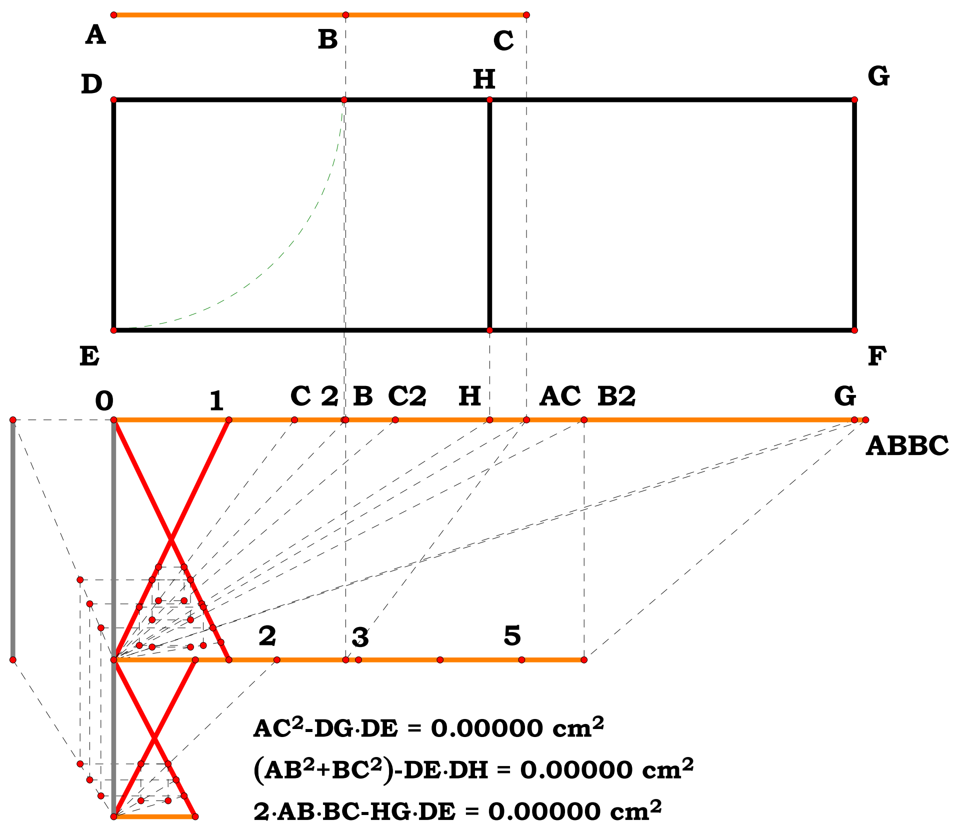
AND LET,

IT BE CALLED A **FIRST BIMEDIAL** STRAIGHT LINE.

Q. E. D.

PROPOSITION 38.

IF TWO MEDIAL STRAIGHT LINES COMMENSURABLE, IN SQUARE, ONLY AND CONTAINING A MEDIAL RECTANGLE BE ADDED TOGETHER, THE WHOLE IS IRRATIONAL; AND LET IT BE CALLED A **SECOND BIMEDIAL** *STRAIGHT LINE.*



FOR LET,

TWO MEDIAL STRAIGHT LINES, *AB*, *BC*,
COMMENSURABLE, IN SQUARE, ONLY, AND
CONTAINING A MEDIAL RECTANGLE BE ADDED TOGETHER;

I SAY THAT;

AC IS IRRATIONAL.

FOR LET,

A RATIONAL STRAIGHT LINE, *DE*, BE SET OUT,

[I. 44]

AND LET,

THE PARALLELOGRAM, *DF*, EQUAL TO
THE SQUARE, ON *AC*, BE APPLIED TO *DE*,
PRODUCING *DG*, AS BREADTH.

[II. 4]

THEN, SINCE,

THE SQUARE, ON *AC*, =
THE SQUARES, ON *AB*, *BC*, AND

TWICE THE RECTANGLE, AB, BC,

LET,

EH, EQUAL TO THE SQUARES, ON AB, BC,
BE APPLIED TO DE;

THEREFORE,

THE REMAINDER,
HF = TWICE THE RECTANGLE, AB, BC.

AND, SINCE,

EACH, OF THE STRAIGHT LINES, AB, BC, IS MEDIAL,

THEREFORE,

THE SQUARES, ON AB, BC, ARE, ALSO, MEDIAL.

BUT, BY HYPOTHESIS,

TWICE THE RECTANGLE, AB, BC, IS, ALSO, MEDIAL. AND
EH = THE SQUARES, ON AB, BC, WHILE
FH = TWICE THE RECTANGLE, AB, BC;

THEREFORE,

EACH, OF THE RECTANGLES, EH, HF, IS MEDIAL. AND
THEY ARE APPLIED TO THE RATIONAL STRAIGHT LINE, DE;

[X. 22]

THEREFORE,

EACH, OF THE STRAIGHT LINES, DH, HG, IS RATIONAL AND
INCOMMENSURABLE, IN LENGTH, WITH DE.

SINCE THEN,

AB IS INCOMMENSURABLE, IN LENGTH, WITH BC, AND
AS AB IS TO BC,
SO IS THE SQUARE, ON AB, TO THE RECTANGLE, AB, BC,

[X. 11]

THEREFORE,

THE SQUARE, ON AB, IS INCOMMENSURABLE WITH
THE RECTANGLE, AB, BC.

[X. 15]

BUT, THE SUM OF

THE SQUARES, ON AB, BC, IS COMMENSURABLE WITH
THE SQUARE, ON AB,

[X. 6]

AND, TWICE

THE RECTANGLE, AB, BC, IS COMMENSURABLE WITH
THE RECTANGLE AB, BC.

[X. 13]

THEREFORE, THE SUM OF

THE SQUARES, ON AB, BC, IS INCOMMENSURABLE WITH TWICE THE RECTANGLE, AB, BC.

BUT,

EH = THE SQUARES, ON AB, BC, AND
HF = TWICE THE RECTANGLE, AB, BC.

THEREFORE,

EH IS INCOMMENSURABLE WITH HF,

[VI. 1, X. 11]

SO THAT,

DH IS, ALSO, INCOMMENSURABLE, IN LENGTH, WITH HG.

THEREFORE,

DH, HG ARE RATIONAL STRAIGHT LINES COMMENSURABLE, IN SQUARE, ONLY;

[X. 36]

SO THAT,

DG IS IRRATIONAL.

[CF. X. 20]

BUT,

DE IS RATIONAL; AND
THE RECTANGLE CONTAINED BY AN IRRATIONAL, AND A RATIONAL STRAIGHT LINE IS IRRATIONAL;

[X. DEF. 4]

THEREFORE,

THE AREA, DF, IS IRRATIONAL, AND
THE SIDE OF THE SQUARE EQUAL TO IT IS IRRATIONAL.

BUT,

AC IS THE SIDE OF THE SQUARE EQUAL TO DF,

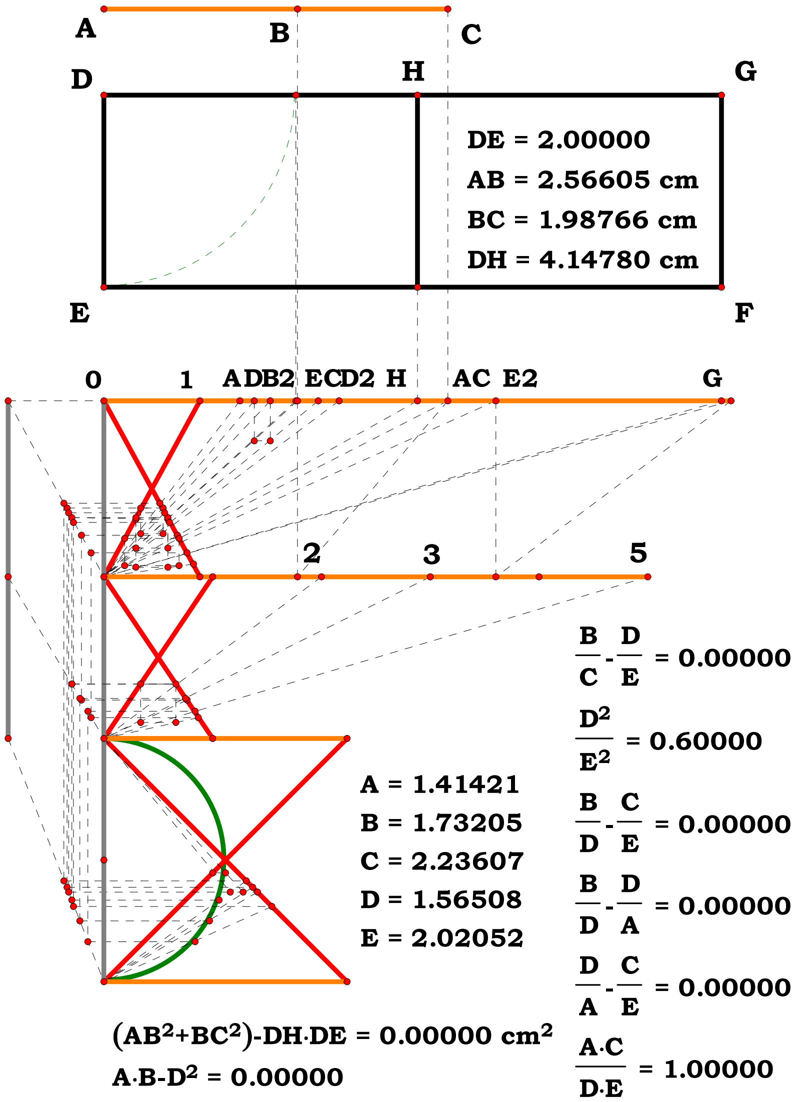
THEREFORE,

AC IS IRRATIONAL.

AND LET,

IT BE CALLED A **SECOND BIMEDIAL** STRAIGHT LINE.

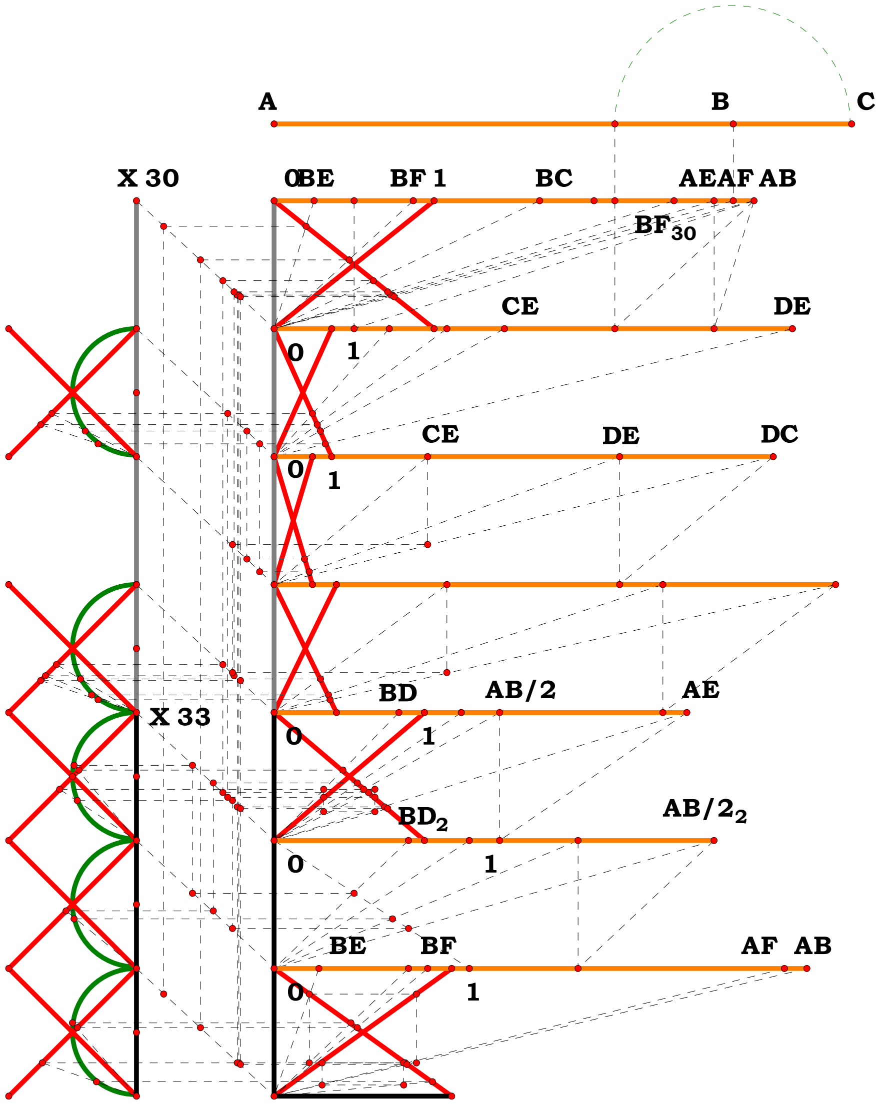
Q. E. D.



A
B
C
D
H
G
DE = 2.00000
AB = 2.56605 cm
BC = 1.98766 cm
DH = 4.14780 cm
E
F
0
1
ADB2 ECD2 H
AC E2
G
2
3
5
$\frac{B}{C}-\frac{D}{E} = 0.00000$
$\frac{D^2}{E^2} = 0.60000$
A = 1.41421
B = 1.73205
C = 2.23607
D = 1.56508
E = 2.02052
$\frac{B}{D}-\frac{C}{E} = 0.00000$
$\frac{B}{D}-\frac{D}{A} = 0.00000$
$\frac{D}{A}-\frac{C}{E} = 0.00000$
$(AB^2+BC^2)-DH \cdot DE = 0.00000\ cm^2$
$A \cdot B - D^2 = 0.00000$
$\frac{A \cdot C}{D \cdot E} = 1.00000$

PROPOSITION 39.

IF TWO STRAIGHT LINES INCOMMENSURABLE, IN SQUARE, WHICH MAKE THE SUM OF THE SQUARES ON THEM RATIONAL, BUT THE RECTANGLE CONTAINED BY THEM MEDIAL, BE ADDED TOGETHER, THE WHOLE STRAIGHT LINE IS IRRATIONAL; AND LET IT BE CALLED **MAJOR**.



[X. 33]

FOR,

LET TWO STRAIGHT LINES, *AB*, *BC*,
INCOMMENSURABLE, IN SQUARE, AND
FULFILLING THE GIVEN CONDITIONS, BE ADDED TOGETHER;

I SAY THAT;

AC IS IRRATIONAL.

[X. 6 AND 23, POR.]

FOR, SINCE,

THE RECTANGLE, *AB*, *BC*, IS MEDIAL, TWICE
THE RECTANGLE, *AB*, *BC*, IS, ALSO, MEDIAL.

BUT, THE SUM OF
THE SQUARES, ON AB, BC, IS RATIONAL;

THEREFORE, TWICE
THE RECTANGLE, AB, BC, IS INCOMMENSURABLE WITH
THE SUM OF THE SQUARES, ON AB, BC,

[X. 16]

SO THAT,
THE SQUARES, ON AB, BC, TOGETHER WITH TWICE
THE RECTANGLE, AB, BC, THAT IS
THE SQUARE, ON AC, IS, ALSO, INCOMMENSURABLE WITH
THE SUM OF THE SQUARES, ON AB, BC;

THEREFORE,
THE SQUARE, ON AC, IS IRRATIONAL,

[X. DEF. 4]

SO THAT,
AC IS, ALSO, IRRATIONAL.

AND LET,
IT BE CALLED **MAJOR**.

Q. E. D.

Proposition 40.

If two straight lines incommensurable, in square, which make the sum of the squares on them medial, but the rectangle contained by them rational, be added together, the whole straight line is irrational; and let it be called the **side of a rational plus a medial area**.

[x. 34]

For let,

two straight lines, AB, BC,
incommensurable, in square, and
fulfilling the given conditions, be added together;

I say that;

AC is irrational.

For, since,

the sum of the squares, on AB, BC, is medial, while
twice the rectangle, AB, BC, is rational,

therefore, the sum of

the squares, on AB, BC, is incommensurable
with twice the rectangle, AB, BC;

[x. 16]

so that,

the square, on AC, is, also,
incommensurable with twice
the rectangle, AB, BC.

But,

twice the rectangle, AB, BC, is rational;

therefore,

the square, on AC, is irrational.

[x. Def. 4]

Therefore,

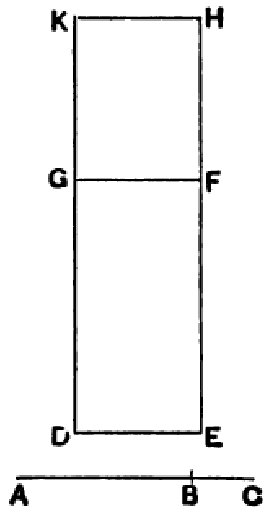
AC is irrational.

And let,

it be called the **side of a rational plus a medial area**.

Q. E. D.

PROPOSITION 41.



IF TWO STRAIGHT LINES INCOMMENSURABLE, IN SQUARE, WHICH MAKE THE SUM OF THE SQUARES ON THEM MEDIAL, AND THE RECTANGLE CONTAINED BY THEM MEDIAL AND, ALSO, *INCOMMENSURABLE WITH THE SUM OF THE SQUARES ON THEM, BE ADDED TOGETHER, THE WHOLE STRAIGHT LINE IS IRRATIONAL; AND LET IT BE CALLED THE* **SIDE OF THE SUM OF TWO MEDIAL AREAS**.

[X. 35]

FOR LET,

TWO STRAIGHT LINES *AB*, *BC*,
INCOMMENSURABLE, IN SQUARE, AND
SATISFYING THE GIVEN CONDITIONS BE ADDED TOGETHER;

I SAY THAT;

AC IS IRRATIONAL.

LET,

A RATIONAL STRAIGHT LINE, *DE*, BE SET OUT,

AND LET, THERE BE APPLIED TO *DE*,

THE RECTANGLE, *DF*, EQUAL TO
THE SQUARES, ON *AB*, *BC*, AND
THE RECTANGLE, *GH*, EQUAL TO TWICE
THE RECTANGLE, *AB*, *BC*;

[II. 4]

THEREFORE,

THE WHOLE, *DH*, = THE SQUARE, ON *AC*.

NOW, SINCE, THE SUM OF

THE SQUARES, ON *AB*, *BC*, IS MEDIAL, AND
= *DF*,

THEREFORE,

DF IS, ALSO, MEDIAL.

[X. 22]

AND,

IT IS APPLIED TO THE RATIONAL STRAIGHT LINE, *DE*;

THEREFORE,

DG IS RATIONAL AND
INCOMMENSURABLE, IN LENGTH, WITH *DE*.

FOR, THE SAME REASON

GK IS, ALSO, RATIONAL, AND

INCOMMENSURABLE, IN LENGTH, WITH *GF*,

THAT IS,

DE.

AND, SINCE,

THE SQUARES, ON *AB*, *BC*, ARE INCOMMENSURABLE WITH

TWICE THE RECTANGLE *AB*, *BC*,

DF IS INCOMMENSURABLE WITH *GH*;

[VI. 1, X. 11]

SO THAT,

DG IS, ALSO, INCOMMENSURABLE WITH *GK*. AND

THEY ARE RATIONAL;

THEREFORE,

DG, *GK* ARE RATIONAL STRAIGHT LINES

COMMENSURABLE, IN SQUARE, ONLY;

[X. 36]

THEREFORE,

DK IS IRRATIONAL AND WHAT IS CALLED BINOMIAL,

BUT,

DE IS RATIONAL;

[X. DEF. 4]

THEREFORE,

DH IS IRRATIONAL, AND

THE SIDE OF THE SQUARE WHICH = IT IS IRRATIONAL.

BUT,

AC IS THE SIDE OF THE SQUARE EQUAL TO *HD*;

THEREFORE,

AC IS IRRATIONAL.

AND LET,

IT BE CALLED THE **SIDE OF THE SUM OF TWO MEDIAL AREAS**.

Q. E. D.

LEMMA.

AND THAT THE AFORESAID IRRATIONAL STRAIGHT LINES ARE DIVIDED ONLY IN ONE WAY INTO THE STRAIGHT LINES OF WHICH THEY ARE THE SUM AND WHICH PRODUCE THE TYPES IN QUESTION, WE WILL NOW PROVE AFTER PREMISING THE FOLLOWING LEMMA.

LET,

THE STRAIGHT LINE, AB, BE SET OUT,

LET,

THE WHOLE BE CUT INTO UNEQUAL PARTS
AT EACH, OF THE POINTS, C, D,

AND LET,

AC BE SUPPOSED GREATER THAN DB;

I SAY THAT;

THE SQUARES, ON AC, CB, ARE GREATER THAN
THE SQUARES, ON AD, DB.

FOR LET,

AB BE BISECTED AT E.

THEN, SINCE,

AC IS GREATER THAN DB,

LET,

DC BE SUBTRACTED FROM EACH;

THEREFORE,

THE REMAINDER, AD, IS GREATER THAN
THE REMAINDER, CB.

BUT,

$AE = EB$;

THEREFORE,

DE IS LESS THAN EC;

THEREFORE,

THE POINTS, C, D, ARE NOT EQUIDISTANT FROM
THE POINT OF BISECTION.

[II. 5]

AND, SINCE,

THE RECTANGLE, AC, CB, TOGETHER WITH
THE SQUARE, ON EC, =
THE SQUARE, ON EB,

[*ID.*]

AND, FURTHER,

THE RECTANGLE, AD, DB, TOGETHER WITH
THE SQUARE, ON DE, =
THE SQUARE, ON EB,

THEREFORE,

THE RECTANGLE, AC, CB, TOGETHER WITH

THE SQUARE, ON *EC*, =
THE RECTANGLE, *AD*, *DB*,
TOGETHER WITH THE SQUARE, ON *DE*. AND OF THESE
THE SQUARE, ON *DE*, IS LESS THAN
THE SQUARE, ON *EC*;

THEREFORE,
THE REMAINDER,
THE RECTANGLE, *AC*, *CB*, IS, ALSO, LESS THAN
THE RECTANGLE, *AD*, *DB*,

SO THAT, TWICE
THE RECTANGLE, *AC*, *CB*, IS, ALSO, LESS THAN TWICE
THE RECTANGLE, *AD*, *DB*.

THEREFORE, ALSO,
THE REMAINDER,
THE SUM OF THE SQUARES, ON *AC*, *CB*, IS GREATER THAN
THE SUM OF THE SQUARES, ON *AD*, *DB*.

Q. E. D.

PROPOSITION 42.

A BINOMIAL STRAIGHT LINE IS DIVIDED INTO ITS TERMS AT ONE POINT ONLY.

LET,

AB, BE A BINOMIAL STRAIGHT LINE
DIVIDED INTO ITS TERMS AT *C*;

[X. 36]

THEREFORE,

AC, *CB* ARE RATIONAL STRAIGHT LINES
COMMENSURABLE, IN SQUARE, ONLY.

I SAY THAT;

AB IS NOT DIVIDED AT ANOTHER POINT INTO
TWO RATIONAL STRAIGHT LINES
COMMENSURABLE, IN SQUARE, ONLY.

FOR, IF POSSIBLE, LET,

IT BE DIVIDED AT *D* ALSO,

SO THAT,

AD, *DB* ARE, ALSO, RATIONAL STRAIGHT LINES
COMMENSURABLE, IN SQUARE, ONLY.

IT IS THEN MANIFEST THAT,

AC IS NOT THE SAME WITH *DB*.

FOR, IF POSSIBLE, LET,

IT BE SO.

THEN,

AD WILL, ALSO, BE THE SAME AS *CB*, AND
AS *AC* IS TO *CB*,
SO WILL *BD* BE TO *DA*;

THUS,

AB WILL BE DIVIDED AT *D*, ALSO,
IN THE SAME WAY AS BY THE DIVISION AT *C*: WHICH
IS CONTRARY TO THE HYPOTHESIS.

THEREFORE,

AC IS NOT THE SAME WITH *DB*.

FOR THIS REASON ALSO,

THE POINTS, *C*, *D*, ARE NOT EQUIDISTANT FROM
THE POINT, OF BISECTION.

THEREFORE, THAT BY WHICH

THE SQUARES, ON *AC*, *CB*, DIFFER FROM
THE SQUARES, ON *AD*, *DB*, IS, ALSO, THAT BY WHICH TWICE

THE RECTANGLE, AD, DB, DIFFERS FROM TWICE
THE RECTANGLE, AC, CB,

[II. 4]

BECAUSE BOTH,
THE SQUARES, ON AC, CB, TOGETHER WITH TWICE
THE RECTANGLE, AC, CB, AND
THE SQUARES, ON AD, DB, TOGETHER WITH TWICE
THE RECTANGLE, AD, DB, ARE EQUAL TO
THE SQUARE, ON AB.

BUT,
THE SQUARES, ON AC, CB, DIFFER FROM
THE SQUARES, ON AD, DB, BY A RATIONAL AREA,

FOR,
BOTH ARE RATIONAL;

THEREFORE, TWICE
THE RECTANGLE, AD, DB, ALSO, DIFFERS FROM TWICE
THE RECTANGLE, AC, CB, BY A RATIONAL AREA,

[X. 21]

THOUGH,
THEY ARE MEDIAL:

WHICH,
IS ABSURD,

[X. 26]

FOR,
A MEDIAL AREA DOES NOT EXCEED
A MEDIAL BY A RATIONAL AREA.

THEREFORE,
A BINOMIAL STRAIGHT LINE IS NOT DIVIDED
AT DIFFERENT POINTS;

THEREFORE,
IT IS DIVIDED AT ONE POINT ONLY.

Q. E. D.

PROPOSITION 43.

A FIRST BIMEDIAL STRAIGHT LINE IS DIVIDED AT ONE POINT ONLY.

[x. 37]

LET,

AB BE A FIRST BIMEDIAL STRAIGHT LINE,
DIVIDED AT C,

SO THAT,

AC, CB ARE MEDIAL STRAIGHT LINES
COMMENSURABLE, IN SQUARE, ONLY, AND
CONTAINING A RATIONAL RECTANGLE;

I SAY THAT;

AB IS NOT SO DIVIDED AT ANOTHER POINT.

FOR, IF POSSIBLE, LET,

IT BE DIVIDED, AT D, ALSO,

SO THAT,

AD, DB ARE, ALSO, MEDIAL STRAIGHT LINES
COMMENSURABLE, IN SQUARE, ONLY, AND
CONTAINING A RATIONAL RECTANGLE.

SINCE, THEN,

THAT BY WHICH TWICE
THE RECTANGLE, AD, DB, DIFFERS FROM TWICE
THE RECTANGLE ,AC, CB, IS THAT BY WHICH
THE SQUARES, ON AC, CB, DIFFER FROM
THE SQUARES, ON AD, DB, WHILE TWICE
THE RECTANGLE, AD, DB, DIFFERS FROM TWICE
THE RECTANGLE, AC, CB, BY A RATIONAL AREA—

FOR,

BOTH ARE RATIONAL—

THEREFORE,

THE SQUARES, ON AC, CB, ALSO, DIFFER FROM
THE SQUARES, ON AD, DB, BY A RATIONAL AREA,

THOUGH,

THEY ARE MEDIAL:

[x. 26]

WHICH,

IS ABSURD.

THEREFORE,

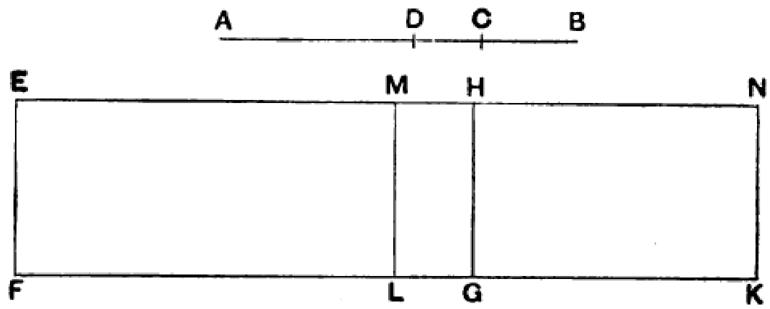
A FIRST BIMEDIAL STRAIGHT LINE IS NOT DIVIDED INTO ITS TERMS AT DIFFERENT POINTS;

THEREFORE,

IT IS SO DIVIDED AT ONE POINT ONLY.

PROPOSITION 44.

A SECOND BIMEDIAL STRAIGHT LINE IS DIVIDED AT ONE POINT ONLY.



[X. 38]

LET,

AB BE A SECOND BIMEDIAL STRAIGHT LINE,
DIVIDED AT C,

SO THAT,

AC, CB ARE MEDIAL STRAIGHT LINES,
COMMENSURABLE, IN SQUARE, ONLY, AND
CONTAINING A MEDIAL RECTANGLE;

IT IS THEN MANIFEST THAT;

C IS NOT AT THE POINT OF BISECTION,

BECAUSE,

THE SEGMENTS ARE NOT COMMENSURABLE, IN LENGTH.

I SAY THAT;

AB IS NOT SO DIVIDED AT ANOTHER POINT.

FOR, IF POSSIBLE, LET,

IT BE DIVIDED AT D ALSO,

SO THAT,

AC IS NOT THE SAME WITH DB,

BUT,

AC IS SUPPOSED GREATER;

[LEMMA]

IT IS THEN CLEAR THAT; AS WE PROVED ABOVE,

THE SQUARES, ON AD, DB, ARE, ALSO, LESS THAN
THE SQUARES, ON AC, CB;

AND SUPPOSE THAT;

AD, DB ARE MEDIAL STRAIGHT LINES
COMMENSURABLE, IN SQUARE, ONLY, AND
CONTAINING A MEDIAL RECTANGLE.

NOW LET,

A RATIONAL STRAIGHT LINE, *EF*, BE SET OUT,

LET,

THERE BE APPLIED, TO *EF*,
THE RECTANGULAR PARALLELOGRAM, *EK*, EQUAL TO
THE SQUARE ,ON *AB*,

AND LET,

EG EQUAL TO THE SQUARES, ON *AC*, *CB*,
BE SUBTRACTED;

[II. 4]

THEREFORE,

THE REMAINDER, *HK*, = TWICE
THE RECTANGLE, *AC*, *CB*.

AGAIN, LET,

THERE BE SUBTRACTED *EL*, EQUAL TO
THE SQUARES, ON *AD*, *DB*,

[LEMMA]

WHICH, WERE PROVED LESS THAN
THE SQUARES, ON *AC*, *CB*;

THEREFORE,

THE REMAINDER, *MK*, = TWICE
THE RECTANGLE, *AD*, *DB*.

NOW, SINCE,

THE SQUARES, ON *AC*, *CB*, ARE MEDIAL,

THEREFORE,

EG IS MEDIAL. AND
IT IS APPLIED TO THE RATIONAL STRAIGHT LINE, *EF*;

[X. 22]

THEREFORE,

EH IS RATIONAL AND
INCOMMENSURABLE, IN LENGTH, WITH *EF*.

FOR THE SAME REASON,

HN IS, ALSO, RATIONAL AND
INCOMMENSURABLE, IN LENGTH, WITH *EF*.

AND, SINCE,

AC, *CB* ARE MEDIAL STRAIGHT LINES,
COMMENSURABLE, IN SQUARE, ONLY,

THEREFORE,

AC IS INCOMMENSURABLE, IN LENGTH, WITH *CB*.

BUT,

as AC is to CB,
so is the square, on AC, to the rectangle, AC, CB;

[x. 11]

therefore,
the square, on AC, is incommensurable with
the rectangle, AC, CB.

But,
the squares, on AC, CB, are commensurable with
the square, on AC;

[x. 5]

for,
AC, CB are commensurable, in square.

[x. 6]

And,
twice the rectangle, AC, CB, is commensurable
with the rectangle, AC, CB.

[x. 13]

Therefore,
the squares, on AC, CB, are, also,
incommensurable with twice
the rectangle, AC, CB.

But,
EG = the squares, on AC, CB, and,
HK = twice the rectangle, AC, CB;

therefore,
EG is incommensurable with HK,

[vi. 1, x. 11]

so that,
EH is, also, incommensurable, in length, with HN.

And,
they are rational;

therefore,
EH, HN are rational straight lines
commensurable, in square, only.

[x. 36]

But,
if two rational straight lines
commensurable, in square, only, be added together,

THE WHOLE IS THE IRRATIONAL WHICH IS CALLED BINOMIAL.

THEREFORE,
EN IS A BINOMIAL STRAIGHT LINE DIVIDED AT *H*.

IN THE SAME WAY,
EM, *MN* WILL, ALSO, BE PROVED TO
BE RATIONAL STRAIGHT LINES
COMMENSURABLE, IN SQUARE, ONLY; AND
EN WILL BE A BINOMIAL STRAIGHT LINE,
DIVIDED AT DIFFERENT POINTS, *H* AND *M*.

AND,
EH IS NOT THE SAME WITH *MN*.

FOR,
THE SQUARES, ON *AC*, *CE*, ARE GREATER THAN
THE SQUARES, ON *AD*, *DB*.

BUT,
THE SQUARES, ON *AD*, *DB*, ARE GREATER THAN TWICE
THE RECTANGLE, *AD*, *DB*;

THEREFORE,
ALSO THE SQUARES, ON *AC*, *CB*,

THAT IS,
EG, ARE MUCH GREATER THAN TWICE
THE RECTANGLE, *AD*, *DB*,

THAT IS,
MK,

SO THAT,
EH IS, ALSO, GREATER THAN *MN*.

THEREFORE,
EH IS NOT THE SAME WITH *MN*.

Q. E. D.

PROPOSITION 45.

A MAJOR STRAIGHT LINE IS DIVIDED AT ONE AND THE SAME POINT ONLY.

[x. 39]

LET,

AB BE A MAJOR STRAIGHT LINE DIVIDED AT *C*,

SO THAT,

AC CB ARE INCOMMENSURABLE, IN SQUARE, AND
MAKE THE SUM OF
THE SQUARES, ON *AC*, *CB*, RATIONAL,

BUT,

THE RECTANGLE, *AC*, *CB*, MEDIAL;

I SAY THAT;

AB IS NOT SO DIVIDED AT ANOTHER POINT.

FOR, IF POSSIBLE, LET,

IT BE DIVIDED AT *D* ALSO,

SO THAT,

AD, *DB* ARE, ALSO, INCOMMENSURABLE, IN SQUARE, AND
MAKE THE SUM OF
THE SQUARES, ON *AD DB*, RATIONAL, BUT
THE RECTANGLE CONTAINED BY THEM MEDIAL.

THEN, SINCE, THAT BY WHICH

THE SQUARES, ON *AC*, *CB*, DIFFER FROM
THE SQUARES, ON *AD*, *DB*, IS, ALSO, THAT BY WHICH TWICE
THE RECTANGLE, *AD*, *DB*, DIFFERS FROM TWICE
THE RECTANGLE, *AC*, *CB*, WHILE
THE SQUARES, ON *AC*, *CB*, EXCEED
THE SQUARES, ON *AD*, *DB*, BY A RATIONAL AREA—

FOR,

BOTH ARE RATIONAL—

THEREFORE, TWICE

THE RECTANGLE, *AD*, *DB*, ALSO, EXCEEDS TWICE
THE RECTANGLE, *AC*, *CB*, BY A RATIONAL AREA,

THOUGH,

THEY ARE MEDIAL:

[x. 26]

WHICH,

IS IMPOSSIBLE.

THEREFORE,

A MAJOR STRAIGHT LINE IS NOT DIVIDED AT DIFFERENT POINTS;

THEREFORE,

IT IS ONLY DIVIDED AT ONE AND THE SAME POINT.

Q. E. D.

PROPOSITION 46.

THE SIDE OF A RATIONAL PLUS A MEDIAL AREA IS DIVIDED AT ONE POINT ONLY.

[X. 40]

LET,

AB BE THE SIDE OF A RATIONAL PLUS
A MEDIAL AREA DIVIDED AT *C*,

SO THAT,

AC, *CB* ARE INCOMMENSURABLE, IN SQUARE, AND
MAKE THE SUM OF THE SQUARES, ON *AC*, *CB*, MEDIAL,

BUT,

TWICE THE RECTANGLE, *AC*, *CB*, RATIONAL;

I SAY THAT;

AB IS NOT SO DIVIDED AT ANOTHER POINT.

FOR, IF POSSIBLE, LET,

IT BE DIVIDED, AT *D*, ALSO,

SO THAT,

AD, *DB* ARE, ALSO, INCOMMENSURABLE, IN SQUARE, AND
MAKE THE SUM OF THE SQUARES, ON *AD*, *DB*, MEDIAL,

BUT, TWICE

THE RECTANGLE ,*AD*, *DB*, RATIONAL.

SINCE THEN, THAT BY WHICH TWICE

THE RECTANGLE, *AC*, *CB*, DIFFERS FROM TWICE
THE RECTANGLE, *AD*, *DB*, IS, ALSO, THAT BY WHICH
THE SQUARES, ON *AD*, *DB*, DIFFER FROM
THE SQUARES, ON *AC*, *CB*, WHILE TWICE
THE RECTANGLE, *AC*, *CB*, EXCEEDS TWICE
THE RECTANGLE, *AD*, *DB*, BY A RATIONAL AREA,

THEREFORE,

THE SQUARES, ON *AD*, *DB*, ALSO, EXCEED
THE SQUARES, ON *AC*, *CB*, BY A RATIONAL AREA,

THOUGH,

THEY ARE MEDIAL:

[X. 26]

WHICH,

IS IMPOSSIBLE.

THEREFORE,

THE SIDE OF A RATIONAL PLUS
A MEDIAL AREA IS NOT DIVIDED AT DIFFERENT POINTS;

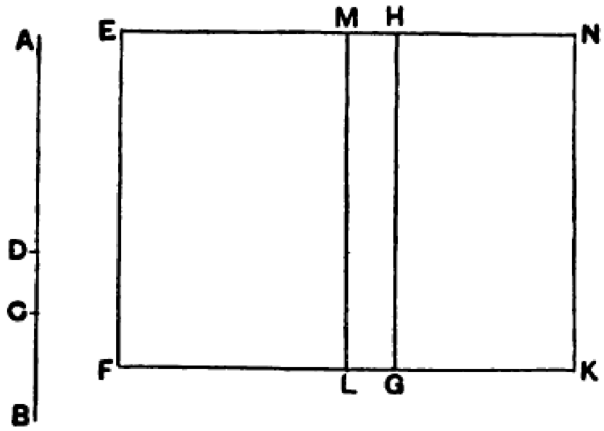
THEREFORE,

IT IS DIVIDED AT ONE POINT ONLY.

Q. E. D.

PROPOSITION 47.

THE SIDE OF THE SUM OF TWO MEDIAL AREAS IS DIVIDED AT ONE POINT ONLY.



LET,

AB BE DIVIDED, AT *C*,

SO THAT,

AC, *CB* ARE INCOMMENSURABLE, IN SQUARE, AND
MAKE THE SUM OF
THE SQUARES, ON *AC*, *CB*, MEDIAL, AND
THE RECTANGLE, *AC*, *CB*, MEDIAL AND, ALSO,
INCOMMENSURABLE WITH THE SUM OF
THE SQUARES ON THEM;

I SAY THAT;

AB IS NOT DIVIDED, AT ANOTHER POINT,
SO AS TO FULFIL THE GIVEN CONDITIONS.

FOR, IF POSSIBLE, LET,

IT BE DIVIDED AT *D*,

SO THAT AGAIN,

AC IS OF COURSE NOT THE SAME AS *BD*,

BUT,

AC IS SUPPOSED GREATER;

LET,

A RATIONAL STRAIGHT LINE, *EF*, BE SET OUT,

AND LET,

THERE BE APPLIED, TO *EF*,
THE RECTANGLE, *EG*, EQUAL TO
THE SQUARES, ON *AC*, *CB*, AND
THE RECTANGLE, *HK*, EQUAL TO TWICE
THE RECTANGLE, *AC*, *CB*;

[II. 4]

THEREFORE,

THE WHOLE, EK, =
THE SQUARE, ON AB.

AGAIN, LET,

EL, EQUAL TO THE SQUARES, ON AD, DB,
BE APPLIED TO EF;

THEREFORE,

THE REMAINDER,
TWICE THE RECTANGLE, AD, DB, =
THE REMAINDER, MK.

AND SINCE, BY HYPOTHESIS,

THE SUM OF THE SQUARES, ON AC, CB, IS MEDIAL,

THEREFORE,

EG IS, ALSO, MEDIAL.

AND,

IT IS APPLIED TO THE RATIONAL STRAIGHT LINE, EF;

[X. 22]

THEREFORE,

HE IS RATIONAL AND
INCOMMENSURABLE, IN LENGTH, WITH EF.

FOR THE SAME REASON,

HN IS, ALSO, RATIONAL AND
INCOMMENSURABLE, IN LENGTH, WITH EF.

AND, SINCE, THE SUM OF

THE SQUARES, ON AC, CB,
IS INCOMMENSURABLE WITH TWICE
THE RECTANGLE, AC, CB,

THEREFORE,

EG IS, ALSO, INCOMMENSURABLE WITH GN,

[VI. 1, X. 11]

SO THAT,

EH IS, ALSO, INCOMMENSURABLE WITH HN.
AND THEY ARE RATIONAL;

THEREFORE,

EH, HN ARE RATIONAL STRAIGHT LINES,
COMMENSURABLE, IN SQUARE, ONLY;

[X. 36]

THEREFORE,

EN IS A BINOMIAL STRAIGHT LINE DIVIDED, AT H.

SIMILARLY, WE CAN PROVE THAT;

IT IS, ALSO, DIVIDED AT *M*. AND
EH IS NOT THE SAME WITH *MN*;

THEREFORE,

A BINOMIAL HAS BEEN DIVIDED AT DIFFERENT POINTS:

[X. 42]

WHICH,

IS ABSURD.

THEREFORE,

A SIDE OF THE SUM OF TWO MEDIAL AREAS IS NOT DIVIDED AT DIFFERENT POINTS;

THEREFORE,

IT IS DIVIDED AT ONE POINT ONLY.